Can Top Stars tell a Black-ball from a Fuzz-hole?

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Talk at "Christmas Meeting"
IFT, Madrid

Based on a several papers with D. Consoli, A. Grillo, J. F. Morales, G. Di Russo, D. Bini, T. Ikeda, P. Pani, G. Raposo, G. Sudano, C. Di Benedetto, G. Dibitetto, A. Ruiperez-Vicente, F. Riccioni, C. Gambino, V. Zevola ...
See also related work by F. Fucito, R. Poghossian, H. Poghosyan, A. Cipriani, R. Russo and the original work by Aminov, Grassi, Hatsuda and then Bonelli, Iossa, Panea-Lichtig, Tanzini, ... Fioravanti,

Plan of the Talk

- Black Holes in GR and Information Paradox
- String Theory and the Fuzz-ball Proposal
- From branes to BHs and back via quantum Seiberg-Witten curves
- BH and fuzzball perturbation theory
- Discriminating fuzz balls and other Exotic Compact Objects (ECO's) from BHs ... Top(ological) Stars and W-solitons
- Summary, conclusions and future directions

BHs in GR

BHs are very simple and symmetric objects, too simple to be compatible with ... the complexity of our Universe ...

- Singularity Theorems/Gravitational collapse: Trapped Surface $\Rightarrow$ Singularity *"Black Holes (formation) ... a robust prediction of the general theory of relativity"* (Nobel Prize 2020 to Penrose)
- Cosmic Censorship: Singularity $\Rightarrow$ Horizon
- Area Theorem: $\delta A_H \geq 0$ ... recently tested by LIGO/Virgo/KAGRA
- No Hair theorem: stationary, asymptotically flat BH's in $D = 4$ mass M, charge Q, angular momentum J (Kerr-Newman)

Black Hole Thermodynamics

Black Hole as a black body ($k_B = 1$):

$$dM = \frac{\kappa}{8\pi} dA + ...$$

$\kappa =$ surface gravity, constant on (Killing) horizon

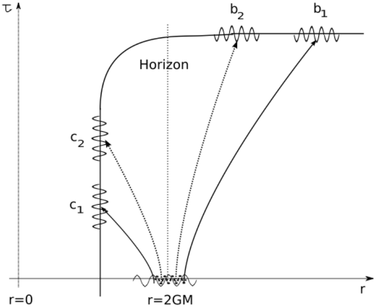
$$T_{BH} = \frac{\kappa}{2\pi} = \frac{1}{8\pi M} \quad , \quad S_{B-H} = \frac{1}{4} A$$

Negative specific heat?

$$S_{GR+QFT}(BH) = log(N_{micro-states}) = log(1) = 0 \ \ (!!??)$$

Where are the micro-states?
In GR a BH does not emit.
Semi-classically: Hawking radiation, a BH evaporates!

Information Paradox



- Pure state enters/forms a BH.
- Emitted radiation is thermal (no information), but entangled with BH.
- Emitted particles do not depend on the state of earlier produced pairs.
- BH completely evaporates: there is nothing to be entangled with.
- At the end, only radiation in a mixed state ⇒ loss of unitarity.

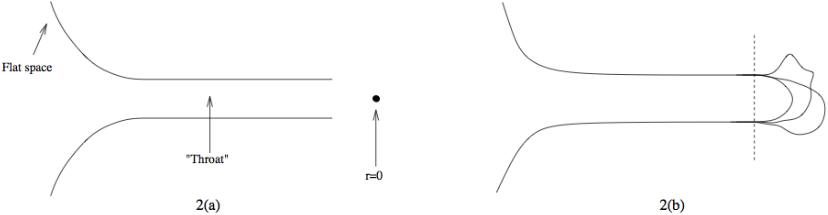
Information Paradox: Possible Resolutions

The paradox cannot be solved by adding small corrections to the semi-classical computation and information cannot be recovered at the last stages of evaporation [Lunin, Mathur, ...]

- Unitarity is lost! [Hawking]
- Remnants, Baby Universe [Susskind]
- Non Local BH-radiation interactions / Information islands [Maldacena-Susskind; Raju-Papadodimas; Almehiri, Mahajan, Maldacena, Zhao]
- Firewall / BH Butterfly effect [Almehiri, Marolf, Polchinski, Sully; Shenker, Stanford]
- Soft hairs in the asymptotic structure of space-time [Hawking, Perry, Strominger; Dvali, Gomez, Lüst], ...
- BH - string correspondence and fuzzball proposal [Horowitz, Polchinski; Damour, Veneziano; Lunin, Mathur; ... Chen, Maldacena, Witten]
- Highly Excited Strings ... chaotic scattering and decay amplitudes [Gross, Rosenhaus; MB, Firrotta, Sonnenschein, Weissman; ...]

Fuzz-ball Proposal [Lunin, Mathur; Bena, Giusto, Russo, Shigemori, Skenderis, Taylor, Martinec, Turton, Warner, ...]

(BPS) Black-Hole micro-states dual to smooth, horizon-less (super)gravity solutions ... NO singularity ... NO horizon!
Quantum Gravity effects are horizon-sized due to huge phase space. Many 'new' light d.o.f.'s. Would-be horizon carries information ... the paradox is solved.



Far away fuzz-balls resemble BHs: every micro-state same asymptotic charges (M, J, Q) as would-be BH.
micro-states start to differ from BH around $r \sim R_H$ [S. Mathur (2005)] ... breaking of effective field theory / supergravity (closed-strings) due to new light d.o.f.'s (open-strings)
Classical BH as "coarse-graining" of geometry outside would-be "horizon"

BHs in String Theory: The Naive D1-D5

BHs in string theory as bound states of intersecting branes.
E.g. 'small' BPS BH in $D = 5$ with $d(Q_1, Q_5) = \exp 2\pi\sqrt{Q_1 Q_5}$

Brane	t	x_1	x_2	x_3	x_4	y_5	y_6	y_7	y_8	y_9
$D1$	$-$	.	.	.	.	$-$	.	.	.	.
$D5$	$-$	.	.	.	.	$-$	$-$	$-$	$-$	$-$

Harmonic function superposition rule

$$ds^2 = (H_1 H_5)^{-1/2}(-dt^2 + dy_5^2) + (H_1 H_5)^{1/2}(dx_1^2 + \ldots dx_4^2)$$

$$+H_1^{1/2} H_5^{-1/2}(dy_6^2 + \ldots dy_9^2)$$

$$H_i = 1 + \frac{Q_i}{r^2} \quad , \quad r^2 = x_1^2 + ... + x_4^2$$

The D1-D5 system is U-dual to F1-P or D3-D3' or ...

Naive D1-D5-P: Strong vs Weak Coupling

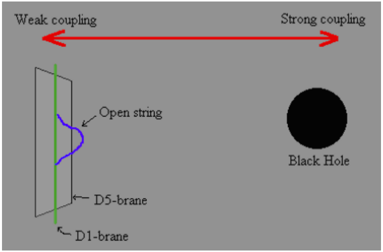
Adding KK momentum → "large" (3-charge) BH in $D = 5$

$$ds^2 = (H_1 H_5)^{-1/2}[-dt^2 + dy_5^2 + (H_P - 1)(dt + dy_5)^2]$$

$$+(H_1 H_5)^{1/2}(dx_1^2 + \dots dx_4^2) + H_1^{1/2} H_5^{-1/2}(dy_6^2 + \dots dy_9^2)$$

Strong coupling $g_s Q >> 1$: small curvature at the horizon
Macroscopic (geometric) entropy $S_{BH} = 2\pi\sqrt{Q_1 Q_5 Q_P}$ Weak Coupling $g_s Q << 1$: D-branes and open strings .
Effective $\mathcal{N} = (4,4)$ CFT with central charge $c = 6N_1 N_5$ from $(1,5)$ strings.
For large charges, C(H)ardy-Ramanujan formula:

$$d(Q_P) \sim e^{2\pi\sqrt{cQ_P/6}} \Rightarrow S_{micro} = log(d(Q_P)) = S_{MACRO}$$ [Strominger, Vafa; ...]



D1-D5 (Circular) Fuzz-ball

Exposing the micro-states

$$ds^2 = -(H_1 H_5)^{-1/2}[(dt + A_i dx^i)^2 - (dy_5 + B_i dx^i)^2]$$

$$+(H_1 H_5)^{1/2} \sum_{i=1}^{4} dx_i^2 + (H_1/H_5)^{1/2} \sum_{a=1}^{4} dy_a^2$$

$$H_1 = 1 + \frac{Q_1}{\ell} \int_0^\ell \frac{dv}{|\vec{x} - \vec{F}(v)|^2} \quad H_5 = 1 + \frac{Q_1}{\ell} \int_0^\ell \frac{dv|\dot{\vec{F}}(v)|^2}{|\vec{x} - \vec{F}(v)|^2}$$

$$A_i = \frac{Q_1}{\ell} \int_0^\ell \frac{dv \dot{F}_i(v)}{|\vec{x} - \vec{F}(v)|^2} \quad dB = \star_4 dA \quad v = t - y_5$$

E.g. circle: $F_1 = cos(2\pi v/\ell)$, $F_2 = sin(2\pi v/\ell)$, $F_3 = F_4 = 0$
Regular geometry! No horizon!
Coordinate singularity at $x^i = F^i(v)$ resolved into K-K monopole (super-tube)
Throat ends in smooth "cap": different profiles, different micro-states ('hairs').
Entropy $S_{micro} = 2\sqrt{2}\pi\sqrt{Q_1 Q_5}$ from CFT or from 'geometric quantization'
But what are the micro-states in the gravity picture?

Stringy Origin of 4d BPS Black Holes Micro-states

Enormous progress in 5-d ... 'Super-strata' [Bena, Giusto, Gibbons, Martinec, Russo, Shigemori, Warner, ...] ... with holographic support AdS_3/CFT_2!

Much less is known in 4-d ! NON-typical micro-states ... yet Top(ological) Stars [Bah, Heidmann; ...] and W-solitons [Dima, Heidmann, Melis, Pani, Patashuri; ...]

The goal: recover micro-state geometries in supergravity from string theory amplitudes

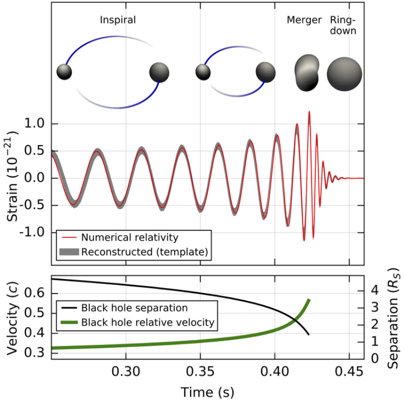
E.g. STU(R)BHs = bound-states of 4 stacks of D3-branes on T^6 in Type IIB [Cvetic, Youm; Callan, Maldacena, Strominger; Larsen; ...]

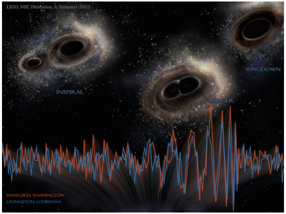
Brane	t	x_1	x_2	x_3	y_1	$\tilde{y}_1$	y_2	$\tilde{y}_2$	y_3	$\tilde{y}_3$
$D3_0$	—	.	.	.	—	.	—	.	—	.
$D3_1$	—	.	.	.	—	.	.	—	.	—
$D3_2$	—	.	.	.	.	—	—	.	.	—
$D3_3$	—	.	.	.	.	—	.	—	—	.

One-to-one relation between open-string condensates and fields in the bulk for a large class of 4d BPS BH's [Giusto, Russo, Turton ... + Bufalini; MB, Morales, Pieri;... +Zinnato, ...]

How do we ‘observe’ BHs or ECOs and fuzz-balls
BH and Fuzz-ball Perturbation Theory

Gravitational Waves from BH mergers





LIGO/Virgo collaboration (GW150914, ...)
Inspiral, Merger, Ring-down ... QNMs, ... echoes
LISA: EMRI's (Extreme Mass-Ratio Inspirals) ... Self-force approach

Event Horizon Telescope

Long baseline precision spectroscopy

'Image' of M87* ... for sure we don't see the 'horizon'
if it exists (as in GR), no signal can escape from it
if it does not (as in String Theory Fuzzball) ... we see light-rings / plasma emission (environment)

Black Hole and fuzzball perturbation theory

- Choose your preferred BH or fuzzball: $D = 4, 5, 6$, flat or (A)dS, ... and your preferred perturbation: massless/massive; spin $s = 0, 1, 2$, ...
- Linearized wave-equation (e.g. $s = 0$)

$$\Box\Phi = \mu^2\Phi$$

$$\Phi = e^{-i\omega t}e^{im_\phi\phi}R(r)S(\theta) \times e^{im_\psi\psi} \times e^{ipz} \times e^{iP_iZ^i}$$

- Separate radial (r) and angular (θ) dynamics à la Carter

$$K^2 = \ell(\ell+D-3) + ...$$

- In canonical form: Schrödinger-like equation(s)

$$\Phi''(y) + Q_{BH}(y)\Phi(y) = 0 \quad , \quad Q_{BH} \approx \omega^2 - \mathcal{V}_{BH}$$

From Regge-Wheeler-Zerilli to Teukolsky

Radial dynamics: CHE = Confluent Heun Equation

- one irregular singularity ∞: $R \sim e^{i\omega r}$
- two regular singularities $r = r_H$, $r = 0/r_-$: $R \sim (r - r_s)^\sigma$

Angular dynamics

- Schwarzschild (or RN, or Top Stars): Spherical harmonics
  $K^2 = \ell(\ell + D - 3) = K_0^2$ independent of ω
- Kerr(-Newman) or Rotating Top Stars: Spheroidal harmonics
  $K^2 = K_0^2 + \mathcal{O}(a_J \omega)$ ... CHE ... intertwined with radial

For extremal BHs $r_+ = r_-$ irregular singularity ... further confluence

Observables

- Quasi Normal Modes (QNMs): prompt $\sim$ photon-rings, late ... echoes

$$\omega_{QNM}^{WKB} = \omega_c(\ell, ...) - i(2n+1)\lambda$$

  ω_c: frequency of (un)stable circular orbits
  λ: Lyapunov exponent, exponentially growing geodesic deviation ...
  - bound on chaos $\lambda \leq 2\pi\kappa_B T_{BH}/\hbar$ [Maldacena, Shenker, Stanford]
  - violation near extremality $0 \sim 2\pi\kappa_B T_{BH}/\hbar < \lambda \leq c/b_{c,min}$ [MB, Grillo, Morales + Consoli]
- Echoes ... Long-lived 'meta-stable' QNMs from 'internal' light-rings
- Numerical methods: Leaver continuous fractions ... 'exact' results
- Expansion in series of (confluent) hypergeometric functions [Mano, Suzuki, Takasugi; ...]
  'renormalized angular momentum' $\nu = \ell + ... \equiv a_{qSW} - \frac{1}{2}$ [H. Poghosyan; ...]
- (Near) Super-radiant (N-SR) modes $\mathrm{Im}\omega \approx 0$, amplification factors $Z_{\ell,m}(\omega)$
- Grey-body factors, absorption cross actions $\sigma_{abs}(\omega)$
- Tidal Love Numbers: static $L(0)$ vs dynamical $L(\omega)$.
  In $D = 4$: $L_{BH}(0) = 0$ and $L_{BH}(\omega)$ purely imaginary ... dissipation

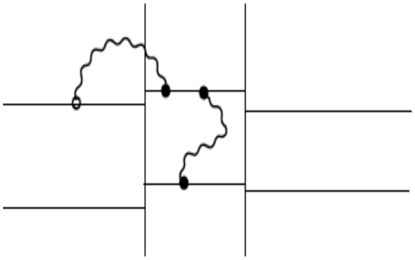
Need connection formulae ... use branes but from a different vantage point!

A novel approach to
string/brane - BH/fuzzball correspondence
... and cosmological perturbations

Hanany-Witten brane setup for $\mathcal{N} = 2$ SYM theory

$\mathcal{N} = 2$ SYM with $G = (S)U(2)$ and N_f hypers
In the Coulomb-branch: $\phi = a\sigma_3$, $q = e^{2\pi i \tau}$ ... analytic prepotential

$$\mathcal{L} = \int d^4\theta \mathcal{F}(\Phi) \quad , \quad \mathcal{F}(a; m_f; q) = \frac{1}{2} a^2 \log q + \ldots \quad , \quad u = \langle Tr\phi^2 \rangle = a^2 + \ldots$$



$N_c = 2$ colour and $N_f = (2, 2)$ flavour D4-branes suspended on NS5-branes
$\beta = 0$, yet $m_f \neq 0$, later on decoupling

Hanany-Witten for 'quantum' Seiberg-Witten

- 'Classical' Seiberg-Witten elliptic curve

$$qy^2 P_L(x) + yP_c(x) + P_R(x) = 0 \quad , \qquad q = \Lambda^\beta$$

$$P_L{=}(x{-}m_1)(x{-}m_2) \quad , \quad P_C{=}(x{-}a_1)(x{-}a_2) \quad , \quad P_R{=}(x{-}m_3)(x{-}m_4)$$

- Periods

$$a(u, m_f, q) = \oint_A \lambda_{SW} \ , \quad a_D(u, m_f, q) = \oint_B \lambda_{SW} = \frac{1}{2\pi i}\frac{\partial \mathcal{F}}{\partial a} \ , \quad \lambda_{SW} = \frac{dx}{y(x)}$$

- 'Quantize' à la Nekrasov-Shatasvili: $\varepsilon_1 = `\hbar', \varepsilon_2 = 0 \sim \Omega$ background

$$\hat{x} = \hbar y \partial_y \quad , \quad \hat{y} = y$$

- 'Quantum' SW curve ... 2nd order diff eq

$$\left[A(y)\hat{x}^2 + B(y)\hat{x} + C(y)\right] U(y) = 0$$

- NS prepotential vs instanton partition function

$$\mathcal{F}(a, m_f; q, \hbar) = \mathcal{F}_{tree} + \mathcal{F}_{1-loop} + \mathcal{F}_{inst} = \lim_{(\varepsilon_1,\varepsilon_2)\to(\hbar,0)} \varepsilon_1\varepsilon_2 \mathcal{Z}(a, m_f; q; \varepsilon_1\varepsilon_2)$$

- Decouple flavours $N_f \to N_f{-}1$ by double scaling
$q_{N_f} \to 0$, $m_{N_f} \to \infty$ with $q_{N_f-1} = q_{N_f} m_{N_f}$ fixed

Gauge/gravity encyclopedia (Radial)

Same structure as BH perturbations (Heun) ... same physics (!?)
Radial equation

- D3-branes, D1-D5 (D3-D3') small BHs, ... $N_f = (0,0)$ DRDCHE (Mathieu)
- BMPV BHs in 5-d (SUSY, extremal), ... $N_f = (0,1) \sim (1,0)$ RDCHE
- Intersecting D3's (4-charge BHs in 4-d), eKN, eSTURBHs ... $N_f = (1,1)$ DCHE
- CCLP (general 5-d charged and rotating BHs), D1-D5 (D3-D3') fuzzball (smooth, horizonless 6-d), JMaRT (smooth, horizonless 6-d), ... $N_f = (0,2) \sim (2,0)$ RCHE
- KN BHs in 4-d, STURBHs ... $N_f = (2,1) \sim (1,2)$ CHE
- KN BHs in AdS_4 with $\mu_\Phi^2 L^2 = -2$ ($\Delta = 1,2$) $N_f = (2,2)$ HE

H=Heun, E=Equation, C=confluent, R=reduced, D=doubly
All with $N_c = 2$ and $N_L, N_R \leq 2$!
Some enjoy generalized Couch-Torrence (CT) conformal inversions [MB, Di Russo; Akhond, MB, Cristofaro, Riccioni w.i.p.]

Gauge/gravity encyclopedia and (Angular) dictionary

Angular equation $\rightarrow$ 'spheroidal' harmonics

- All 4-d geometries (S^2): $N_f = (1,2)$ CHE
- All 5-d and (5-d)$\times S^1$ geometries (S^3): $N_f = (0,2)$ RCHE

More precisely

- RG-scale / instanton counting parameter $q = \Lambda^\beta \sim (\omega a_J)^\beta$, $\beta = 4 - N_f$, $a_J \sim J/M$ BH/fuzz spin
- Coulomb-branch variable $u = \langle Tr\phi^2 \rangle = a^2 + ... \sim K^2$ (Carter) separation constant
- Masses of Hypermultiplets $m_f \sim m_\phi, m_\psi$ angular momenta
- QNM quantization condition ('easy')

$$a = \hbar(\ell + \frac{D-3}{2})$$

- 'Straightforward' determination of K^2

$$\frac{1}{4}(1 + K^2) = u = -q\partial_q \mathcal{F}_{NS}(a = \ell + \frac{1}{2}, m_f, q; \hbar) = a^2 + ...$$

using 'quantum' Matone relation

Radial dictionary

- RG-scale / instanton counting parameter $q = \Lambda^\beta \sim (\omega M)^\beta$, $\beta = 4 - N_f$, M BH/fuzz mass-scale
- Coulomb-branch variable $u = \langle Tr\phi^2 \rangle = a^2 + ... \sim K^2 + \delta K^2$ shifted (Carter) separation constant, 'renormalized angular momentum' $\nu_{_{MST}} \equiv a_{_{qSW}} - \frac{1}{2}$ [H. Poghosyan; ...]
- Masses of Hypermultiplets $m_f \sim m_\phi + \delta m_\phi, m_\psi + \delta m_\psi$ shifted angular momenta (impact parameters)
- QNM quantization condition ('hard'): n 'overtone number'

$$a_D = -\frac{1}{2\pi i}\partial_{a_R}\mathcal{F}_{NS}(a, m_f, q; \hbar) = n\hbar$$

NOT straightforward: a_γ ... photon-rings ... critical geodesics .. WKB

- Invert quantum Matone relation or use 'difference' equation ($\hat{x} \leftrightarrow \hat{y}$)

$$u = -q\partial_q\mathcal{F}_{NS}(a, m_f, q; \hbar) = a^2 + ...$$

to get $a(u)$ with $u_R = u_A + \delta u_{AR}$, then plug into expression for a_D

- Solve for ω in terms of n, ℓ, m's, M, Q, J's ... compare with WKB, Leaver

The AGT picture

AGT duality [Alday, Gaiotto, Tachikawa]: 4-d $\mathcal{N}=2$ quiver theories $\sim$ 2-d Liouville CFT with $c=1+6Q^2$, $Q=b+\frac{1}{b}$, $b=\sqrt{\varepsilon_1/\varepsilon_2}$
Conformal blocks $\sim$ (ratio of) NS quiver partition functions

$$\mathcal{C}^{\alpha_1\cdots\alpha_{n+1}}_{p_0\cdots p_{n+1}}\left(\{z_i\}\right)\prod_{j=1}^{n} z_j^{-\Delta_{p_j}+\Delta_j+\Delta_{p_j+1}} = \frac{Z_{\text{inst}}\left(\{\vec{a}_i\},\{q_i\}\right)}{Z_{U(1)}(\{q_i\})}$$

Consider $SU(2)\times SU(2)$ quiver ($n=2$), 'wave-function'

$$\Psi(\{z_i\}) = \mathcal{C}^{\alpha_1\cdots\alpha_3}_{p_0\cdots p_3}\left(\{z_i\}\right)$$

For $\alpha_3=-b/2$ $\left(L_{-1}^2+b^2L_{-2}\right)V_{\alpha_3}\sim 0$ (null): BPZ equation

$$\Psi''(\{z_i\}) + b^2\sum_{i\neq 1}^{4}\left[\frac{\Delta_i}{(z-z_i)^2}+\frac{1}{z-z_i}\partial_{z_i}\right]\Psi(\{z_i\})=0$$

set $z_0=\infty$, $z_1=1$, $z_2=q$, $z_3=y$, $z_4=0$,
double scaling limit $b\to 0$ ($\varepsilon_1=0$ NS Ω-background) with fixed

$$b^2\Delta_i=\delta_i\,,\quad b^2 c_i=\nu_i\,,\quad \partial_{z_i}\Psi(y;\{z_i\})=c_i\Psi(y;\{z_i\})$$

get quantum SW curve for $\Psi(y)$ with $N_f=4\sim$ KN BH in AdS

Connection formulae from fusing and braiding

Solution near $z_1 = 1$ (‘horizon’ or ‘cap’)

$$\Psi^{(1)}(z) = \sum_{\alpha} C_{\alpha}^{(1)} F_{\alpha}^{(1)}(1-z)$$

- BHs: $C_{+}^{(1)} = 0$ no outgoing signal
- fuzzballs/ECO: $C_{-}^{(1)} = 0$ regularity / no ingoing signal

Solution near $z_2 = 0$ (‘infinity’)

$$\Psi^{(2)}(z) = \sum_{\alpha} C_{\alpha}^{(2)} F_{\alpha}^{(2)}(z)$$

For QNMs: $C_{-}^{(2)} = 0$, only outgoing
Connection formulae for D-R-D-C-HE (... so far largely unknown)

$$F_{\alpha}^{(2)}(z) = \sum_{\beta} M_{\alpha\beta} F_{\beta}^{(1)}(1-z)$$

using fusing and braiding matrices [Bonelli, Iossa, Panea-Lichtig, Tanzini; Consoli, Fucito, Morales, Poghossian; ....]

E.g. connection formulae for RCHE $\sim N_f = (2,0)$

For instance (e.g. circular fuzzball, ...), RCHE $\sim N_f = (2,0)$ qSW ($\beta_{SYM} = 2$)

$$\frac{d^2W}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1}\right)\frac{dW}{dz} + \frac{\beta z - \zeta}{z(z-1)} W = 0$$

gauge / Heun dictionary

$$u = \zeta - \beta + \frac{1}{4}(\gamma + \delta - 1) = a^2 + \dots , \quad \beta = q = \Lambda^2 , \quad \gamma = \mu_\psi \sim a_0 , \quad \delta = \mu_\phi \sim a_1$$

From $z = 0$ (regular) to $z = \infty$ (irregular) (e.g. $z = -\rho^2/a_f^2$)

$$\sqrt{\pi} z^{+\frac{1}{4}+\frac{\gamma+\delta}{2}} e^{\pm i\delta\frac{\pi}{2}} W_0(\zeta, \beta, \gamma, \delta; z) =$$

$$\sum_{\sigma=\pm} \frac{a(\zeta)\Gamma(2\sigma a(\zeta))^2\Gamma(\gamma)(e^{i\pi}\beta)^{-\frac{1}{4}-\sigma a(\zeta)} e^{-\frac{1}{2}\partial_{a_0}\mathcal{F} + \frac{\sigma}{2}\partial_a\mathcal{F}}}{\sigma\Gamma\left(\frac{\gamma-\delta+1}{2} + \sigma a(\zeta)\right)\Gamma\left(\frac{\gamma+\delta-1}{2} + \sigma a(\zeta)\right)} e^{2i\sqrt{\beta z}} W_\infty(\zeta, \beta, \gamma, \delta; \frac{1}{\sqrt{z}})$$

$+$

$$\sum_{\sigma=\pm} \frac{a(\zeta)\Gamma(2\sigma a(\zeta))^2\Gamma(\gamma)(e^{-i\pi}\beta)^{-\frac{1}{4}-\sigma a(\zeta)} e^{\frac{\sigma}{2}\partial_a\mathcal{F} - \frac{1}{2}\partial_{a_0}\mathcal{F}}}{\sigma\Gamma\left(\frac{\gamma-\delta+1}{2} + \sigma a(\zeta)\right)\Gamma\left(\frac{\gamma+\delta-1}{2} + \sigma a(\zeta)\right)} e^{-2i\sqrt{\beta z}} W_\infty(\zeta, e^{2\pi i}\beta, \gamma, \delta; \frac{1}{\sqrt{z}})$$

where $\mathcal{F}(a, q, m_f; \hbar)$ NS pre-potential

Other observables from ‘new’ connection formulae

For real ω $Q = Q^*$: conserved ‘current’ [Bonelli, Iossa, Panea-Lichtig, Tanzini; Consoli, Fucito, Morales, Poghossian]

$$\mathcal{J} = \mathrm{Im}[\Psi^* \partial_z \Psi] = k_H(|C_+^{(1)}|^2 - |C_-^{(1)}|^2) = k_\infty(|C_+^{(2)}|^2 - |C_-^{(2)}|^2)$$

$$\mathcal{J}_{abs} = -k_H |C_-^{(1)}|^2 \quad , \quad \mathcal{J}_{ref} = +k_H |C_+^{(1)}|^2$$

$$\mathcal{J}_{in} = -k_\infty |C_-^{(2)}|^2 \quad , \quad \mathcal{J}_{out} = +k_\infty |C_+^{(2)}|^2$$

- Absorption cross section / grey body factor

$$\sigma_{abs}(\omega) = \frac{|\mathcal{J}_{abs}(\omega)|}{|\mathcal{J}_{in}(\omega)|} = \frac{k_H |C_-^{(1)}|^2}{k_\infty |C_-^{(2)}|^2} \sim \exp(-a_D/\hbar)$$

- Echoes: present when reflectivity $\mathcal{R} \neq 0$ and $\mathcal{E} = \mathcal{R}/(C_+^H - \mathcal{R} C_-^H) \neq 0$

  $$G(z, z') = \theta(z - z')\psi_{in}(z)\psi_{out}(z') + \theta(z' - z)\psi_{in}(z')\psi_{out}(z) + \mathcal{E}\psi_{out}(z)\psi_{out}(z')$$

  with solutions of homogeneous eq: $\psi_{in}|_H = F_-$ and $\psi_{out}|_\infty = \tilde{F}_+$

- Amplification factor (super-radiance)

$$Z_{\ell,m,s}(\omega) = \frac{|\mathcal{J}_{out}(\omega)|}{|\mathcal{J}_{in}(\omega)|} - 1 = \frac{k_H |C_-^{(1)}|^2}{k_\infty |C_-^{(2)}|^2}$$

Intermezzo: Couch-Torrence conformal inversions

Extremal Reissner-Nordström BHs ($Q = M = r_H$), with $u = r - r_H$ symmetry under conformal inversions [Couch, Torrence]

$$u \to Q^2/u \quad , \quad ds^2_{extr} \to W(u) ds^2_{extr}$$

exchange horizon $u_H = 0$ ($r_H = Q = M$) with null infinity [Aretakis; Lu, Pope; Agrawal, Charalambous, Donnay; Akhond, MB, Cristofaro, Riccioni w.i.p.]

Photon-sphere $u_c = Q$ ($r_c = 2Q$) fixed! [MB, Di Russo]

Deflection angle: scattering = fall! Similar to "B2B" [Kälin, Porto; Di Vecchia, Russo, Veneziano; ...]

BUT no analytic continuation needed [MB, Di Russo]

$$\Delta\phi_{fall}(J,E) = \int_0^{u_i} \frac{J du}{u^2 P_u(u;J,E)} = \int_{Q^2/u_i}^{\infty} \frac{J du}{u^2 P_u(u;J,E)} = \Delta\phi_{scatt}(J,E)$$

Valid for other extremal geometries in $D \geq 4$ and for massive (BPS) probes

- D3-branes $u_c = L$
- D3-D3' (D1-D5) 'small' BHs $u_c^2 = L_3 L_{3'}$
- 4-charge STU BHs with $Q_1 Q_2 = Q_3 Q_4$ or permutations, $u_c^4 = Q_1 Q_2 Q_3 Q_4$

Can we tell a fuzzball from a (putative) BH ?

Fuzzball spectroscopy: Ringdown, QNMs, and echoes

Multi-center micro-states in $D = 4(5)$ much more involved, for STU BHs

$$ds^2 = -e^{2U(x)}(dt + \Sigma_{i=1}^3 \omega_i dx^i)^2 + e^{-2U(x)}\Sigma_{i=1}^3 dx_i^2$$

with $e^{-4U(x)} = q_{abcd}\mathcal{H}^a\mathcal{H}^b\mathcal{H}^c\mathcal{H}^d$ and $*_3 d\omega = h_{ab}\mathcal{H}^a d\mathcal{H}^b$, eight harmonic functions $\mathcal{H}^a = v^a + \sum_{I=1}^N \frac{k_I^a}{|\vec{x}-\vec{x}_I|}$ ... bubble equations [Bena, Warner; ...]

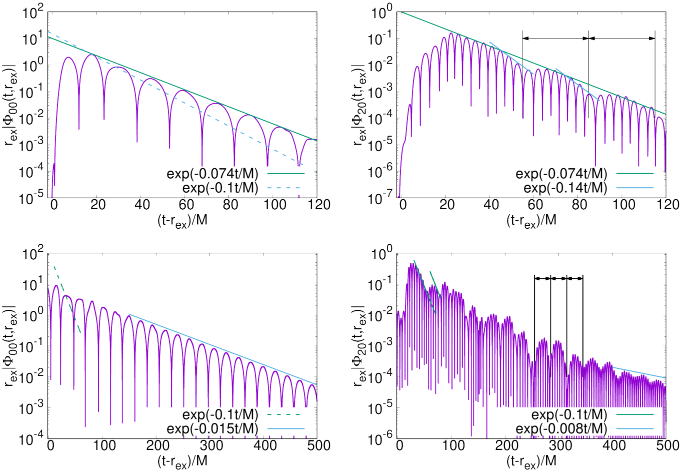
- Generic case, only one isometry $E \sim \partial_t$ ... numerical methods ... Hamiltonian Neural Network [Cipriani, De Santis, Di Russo, Grillo, Tabarroni]
- Axi-symmetric case, $J \sim \partial_\phi$ conserved, yet not separable (r, θ) vs (ρ, z) ... some progress
- Axi-symmetric case with 'equatorial' $z \leftrightarrow -z$ symmetry ... $m = \ell$ tractable ...

Time evolution of a massless scalar Gaussian shell

$$\Phi(t = 0, \vec{x}) = Ae^{\frac{(r-r_0)^2}{\sigma^2}}$$

Intricate mode mixing, issue with 4-d singularities ... numerical analysis: evidence for echoes for $N = 3$! [MB, Consoli, Grillo, Ikeda, Morales, Pani, Raposo]

Time evolution of test scalar field in 3-center micro state geometry



Modes: $\ell = 0, m = 0$ (Left) and $\ell = 2, m = 0$ (Right); $L = 0.67M$, $\kappa = 1$ (Top), and $L = 0.27M$, $\kappa = 2$ (Bottom). Gaussian shell with $\sigma = 0.67M$ (Top) and $\sigma = 0.27M$

Multipolar structure: BHs vs fuzzballs/ECOs

In GR 'no-hair theorem': Kerr(-Newman) BH $J = Ma$,
very peculiar 'spin-induced' multipolar structure [Geroch, Hanssen, Thorne]

$$\mathcal{M}_\ell + i\mathcal{S}_\ell = M(ia)^\ell \quad : \quad \mathcal{M}_{2\ell+1} = 0 \ , \quad \mathcal{S}_{2\ell} = 0$$

ECOs/fuzzballs not excluded by GW observations ... even favoured !
For Kerr-Newman $\mathcal{M}_2 = -Ma^2 < 0$ (oblate), while "Tests of General Relativity with GWTC-3" (November 2025) seems to favour $\mathcal{M}_2 > 0$ (prolate)
Generic micro-state geometries: no axial, no equatorial symmetry ... much richer and involved multipolar structure ... see plots

$$\mathcal{M}_{2\ell+1} \neq 0 \quad , \quad \mathcal{S}_{2\ell} \neq 0$$

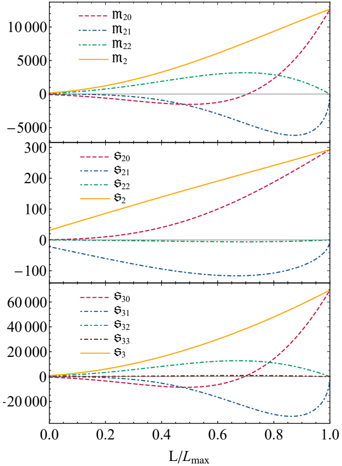
Breaking of equatorial symmetry [Bena, Mayerson; MB, Consoli, Grillo, Morales, Pani, Raposo; ...]
The big issues

- Only a (small) fraction of (a-typical) micro-states known
- Averaging procedure ... BH results (?)

... measure ... toy models: 2-charge circular fuzz balls [MB, Di Russo], general 'spin-induced' structure [Heynen, Mayerson; Gambino, Pani, Riccioni + MB]

E.g. 3-center microstates in STU SUGRA

Comparison for 'representative' $\kappa_i = (325, 751, 798, 272)$, $L_{Max} = 79.3361$



Invariants $\widehat{\mathcal{M}}_2$, $\widehat{\mathcal{M}}_{2,m}$ (top), $\widehat{\mathcal{S}}_2$, $\widehat{\mathcal{S}}_{2,m}$ (middle), $\widehat{\mathcal{S}}_3$, $\widehat{\mathcal{S}}_{3,m}$ (bottom) vs $L/L_{Max} \leq 1$

For fuzz balls: larger than Kerr for $L \sim L_{Max}$, smaller for $L << L_{Max}$.

$L \to 0$ (non-rotating) limit small but non-vanishing

Top(ological) Stars and W-solitons
... at last

Top(ological) Stars 1: basic properties

Smooth horizonless solutions of 5-D Einstein-Maxwell [Bah, Heidemann]
NON-rotating case: astonishingly simple!!!

$$ds^2 = -f_s(r)dt^2 + \frac{dr^2}{f_s(r)f_b(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + f_b(r)dy^2$$

$f_{s,b}(r) = 1 - \frac{r_{s,b}}{r}$, $y \sim y + 2\pi R_y$
'magnetic' F_2 / 'electric' H_3
Regularity condition $r_s = r_b\left(1 - \frac{4r_b^2}{R_y^2}\right)$
No thermodynamical or Gregory-Laflamme instabilities

$$r_s < r_b < 2r_s$$

'cap' at $r = r_b$ NO singularity, NO horizon, 4-d mass $(G_4 = G_5/2\pi R_y = 1)$

$$M_{TS} = \frac{1}{2}r_s + \frac{1}{4}r_b$$

Two classes: TS1 $r_b > 3r_s/2$; TS2 $r_b < 3r_s/2$
For $r_b = 0$ 'singular' solution = Schwarzschild BH times a circle.

Top Stars 2: (critical) geodesics and ISCO

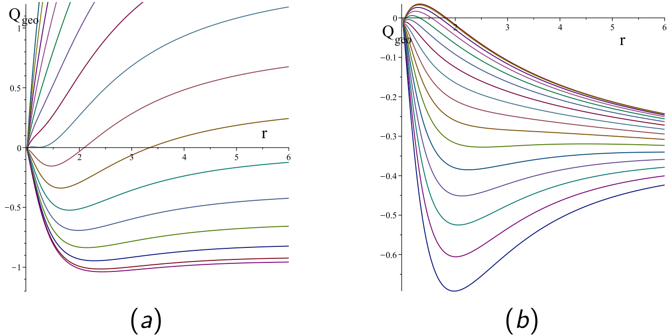
Spherical symmetry: planar (equatorial) geodesics
E, p_y, J conserved, radial momentum $P_r^2 = Q_{\rm geo}(r)$ (see plot)
Depending on impact parameter $b = J/|p_\infty|$: unbound or bound orbits
Critical geodesics $P_r = 0 = P_r'$: photon-spheres / light-rings, ...

$$r_{c1} = r_b < r_{c2} < r_{c3}$$

For massive probes ISCO at $r_0 = r_{c3} > 3r_s/2$ with

$$\Omega = \sqrt{\frac{r_s}{2r_0^3}}\,,\ \gamma = \frac{1}{\sqrt{1-\frac{3r_s}{2r_0}}}\,,\ E = \mu\frac{1-\frac{r_s}{r_0}}{\sqrt{1-\frac{3r_s}{2r_0}}}\,,\ J = \frac{\mu M}{\sqrt{\frac{r_s}{2r_0}\left(1-\frac{3r_s}{2r_0}\right)}}$$

Top Stars 3: effective potential



$P_r^2 = Q_{\rm geo}(r) = E^2 - V_{\rm eff}(r)$ ($M_{TS} = 0.65$) with
$2r_s = 1.6 > 3r_s/2 = 1.2 > r_b = 1 > r_s = 0.8$, $p_y = 1/4$, $\mu = 1$
(a) fixed $J = 3$, varying E (b) fixed $E = 0.8$, varying J

Top Stars 4: Remarkable results

Various approaches ... perfect agreement:

- Two branches of stable ($\mathrm{Im}\omega < 0$!!!) QNMs:
  - 'prompt' ring-down $\sim$ BH: $\omega_{\mathrm{prompt}} = \omega_c(\ell) - i(2n+1)\lambda$
  - long-live 'metastable' $\neq$ BH: $\omega_{\mathrm{meta}} = \omega_{int}(\ell) - ie^{-\mathcal{S}}$
- Non-zero (static) tidal deformability $\neq$ BH:

$$\mathcal{L}(\omega) = \frac{\Gamma(-2a)\Gamma(\frac{1}{2}+m_1+a)\Gamma(\frac{1}{2}+m_2+a)}{\Gamma(2a)\Gamma(\frac{1}{2}+m_1-a)\Gamma(\frac{1}{2}+m_2-a)} \cdot e^{\partial_a \mathcal{F}_I(a)}$$

  $a \sim \ell + ...$ 'renormalized' angular momentum, poles $\sim$ QNMs

- 'Self-force' $\sim \mu << M$: massless (scalar) radiation from bound ('elliptic'/'circular' EMRI) and unbound 'hyperbolic' orbits [MB, D. Bini, G. Di Russo]

Built-in PN and PM expansions: $\omega r \sim v$, $M/r \sim v^2$, $\omega M \sim v^3$

Rotating Top Stars

... not a free lunch ... simple version with 'twist' [MB, Dibitetto, Morales, Ruiperez]

$$ds^2 = -\left(1 - \frac{r_s r}{\Sigma} + r_s r_b a_s^2 \frac{\chi^2}{\Sigma^2}\right) dt^2 + \left(1 - \frac{r_b r}{\Sigma} - r_s r_b a_b^2 \frac{\chi^2}{\Sigma^2}\right) dy^2 + \Sigma\left(\frac{dr^2}{\Delta} + \frac{d\chi^2}{1-\chi^2}\right)$$

$$+\frac{1-\chi^2}{\Sigma}\left[\left(r^2+a^2\right)^2 - a^2\left(1-\chi^2\right)\Delta + r_s r_b a^2 \frac{r^2\left(1-\chi^2\right)}{\Sigma}\right] d\phi^2 + 2 r_s r_b a_s a_b \frac{\chi^2}{\Sigma^2} dt dy$$

$$-2\frac{1-\chi^2}{\Sigma}\left[r_s a_s\left(r - r_b a^2 \frac{\chi^2}{\Sigma}\right) dt d\phi - r_b a_b \left(r - r_s a^2 \frac{\chi^2}{\Sigma}\right) dy d\phi\right]$$

$$A = \sqrt{3 r_s r_b}\frac{\chi}{\Sigma}\left(a_s dt - \left(r^2+a^2\right) d\phi - a_b dy\right)$$

where $\chi = \cos\theta$, $\Sigma = r^2 + a^2\chi^2$, $\Delta = (r - r_s)(r - r_b) + a^2 = (r - r_+)(r - r_-)$
with $r_\pm = \frac{r_b + r_s}{2} \pm \sqrt{\left(\frac{r_b - r_s}{2}\right)^2 - a^2}$
Relations among the parameters

$$a^2 = a_s^2 - a_b^2 = a_s^2\left(1 - \frac{r_b}{r_s}\right) = -a_b^2\left(1 - \frac{r_s}{r_b}\right)\ ,\ a_s^2 = -\frac{a^2 r_s}{r_b - r_s}\ ,\ a_b^2 = -\frac{a^2 r_b}{r_b - r_s}$$

‘Phase diagram’

- Naked singularity (NS): $r_s > r_b$, $a^2 > (\frac{r_s - r_b}{2})^2$.
- Rotating black string (RBS): $r_s > r_b$, $0 < a^2 \leq (\frac{r_s - r_b}{2})^2$.
- Rotating Topological Star (RTS): $r_b > r_s$, $a^2 < 0$.
- Extreme rotating black string (ERBS): $r_s = r_b$, $a^2 = 0$.

Regularity at the cap

$$t' = t - \xi^t y \, , \; \phi' = \phi - \xi^\phi y \, , \; y' = y \, , \; \varrho^2 = 4(r - r_+)$$

reduction along $\xi = \partial_{y'}$ ... identifications $\phi' \sim \phi' + 2\pi$, $y' \sim y' + 2\pi R_y$

$$R_y = \frac{2\, r_b^{3/2}\, (r_+ - r_s)}{\sqrt{r_b - r_s}\, (2r_+ - r_b - r_s)}$$

Asymptotic geometry

To avoid time-like identifications ... boost $(0 < \xi^t < 1)$

$$\hat{t} = \frac{t - \xi^t\, y}{\sqrt{1-\xi^{t2}}}\,,\; y = \frac{y - \xi^t\, t}{\sqrt{1-\xi^{t2}}}$$

with $\xi^t = \frac{a_s\, r_s}{a_b\, r_b}\, \frac{r_+ - r_b}{r_+ - r_s} = \frac{r_s^{3/2}(r_+ - r_b)}{r_b^{3/2}(r_+ - r_s)}$
'Twisted' identifications

$$(\phi, \hat{y}) \sim (\phi + 2\pi, \hat{y}) \sim \left(\phi + 2\pi R_y\, \xi^\phi, \hat{y} + 2\pi R_y \sqrt{1-\xi^{t2}}\right)$$

with $\xi^\phi = -\frac{r_+ - r_b}{a_b\, r_b} = -\sqrt{\frac{(r_+ - r_b)(r_b - r_s)}{r_b^3 (r_+ - r_s)}}$, $R_y \xi^\phi = -\frac{h}{q}$ with $q > h > 0$ coprime
and

$$R^2_{S^1} = \frac{c_+\, r_b^3 - c_- r_s^3}{r_b - r_s} > r_s \quad , \quad c_\pm = 2q^2 - h^2 \pm 2\sqrt{q^2 - h^2}\,.$$

NO scale separation ... yet stability, deformability ... CHE strikes back
... Untwisting possible starting from Kerr-Bolt [Heidmann, Pani, Santos; Bena, Lochet]
$2a = k(R_+ - r_-)$, $2P_0 = NR_y$, $k\omega_y(r_+) = \ell R_y$

W-solitons ... the Great Impersonators

Prototypical BH micro-states [Dima, Heidmann, Melis, Pani, Patashuri]

$$M = \frac{4k^2 + N^2}{8\sqrt{2}k} \quad Q = \frac{N}{2} R_y \quad 2k \geq |N|$$

$|Q| \leq \sqrt{2}M$, $M \geq \frac{R_y}{2\sqrt{2}}$ (parametrically larger for $k >> |N|$)
Pro's (wrt Top Stars)

- Spin structure (local Taub-bolt geometry)
- Non-singular / smooth extremal limit (GH center)
- $M_4 = M_5$ since $G_{yy} \approx 1 - \frac{4M^2}{r^2} + ...$

Neutral case $Q = 0 \sim$ Schwarzschild as $r \to \infty$ (far region)

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} + f(r)\, r^2 (d\theta^2 + \sin^2\theta d\phi^2)\,,$$

with

$$f(r) = \sqrt{1 - \frac{4M}{r}} \sim 1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\,.$$

Good candidates for Schwarzschild BH micro-states!!!

W-soliton: source and 5-D uplift

Sourced by complex scalar field

$$z = \frac{1}{\left(1 - \frac{2M}{r}\right)} \left(\frac{2M}{r} + i\sqrt{1 - \frac{4M}{r}} \right) ,$$

with non-canonical kinetic term $\mathcal{L} = -\frac{3\partial z \cdot \partial \bar{z}}{4\mathrm{Im} z^2}$
Non-singular 5-D uplift $f_s = 1 - \frac{2M}{r}$, $f_4 = 2f_s - 1$ 'cap' at $r = 4M$

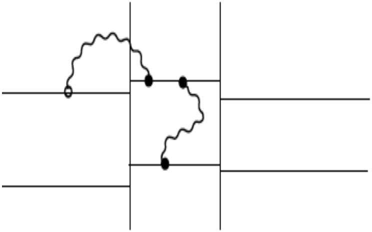
$$ds^2 = -f_s\, dt^2 + \frac{f_s\, dr^2}{2f_s - 1} + f_s\, r^2 d\Omega_2^2 + \frac{(2f_s - 1)}{f_s^2} dy^2$$

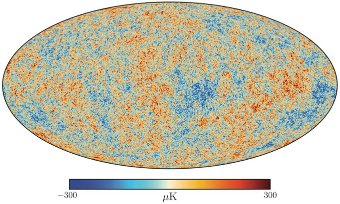
- (Critical) geodesics and single unstable light-ring (like BHs)
- (Scalar) QNMs
- (Scalar) self-force
- Energy and Angular momentum loss

[MB, Bini, Di Russo]

Take-home message

What do they have in common?





Everything ... if M87* were a Kerr BH or a Rot Top Star (in AdS) and the Universe were filled in only with matter and radiation

Everything ... if M87* were a Kerr BH or a Rot Top Star (in AdS) and the Universe were filled in only with matter and radiation

Not much if M87* is a fuzzball or an Exotic Compact object and the Universe contains also dark energy

Conclusions and Outlook

- BH (and cosmological!) perturbations and qSW curves are one and the same thing ... 'true' fuzzballs/ECOs and our 'complex' Universe are another story.
- Mathematics? Physics? Duality between M5's and M2's: wrapping M5 on $\mathbf{R}^4_\Omega \times \Sigma$ AGT correspondence ... wrapping M2 on Σ (BPS) BH ...
- Can we discriminate fuzz-balls from BHs? YES! Multipolar structure, Ring-down ... echoes, tidal Love numbers ... quadratic QNMs [MB, Grassi, Mauri w.i.p.]
- How to construct and superpose 'typical' micro states? Scale separation?
- 3-charge superstrata [Bena, Giusto, Russo, Shigemori, Warner] NOT integrable in general
- 4-charge multi-center, breaking of equatorial symmetry [Bena, Mayerson, ..], keeping axial symmetry may help ...
- Neutral micro-states (with dipole moments) [Bah, Heidmann, Weck; Dima, Heidmann, Melis, Pani, Patashuri] ... very promising W-solitons
- Rotating Top(ological) Stars ... positive $\mathcal{M}_2$