Low-T Spectrum for Charge Diffusion in Holographic Matter

Clément Supiot

Centre de PHysique Théorique (CPHT)

June 18$^{\text{th}}$, 2025

[Goutéraux, Sanchez-Garitaonandia, Ramirez, CS (2506.11974)]

AdS Extremal Black Hole

AdS_{d+2}

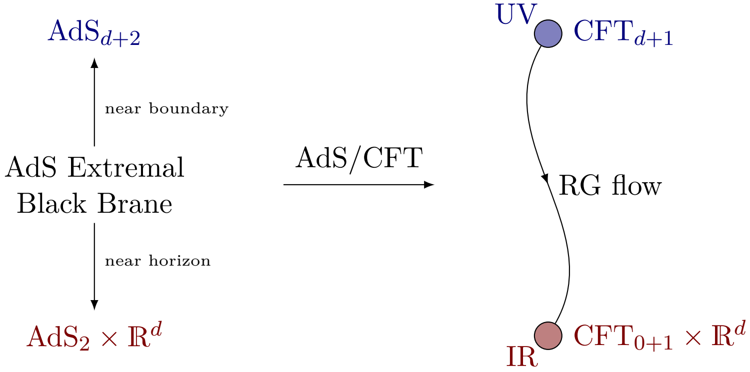
↑ near boundary

AdS Extremal
Black Brane

↓ near horizon

$\mathrm{AdS}_2 \times \mathbb{R}^d$

AdS Extremal Black Hole and RG Flow



Holographic Setup: A Simple Case

4D gravity with massless scalars

$$S_{\mathrm{bg}} = \int \mathrm{d}x^4 \sqrt{-g} \left[R - 2\Lambda + \sum_{i=1}^{2} (\partial \psi_i)^2 \right]$$

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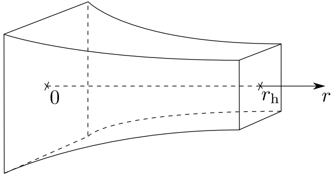

$$\mathrm{d}s^2 = \frac{1}{r^2}\left[-f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + \mathrm{d}x^2 + \mathrm{d}y^2\right]$$

$$r \in (0, r_{\text{h}}]$$

$$f(r) = 1 - \frac{m^2 r^2}{2} - \left(1 - \frac{m^2 r_{\text{h}}^2}{2}\right)\frac{r^3}{r_{\text{h}}^3}, \qquad \psi_i = m x^i$$

[Andrade, Withers '14]

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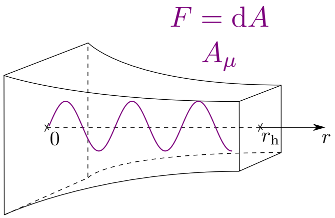

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$U(1)$ perturbation

$$S = -\frac{1}{4}\int \mathrm{d}x^4 \sqrt{-g} F^2$$

$\overset{\text{dual}}{\longleftrightarrow}$

Conserved current

$$\partial_\mu J^\mu = 0$$

Hydrodynamics and the Charge Diffusion Pole

Long-time, long-distance effective description at $T \neq 0$

$$G^R_{\mu\nu}(x-y) = -i\theta(x^0-y^0)\langle[J_\mu(x), J_\nu(y)]\rangle$$

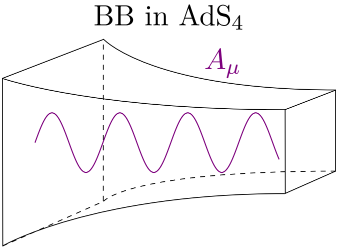
Fourier space: $\omega, k \ll T$, $\boldsymbol{k} = k\boldsymbol{e}_x$

$$\text{Longitudinal:} \quad G^R_{xx} \propto \left(-i\omega + Dk^2 + \mathcal{O}(k^4)\right)^{-1}$$

Charge Diffusion Pole [Kovtun, Starinets '05]

$$\omega = -iDk^2 + \mathcal{O}(k^4)$$

IR Spectrum: a Tower of Critical Modes



Quasi-Normal Mode Spectrum

$\rightarrow$ Ingoing at horizon

$\leftarrow$ Vanishing at boundary

$\Rightarrow$ Complex frequency spectrum

$\underset{\text{AdS/CFT}}{\Leftrightarrow}$ Poles of $G^R_{\mu\nu}(\omega, k)$

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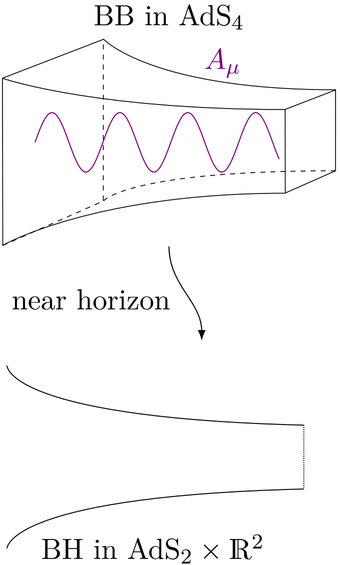

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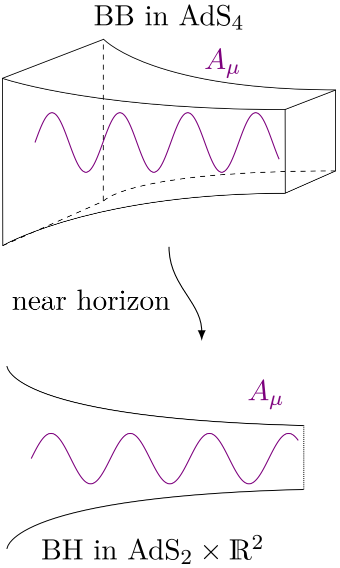

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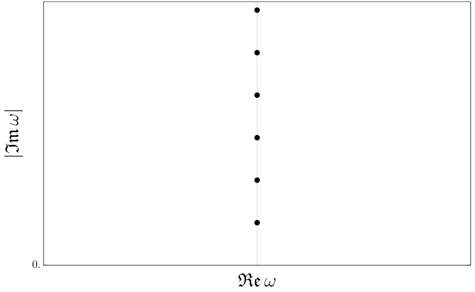

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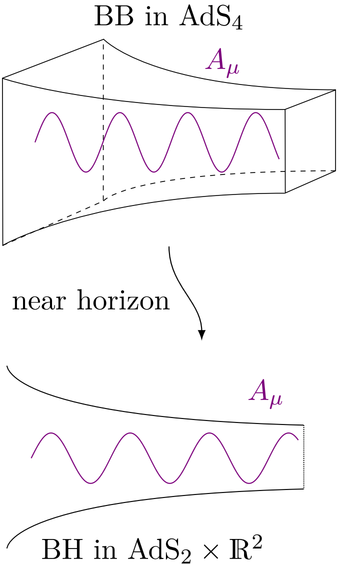

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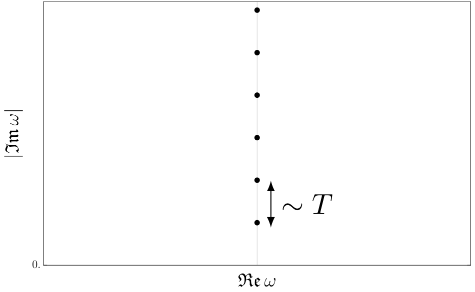
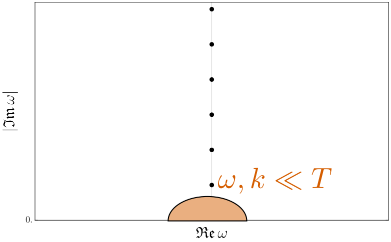


Hydrodynamics

◦ $\omega, k \ll T$

Charge Diffusion Pole

$$\omega = -iD(T)k^2 + \mathcal{O}(k^4)$$



Hydrodynamics vs Zero Temperature

- $\omega, k \ll T$

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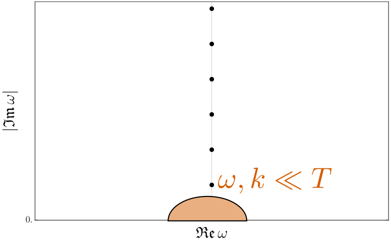

- $\omega, k \ll r_{\mathrm{e}}^{-1}, T = 0$

[Edalati et al. '09] [Davison, Parnachev '13]

Quadratic Pole

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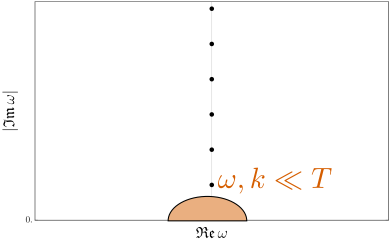

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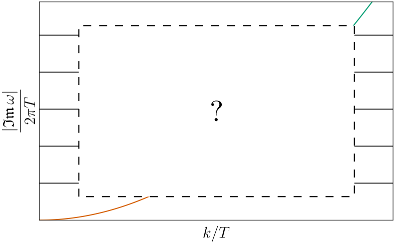


Two regimes not continuously connected

Try to understand diffusion, critical and quadratic modes simultaneously

Equation of Motion for the Longitudinal Sector

To answer this, let's compute G^R_{xx}

Probe perturbation

$$S = -\frac{1}{4}\int \mathrm{d}x^4 \sqrt{-g}F^2 \quad \Rightarrow \quad \nabla_\mu F^{\mu\nu} = 0.$$

$$A_z = 0$$
$$A_\mu = a_\mu(r)\mathrm{e}^{-i\omega t + ikx}$$

Gauge invariant combinations

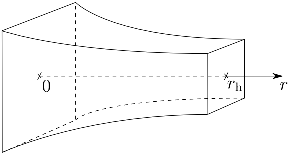
$$E_x = \omega a_x + k a_t$$
$$E_y = \omega a_y$$

E_x, E_y decoupled

Longitudinal EoM

$$\left[\frac{f E_x'}{\omega^2 - k^2 f}\right]' + \frac{E_x}{f} = 0$$

Solve the EoM: Matching Procedure [Davison, Parnachev '13]



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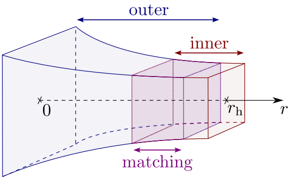

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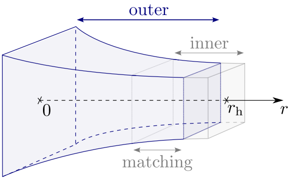

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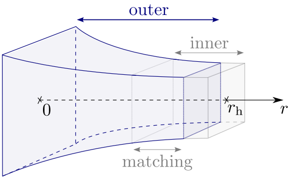

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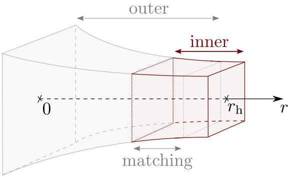

▷ $r_\mathrm{h} - r \ll r_\mathrm{h} \lesssim r_\mathrm{e}$

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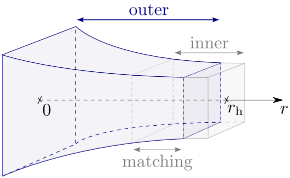

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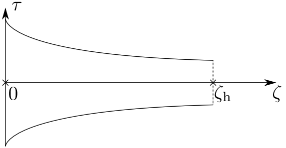

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Zoom in near horizon [Faulkner et al. '09]

$$r = r_\mathrm{e} - \epsilon \frac{r_\mathrm{e}^2}{3\zeta}, \quad t = \epsilon^{-1}\tau$$

$$\mathrm{d}s^2 \underset{\epsilon\to 0}{=} \frac{1}{3\zeta^2}\left[-f_2 \mathrm{d}\tau^2 + \frac{\mathrm{d}\zeta^2}{f_2}\right] + \frac{1}{r_\mathrm{e}^2}\left[\mathrm{d}x^2 + \mathrm{d}y^2\right] + \mathcal{O}(\epsilon)$$

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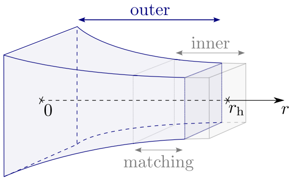

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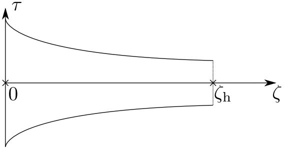

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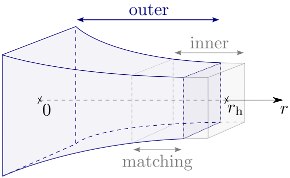

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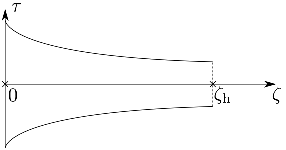

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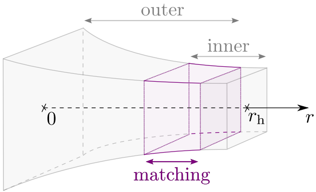
Scaling for quadratic mode beyond hydrodynamics

$$\epsilon \sim k^2 \sim \omega \sim T \ll m$$

Matching and G^R_{xx}



Leading order matching

$$G^R_{xx} = \frac{\omega^2 + \mathcal{O}(\epsilon^3)}{-i\omega + r_{\mathrm{e}} k^2 + \mathcal{O}(\epsilon^2)}$$

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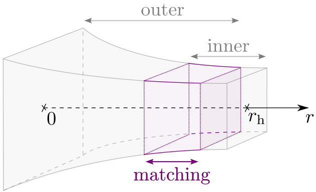

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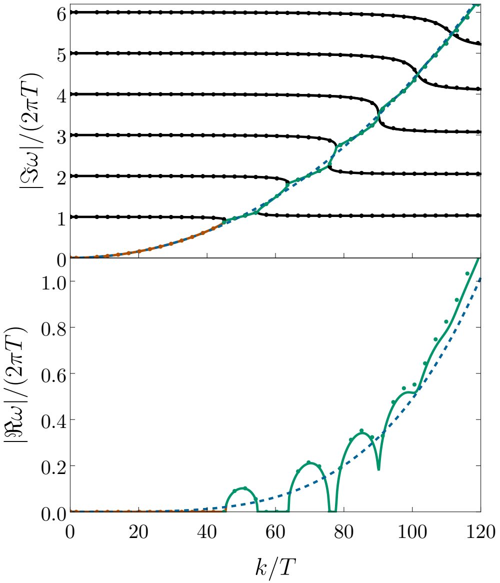
Next-to-leading order matching

$$G^R_{xx} = \frac{\omega^2\left(1 - \frac{k^2 r_\mathrm{e}^2 \mathcal{G}}{3}\right)}{i\omega - k^2 r_\mathrm{e} + \frac{k^2 r_\mathrm{e}}{6}\left(4\pi T r_\mathrm{e} + 2(k^2 r_\mathrm{e}^2 + i\omega r_\mathrm{e})\mathcal{G} + 3k^2 r_\mathrm{e}^2 \log 3\right)}$$

$$\mathcal{G} = \pi \cot\left(\frac{i\omega}{2T}\right) + \gamma + \psi\left(\frac{i\omega}{2\pi T}\right) - \log\left(\frac{9}{4\pi T r_\mathrm{e}}\right)$$

Results: Spectrum of the Longitudinal Sector



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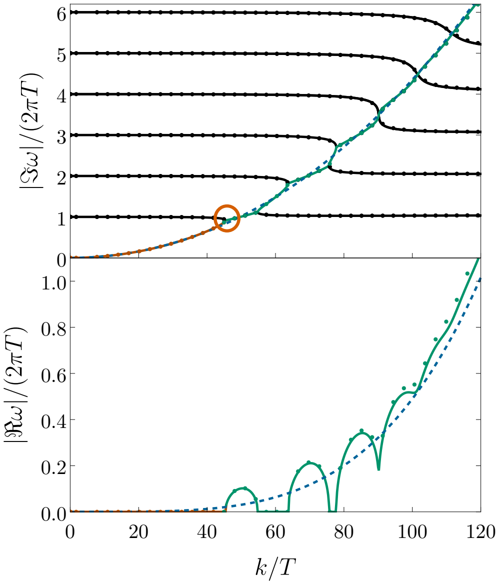

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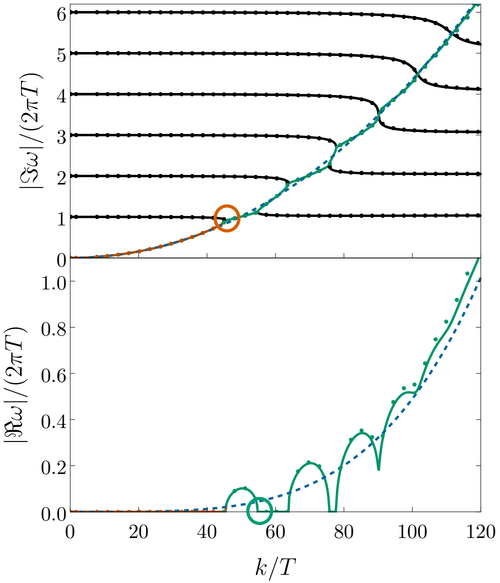

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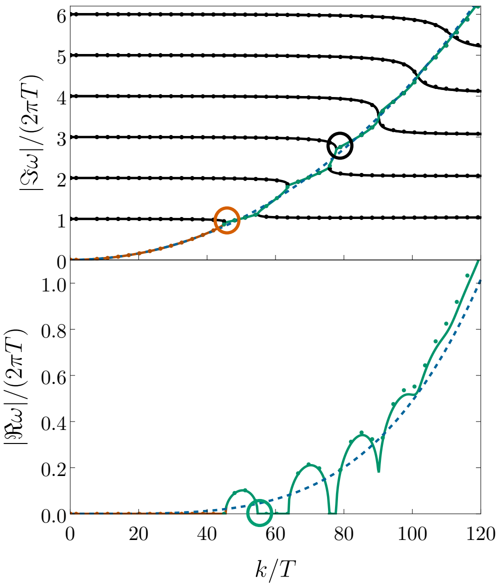

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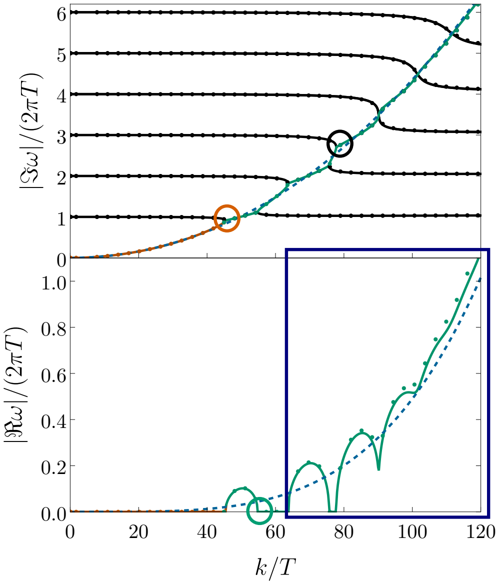

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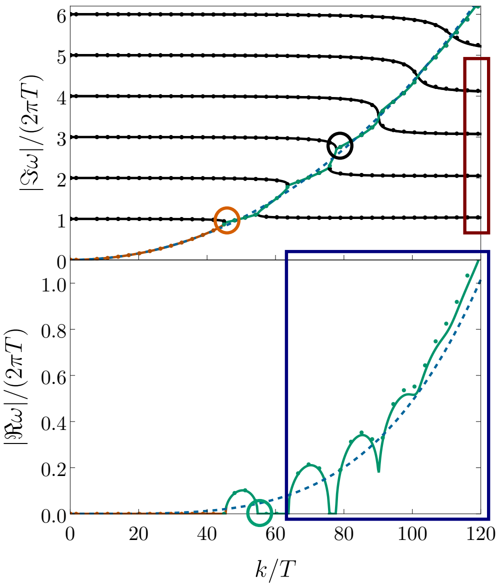

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- □ Spectrum Rearrangement

Conclusion

Summary

- Analytic Green function for $\omega \sim T$
- Irrelevant deformations are crucial
- Low-T quadratic mode very different from Diffusion mode beyond hydrodynamics
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End

Appendix: Outer Region

$$\left[\frac{f E_x'}{\omega^2 - k^2 f}\right]' + \frac{E_x}{f} = 0$$

$$\Downarrow$$

Formal Solution

$$E_x = \mathcal{E}_0 + \frac{\mathcal{E}_1}{k^2 - \omega^2}\left(k^2 r - \omega^2 \int_0^r \frac{\mathrm{d}r_1}{f(r_1)}\right)$$
$$+ \int_0^r \mathrm{d}r_1 \left(k^2 - \frac{\omega^2}{f(r_1)}\right) \int_0^{r_1} \mathrm{d}r_2 \frac{E_x(r_2)}{f(r_2)}$$

$$k^2 \sim \omega \sim T \sim \epsilon \quad E_x^{(\mathcal{O})} = \sum_{n\geq 0} \epsilon^n E_{x,n\geq 0}^{(\mathcal{O})} \quad \mathcal{E}_i = \sum_{n\geq 0} \epsilon^n \mathcal{E}_i^{(n)}$$

Appendix: Inner Region

$$(r, t) \to (\zeta, \tau), \quad k^2 \to \epsilon k^2, \quad E_x^{(\mathcal{I})} = \sum_{n \geq 0} \epsilon^n E_{x,n\geq 0}^{(\mathcal{I})}$$

$$\mathcal{D}E_{x,0}^{(\mathcal{I})} \equiv \left[\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} - \frac{2}{\zeta_\mathrm{h}} \frac{1}{f_2(\zeta)} \frac{\mathrm{d}}{\mathrm{d}\zeta} + \frac{\omega^2}{f_2(\zeta)^2}\right] E_{x,0}^{(\mathcal{I})} = 0$$

$$\mathcal{D}E_{x,n\geq 0}^{(\mathcal{I})} = \mathcal{S}^{(n)}[E_{x,n-1}^{(\mathcal{I})}, \ldots E_{x,0}^{(\mathcal{I})}]$$

$$E_{x,0}^{(\mathcal{I})} = c_+^{(0)} \mathrm{e}^{i\omega\zeta_\mathrm{h} \operatorname{arctanh} \zeta/\zeta_\mathrm{h}} + c_-^{(0)} \mathrm{e}^{-i\omega\zeta_\mathrm{h} \operatorname{arctanh} \zeta/\zeta_\mathrm{h}}$$

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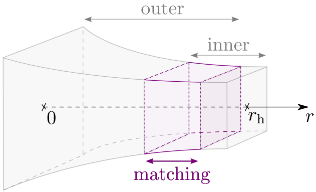
Tortoise coordinates

$$r^\star = \int \frac{\mathrm{d}r}{f(r)} = \epsilon^{-1} \zeta_\mathrm{h} \operatorname{arctanh} \zeta/\zeta_\mathrm{h} + \mathcal{O}(\epsilon^0)$$

Appendix: Matching and G^R_{xx}



$$r = r_\mathrm{e} - \epsilon \frac{r_\mathrm{e}^2}{3\zeta} \Rightarrow \text{Mixing of orders}$$

$$\sum_{n=0}^{1} \epsilon^n E^{(\mathcal{O})}_{x,n} \underset{\epsilon \to 0}{=} \mathcal{E}^{(0)}_0 + \mathcal{E}^{(0)}_1 r_\mathrm{e} - \mathcal{E}^{(0)}_1 \frac{\omega^2}{k^2} \zeta$$

$$E^{(\mathcal{I})}_{x,0} \underset{\omega\zeta \ll 1}{=} c^{(0)}_+ + c^{(0)}_+ i\omega\zeta$$

Leading order matching

$$G^R_{xx} = \frac{\omega^2 + \mathcal{O}(\epsilon)}{-i\omega + r_\mathrm{e} k^2 + \mathcal{O}(\epsilon)}$$