# Pseudospectra of ℂMMs

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Based on

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**New Insights in Black Hole Physics from Holography**

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# Pseudospectra of QNMs

# Setup

$\mathrm{SAdS}_5$ black brane

$$ds^2 = \frac{1}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + d\vec{x}^2\right) \qquad f(z) = 1 - z^4$$

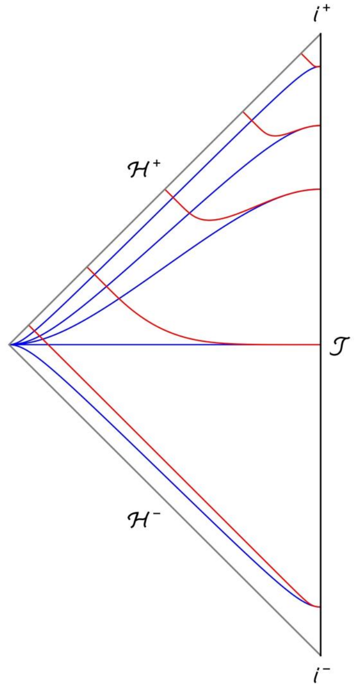
QNMs of a massless scalar field: solutions of $\nabla^2\phi = 0$ with fixed $(\omega, k)$ that are

- Infalling at $z = 1$
- Normalizable at $z = 0$ (sourceless)

# QNMs as eigenfunctions of a Non-Hermitian Operator

By choosing to work in regular coordinates coordinates

$$ds^2 = \frac{1}{z^2}(-f d\tau^2 + (2-f)dz^2 - 2(1-f)d\tau dz + d\vec{x}^2)$$




where

Infalling bcs $\Leftrightarrow$ Regularity at $z = 1$

we can write $\nabla^2\phi = 0$ as

$$\omega\begin{pmatrix}\phi\\ \psi\end{pmatrix} = L\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}0 & i\\ \mathcal{L}_1 & \mathcal{L}_2\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}$$

QNFs $\Leftrightarrow$ $\sigma(L)$

$\psi$ is an auxiliary field ($\psi = -i\omega\phi$)

# QNMs as eigenfunctions of a Non-Hermitian Operator

Defining an inner product such that the norm matches the energy of the QNM evaluated on a $\tau =$ constant surface

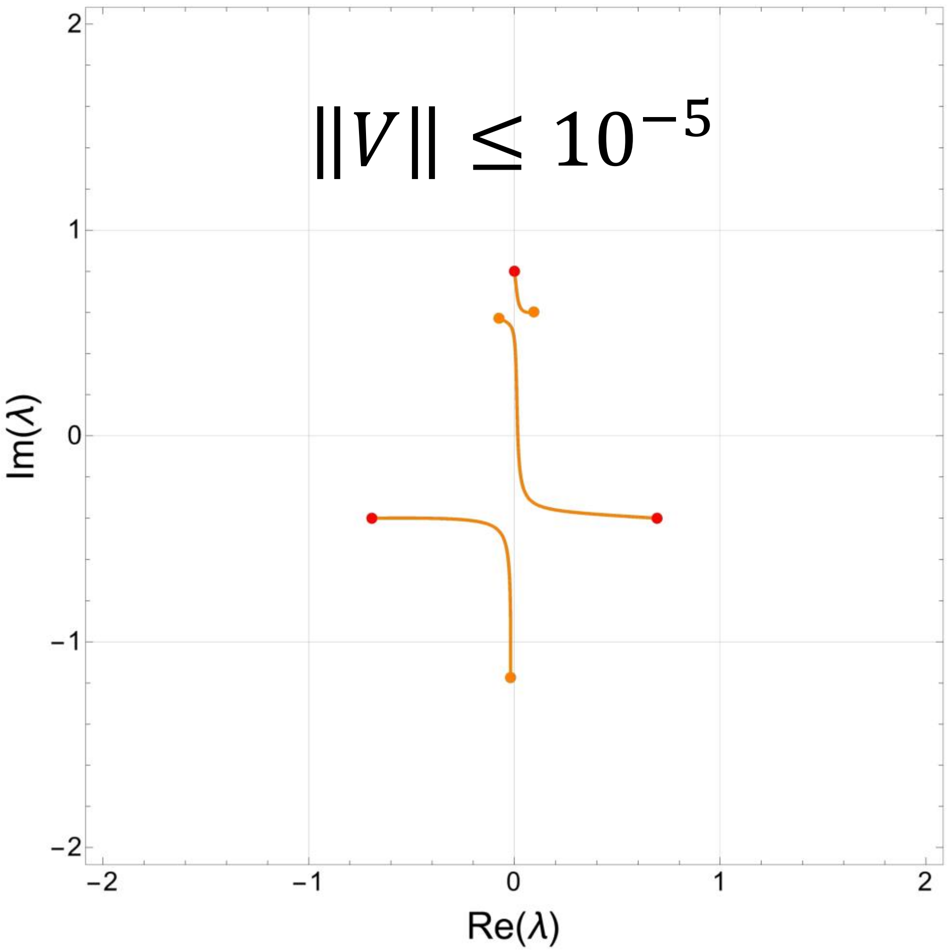
$$L^{\dagger} = L + \#\,\delta(1-z)$$

Horizon $\Rightarrow L^{\dagger} \neq L$

# Spectral (In)stability of Non-Hermitian Operators

$L^\dagger \neq L \Rightarrow \sigma(L)$ can be unstable

Even if $||V||$ is small $\sigma(L+V)$ can look very different from $\sigma(L)$

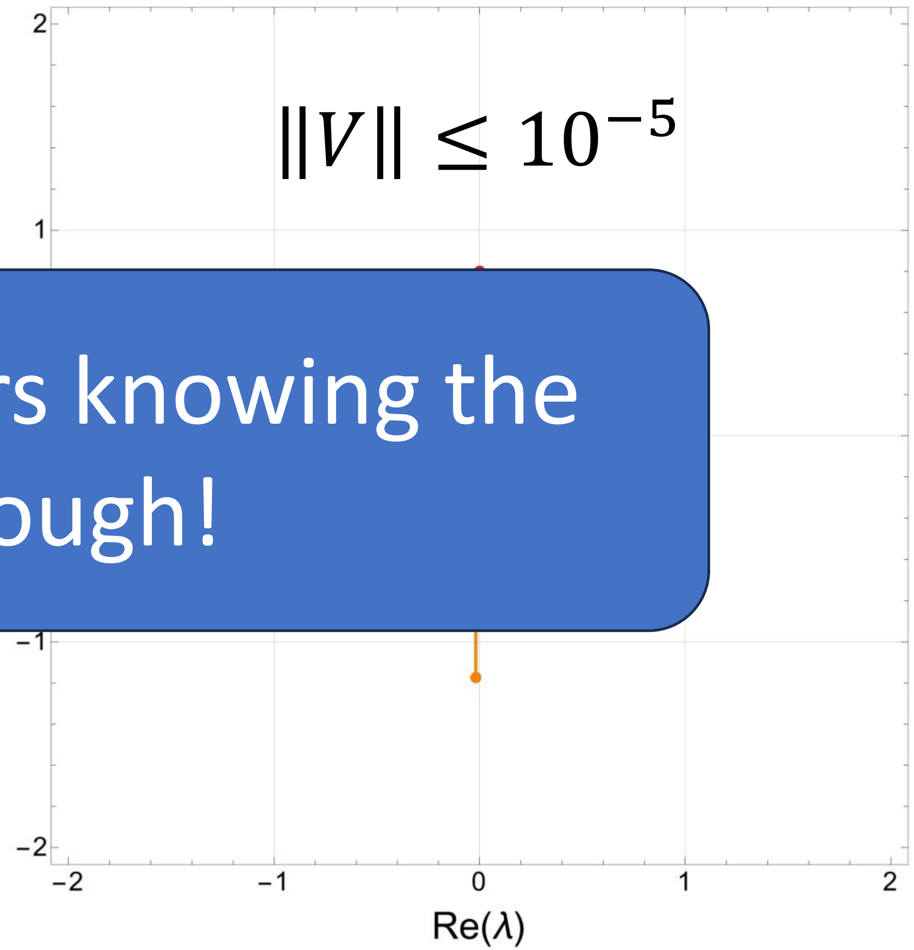


# Spectral (In)stability of Non-Hermitian Operators

$L^\dagger$

Ev

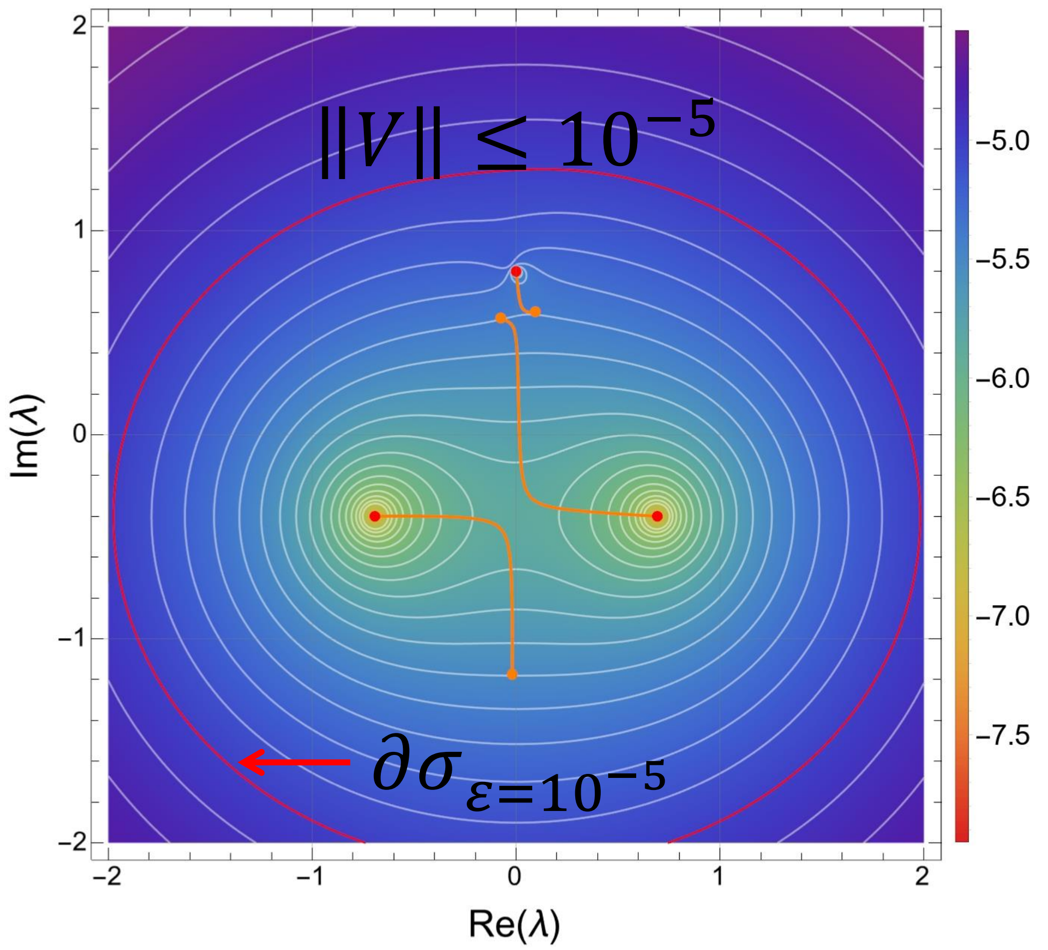
For Non-Hermitian operators knowing the spectrum is not enough!



# Pseudospectrum

$$\sigma_\varepsilon(L) = \{z \in \mathbb{C} : z \in \sigma(L+V)\,, \|V\| < \varepsilon\}$$
$$= \{z \in \mathbb{C} :\ \|(L-z)^{-1}\| < 1/\varepsilon\}$$

$\varepsilon$-pseudospectra allow us to quantify the spectral (in)stability from the knowledge of the resolvent of the operator

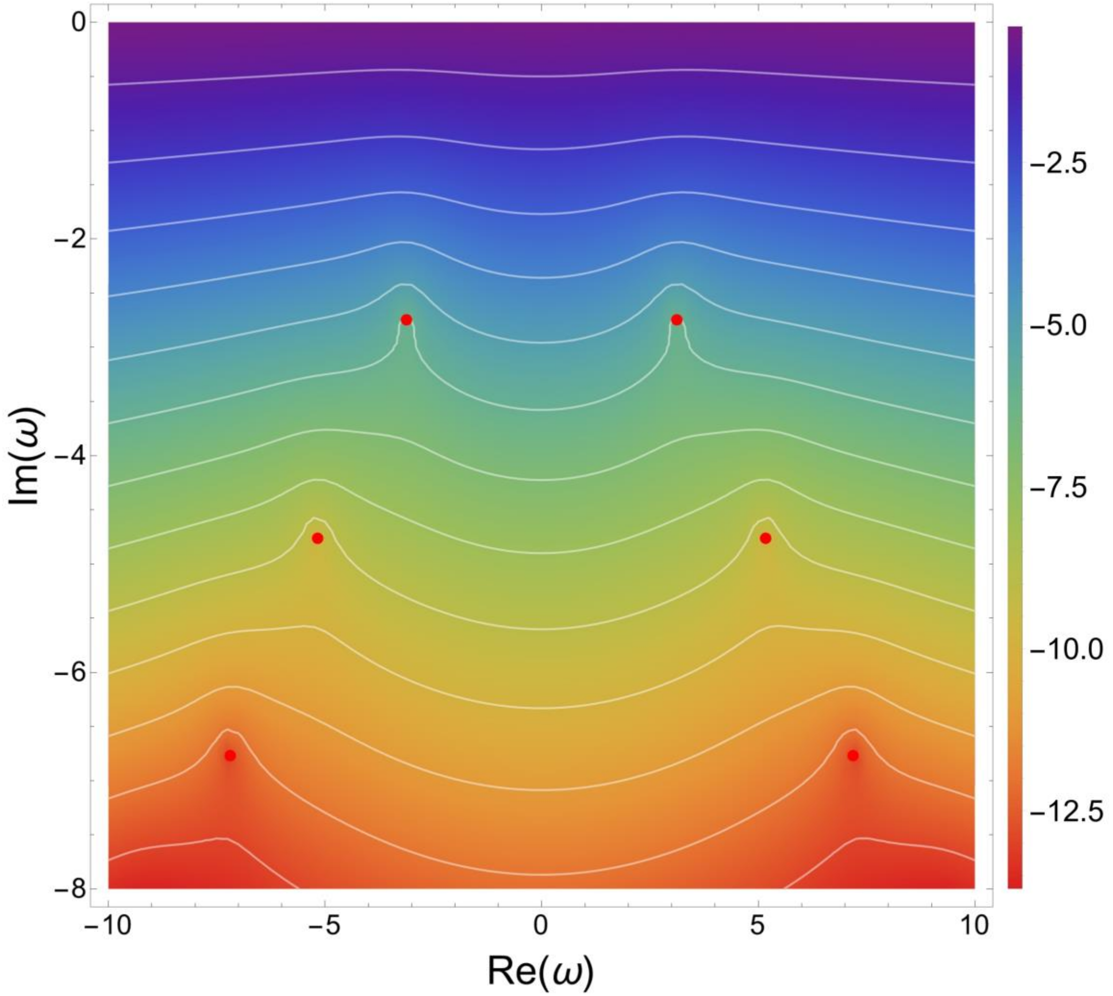


# Pseudospectrum of QNFs ($k = 0$)

[D. Areán, DGF and K. Landsteiner (2023)]

- We truncate Hilbert space by discretizing in a grid of size N
- QNMs are **unstable**
- Instability increases for higher QNMs

**Poles of retarded correlators can change a lot under small perturbations of the theory**



# Convergence

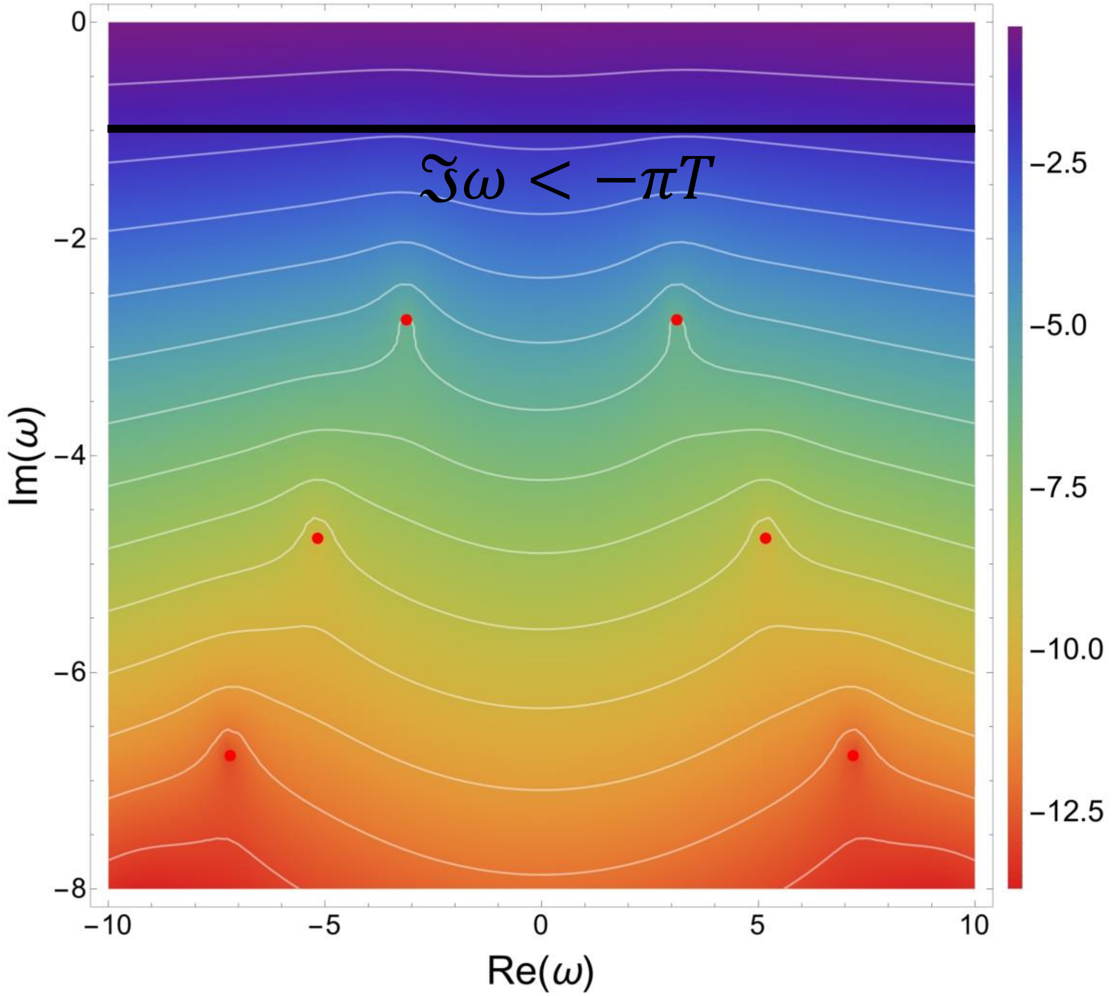
[V. Boyanov et al. (2023)]

**What happens if we increase N?**

Qualitative features **do not change**

but

Pseudospectrum **does not converge** in the region $\Im\omega < -\pi T = -1$



# Convergence

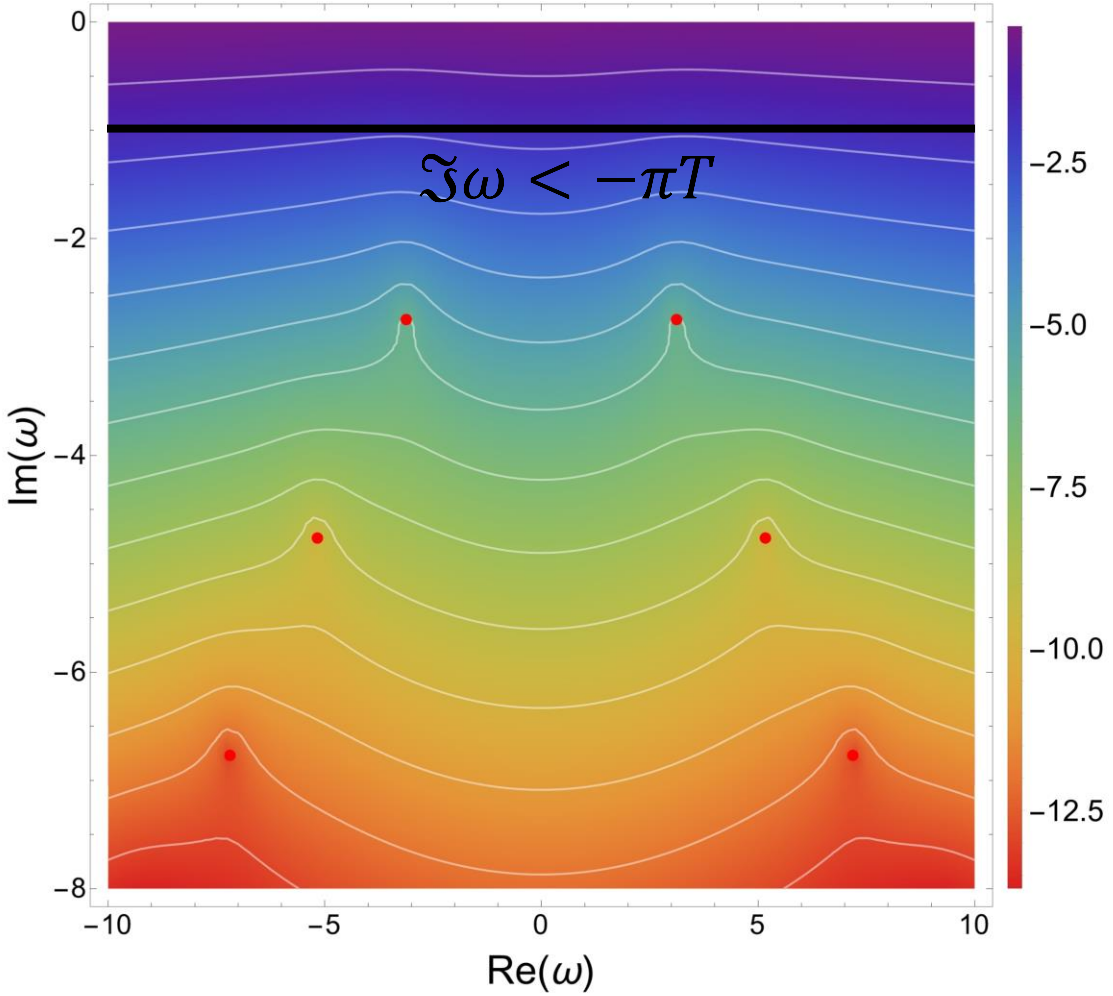
[Warnick (2013)]

Lack of convergence is a fundamental feature of the problem

The spectrum of $L$ in the Hilbert space of functions with **finite energy** is given by

**QNFs + continuum with $\mathfrak{I}\omega < -\lambda_c$**

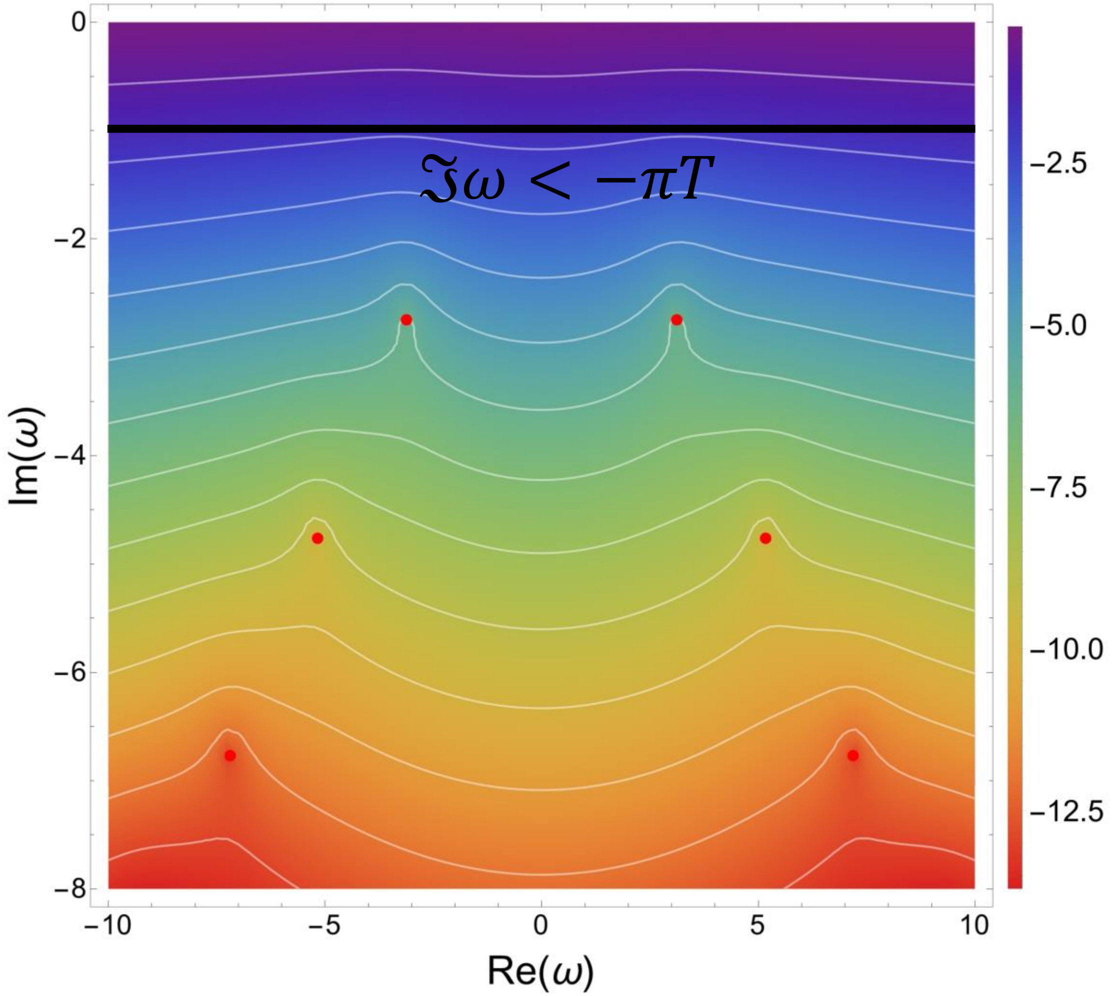
$$\sigma(L) = \lim_{\epsilon\to 0} \lim_{N\to\infty} \sigma_\epsilon(L_N)$$



# Convergence

**Finite energy $\neq$ Regularity**

We need greater control of **higher derivatives** to eliminate outgoing modes for $\Im\omega < -\pi T$



# Convergence

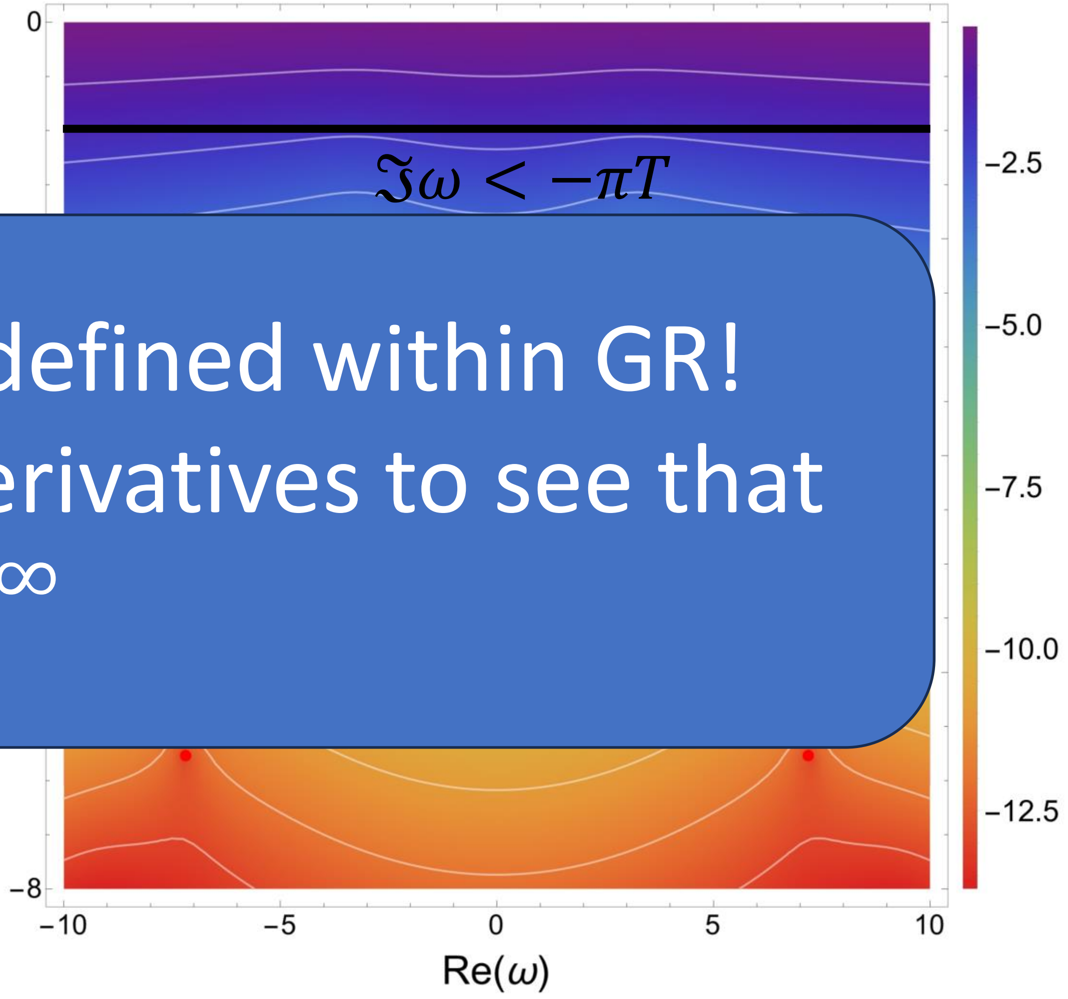
**Finite energy ≠ Regularity**

modes for $\Im\omega < -\pi T$

QNMs cannot be properly defined within GR!

We need access to the nth derivatives to see that

$$\mathbb{C}^{n-1} \neq \mathbb{C}^{\infty}$$



# Physical Picture

**Lack of convergence ⇒ Breakdown of GR EFT**

We can either

- Improve region of convergence by adding higher derivative terms to the norm in an EFT-like expansion [Warnick (2013); V. Boyanov et al. (2023)]
- Keep the grid and consider it as a cutoff telling us the smallest scale we can resolve within our EFT

Takeaway:

- QNFs are unstable, and their instability increases with N ⇒ QNFs can be easily displaced by a perturbation, and the effect of the perturbation increases the more localized it is

# What else can we do?

**Can we find another observable less sensitive to the cutoff?**

Lack of convergence is related to modes with $\Im\omega < -\pi T$. Thus, we want a similar object where we can easily eliminate those modes

# What else can we do?

**Can we find another observable less sensitive to the cutoff?**

Lack of convergence is related to modes with $\Im\omega < -\pi T$. Thus, we want a similar object where we can easily eliminate those modes

We can study the pseudospectrum of $\mathbb{C}$MMs

# Pseudospectrum of $\mathbb{C}$MMs

# What are $\mathbb{C}$MMs?

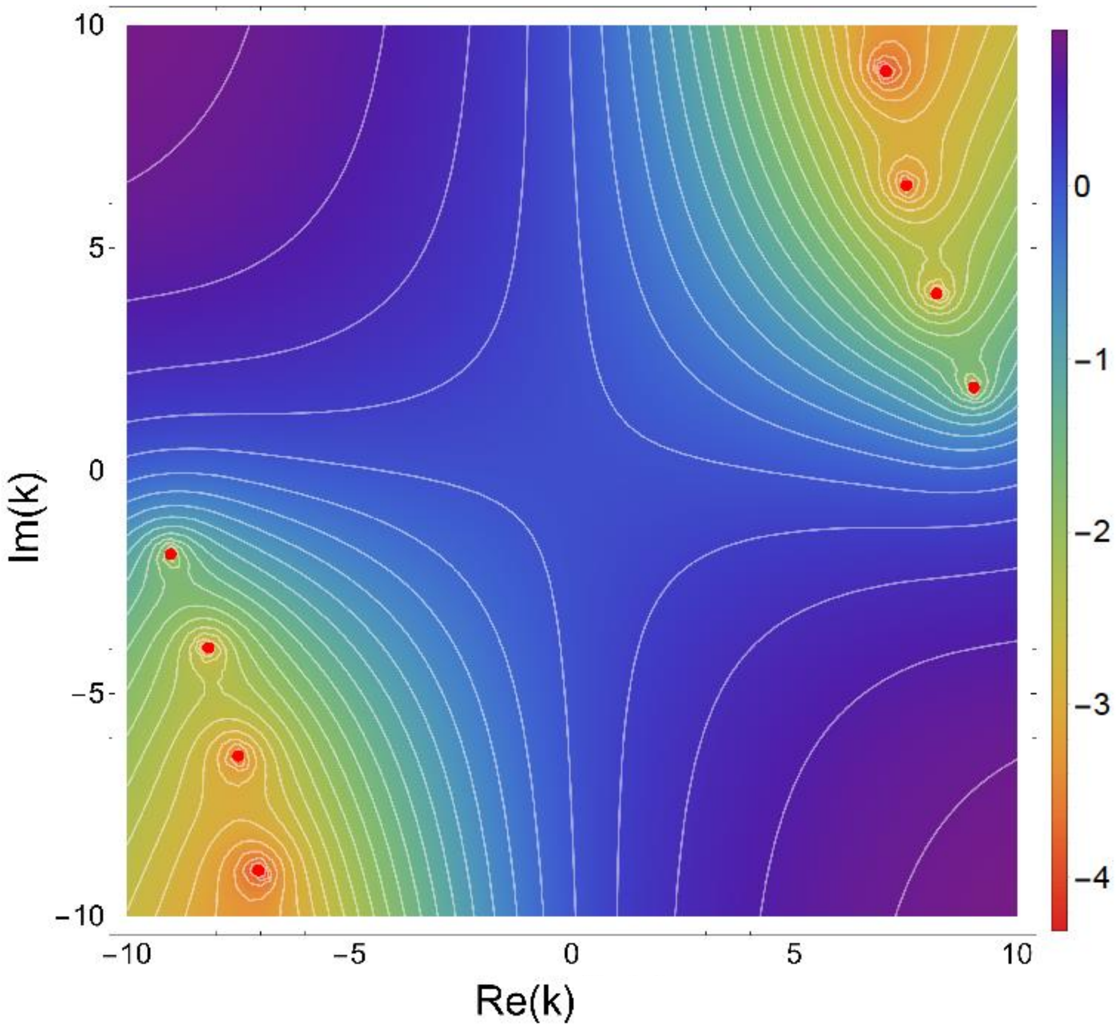
We rewrite the original eigenvalue problema for $\omega(k)$ as

$$k\begin{pmatrix}\phi\\ \tilde{\psi}\end{pmatrix} = \tilde{L}\begin{pmatrix}\phi\\ \tilde{\psi}\end{pmatrix} = \begin{pmatrix}0 & -i\\ \tilde{\mathcal{L}}_1 & \tilde{\mathcal{L}}_2\end{pmatrix}\begin{pmatrix}\phi\\ \tilde{\psi}\end{pmatrix}$$

where now we compute the $\mathbb{C}$LMs $k(\omega)$ and the associated $\mathbb{C}$MMs at fixed $\omega \in \mathbb{R}$

As for QNMs, we define an inner product such that the norm of a $\mathbb{C}$MM is its energy

# Results for $\omega = 10$

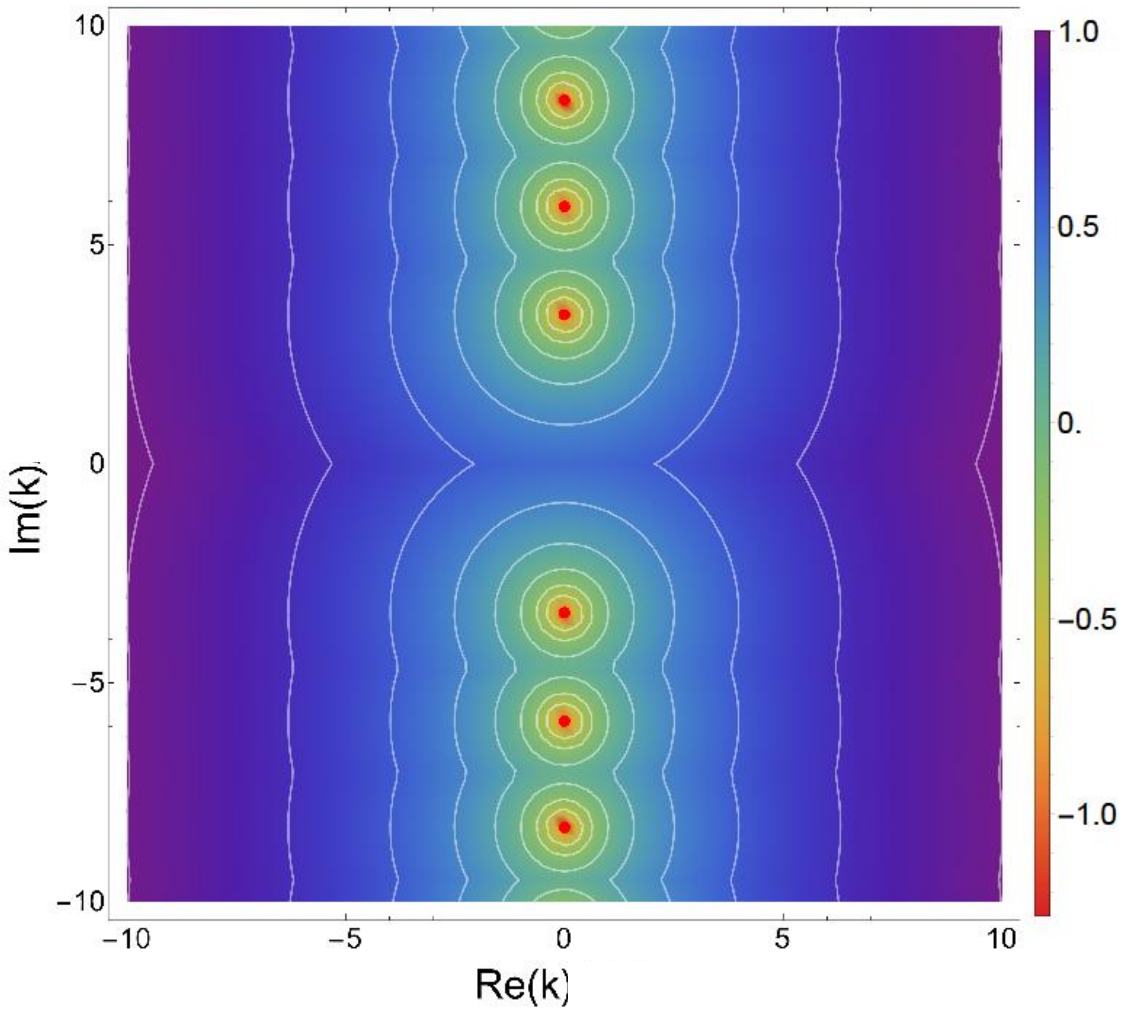



**Pseudospectrum is convergent**
**ℂLMs are unstable**

- Poles $k(\omega)$ are susceptible to migrating a lot for small perturbations of the theory.

- CLMs are more spectrally stable than QNFs

# Results for $\omega = 0$




Pseudospectrum is convergent
ℂLMs are stable

- $\tilde{L}(\omega = 0) = \tilde{L}^{\dagger}(\omega = 0)$
- At $\omega = 0$ ℂLMs are dual to glueball masses of the Hermitian theory resulting from the dimensional reduction on the euclidean thermal circle [Witten, 1998]

# Conclusions

# Main Results

- Non-convergence of pseudospectrum of QNMs is a physical phenomenon
- Pseudospectrum of $\mathbb{C}$MMs converges
- $\mathbb{C}$LMs are less unstable than QNFs
- $\mathbb{C}$LMs are stable at $\omega = 0$ in agreement with dual glueball interpretation

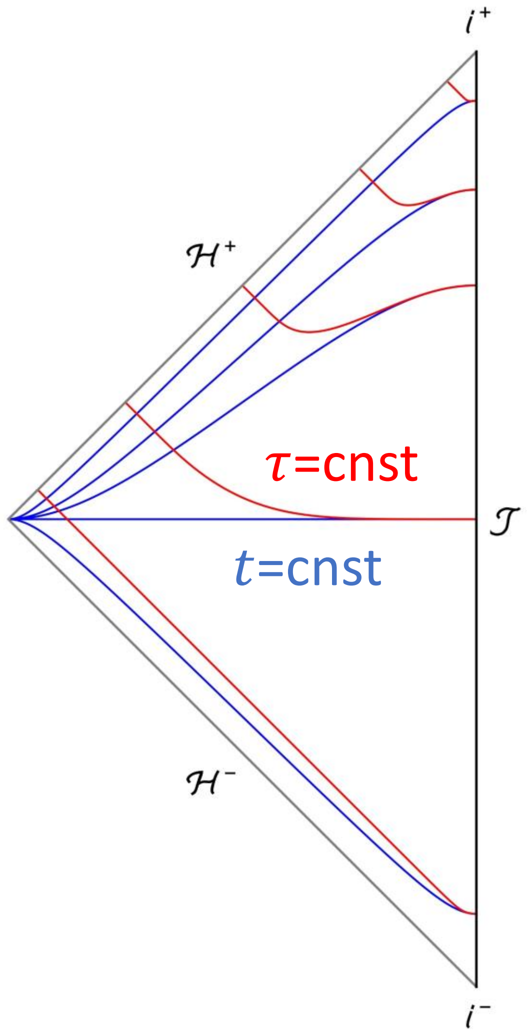
# Thanks For Your Attention!

# Extra Slides

# Regular Coordinates

# Definition of regular coordinates

$$\tau = t + r_\star - (1 - z) = t - \int \frac{dz}{f} - (1 - z)$$



# Explicit forms of the operators

# Operator $L$

$$\omega \begin{pmatrix} \phi \\ \psi \end{pmatrix} = L \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

$$\Downarrow$$

$$\omega\phi = i\psi$$

$$\omega\psi = -i\left(\frac{k^2}{2-f}\phi + \frac{3f - zf'}{z(2-f)}\phi' - \frac{f}{2-f}\phi''\right) + i\left(\frac{3-3f+zf'}{z(2-f)}\psi - \frac{1-f}{2-f}\psi'\right)$$

# More on Pseudospectra

# The three definitions of $\varepsilon$-pseudospectrum

[Trefethen & Embree, 2005]

- Resolvent:

$$\sigma_\varepsilon(\mathcal{L}) = \{z \in \mathbb{C} : \|(\mathcal{L} - z)^{-1}\| < 1/\varepsilon\}$$

- Perturbative:

$$\sigma_\varepsilon(\mathcal{L}) = \{z \in \mathbb{C}, \exists V, \|V\| < \varepsilon : \ z \in \sigma(\mathcal{L} + V)\}$$

- Pseudoeigenvalue:

$$\sigma_\varepsilon(\mathcal{L}) = \left\{z \in \mathbb{C}, \exists u^\varepsilon : \left\|(\mathcal{L} - z)u^\varepsilon\right\| < \varepsilon\left\|u^\varepsilon\right\|\right\}$$

Operator norm

$$\|V\| = \max \frac{\|Vu\|}{\|u\|}$$

# Condition Numbers

[Trefethen & Embree, 2005]

Test non-normality through orthogonality of left- and right-eigenvectors

$$\kappa_i = \frac{\|v_i\| \cdot \|u_i\|}{|\langle v_i, u_i \rangle|}$$

- $\kappa_i = 1 \Rightarrow$ Normal/Stable eigenvalue
- $\kappa_i > 1 \Rightarrow$ Non-normal/Unstable eigenvalue

$$L u_i = \omega_i \, u_i$$
$$L^\dagger v_i = \overline{\omega_i} \, v_i$$

$$|\lambda_i(\varepsilon) - \lambda_i| \leq \varepsilon \kappa_i$$

# $\mathbb{C}$LMs vs QNFs

# Holographic perspective: ℂLMs vs QNFs

ℂLMs

=

poles $k(\omega)$ of the retarded propagator at fixed real $\omega$

ℂLMs describe absorption

QNFs

=

poles $\omega(k)$ of the retarded propagator at fixed real $k$

QNFs describe thermalization

# Holographic perspective: ℂLMs vs QNFs

ℂLMs

=

poles $k(\omega)$ of the retarded propagator at fixed real $\omega$

QNFs

=

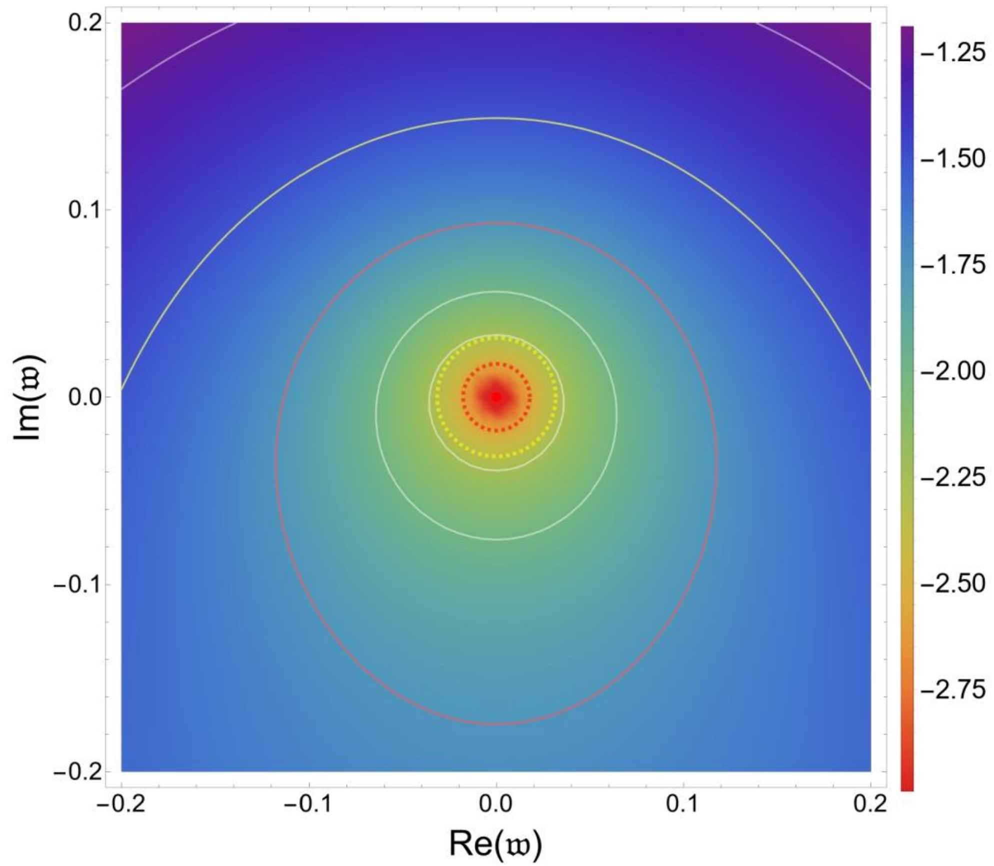
poles $\omega(k)$ of the retarded propagator at fixed real $k$

Fundamentally very different objects

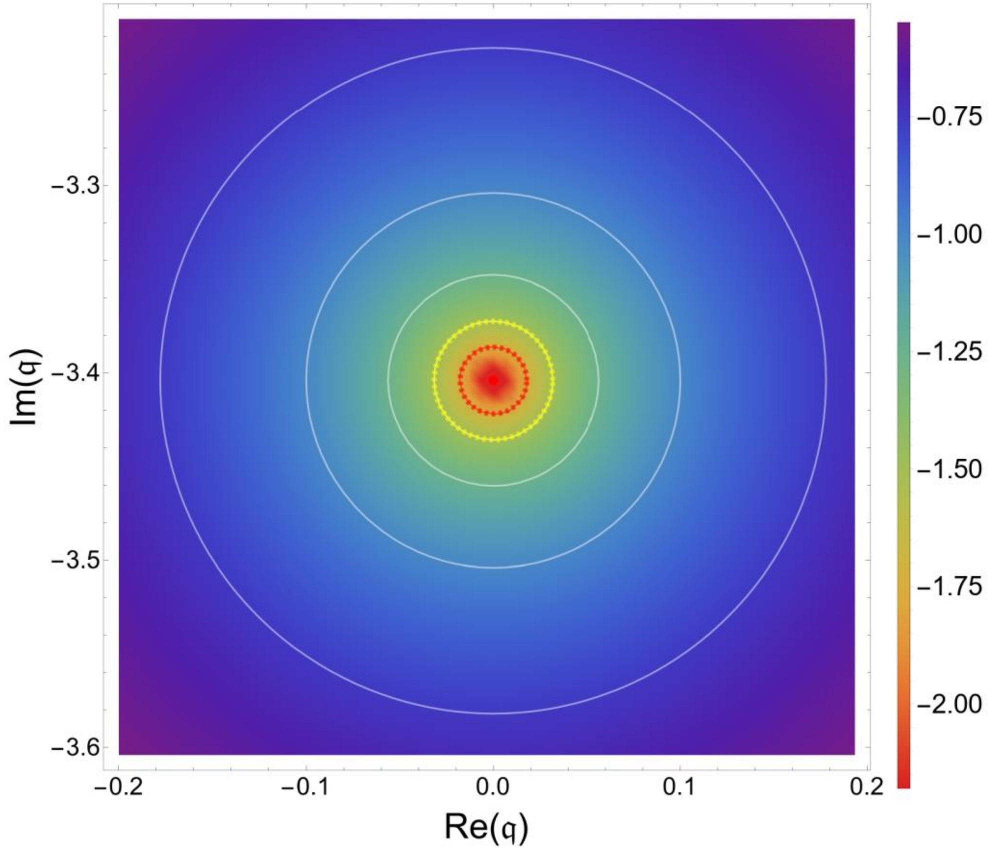
⇓

ℂLMs and QNFs have different stability properties

# Holographic perspective: ℂLMs vs QNFs




QNF $\omega = 0$ at $k = -3.40i$




ℂLM $k = -3.40i$ at $\omega = 0$