QNM orthogonality for AdS black holes

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Based on **2505.04696** with Benjamin Withers and Paolo Arnaudo

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- **This work:** fundamental origin and new relations for AdS black holes

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- Klein-Gordon Φ, $m_\Phi^2 = \Delta(\Delta - 2)$ on non-rotating BTZ $ds^2 = \frac{1}{z^2}\left(-(1-z^2)dt^2 + \frac{1}{(1-z^2)}dz^2 + d\varphi^2\right)$

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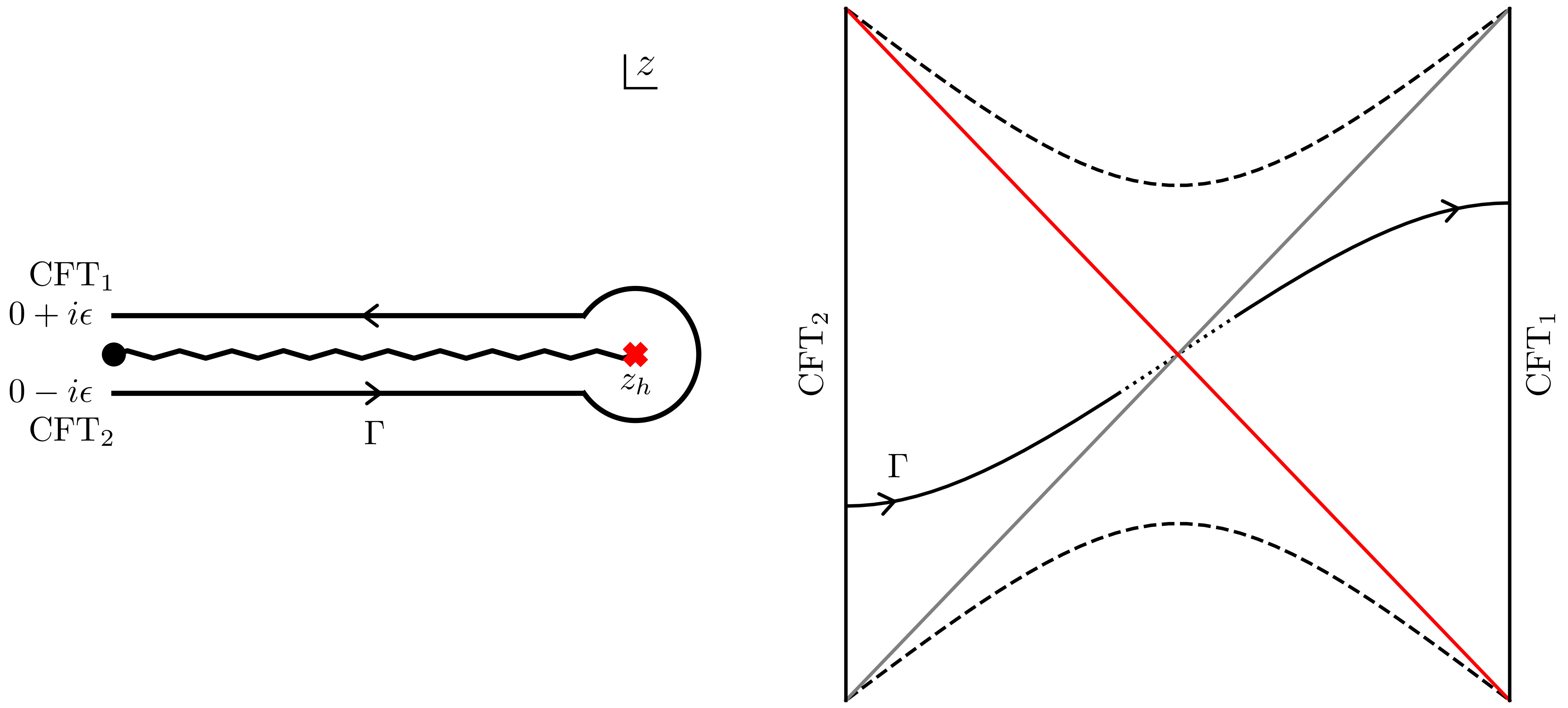
- Starting point: KG product on complex radial contour Γ $\langle a, b\rangle_{\mathrm{KG}} = i\int_0^{2\pi} d\varphi \int_\Gamma dz \frac{1}{z(1-z^2)} a^* \overleftrightarrow{\partial_t} b$



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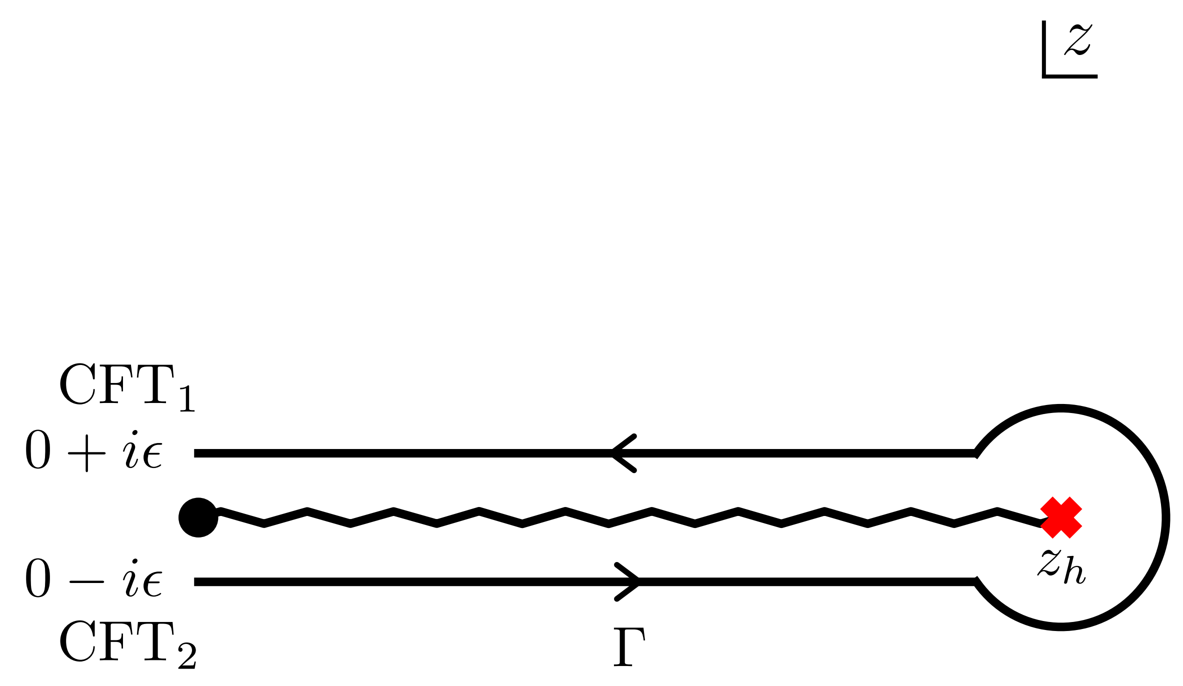

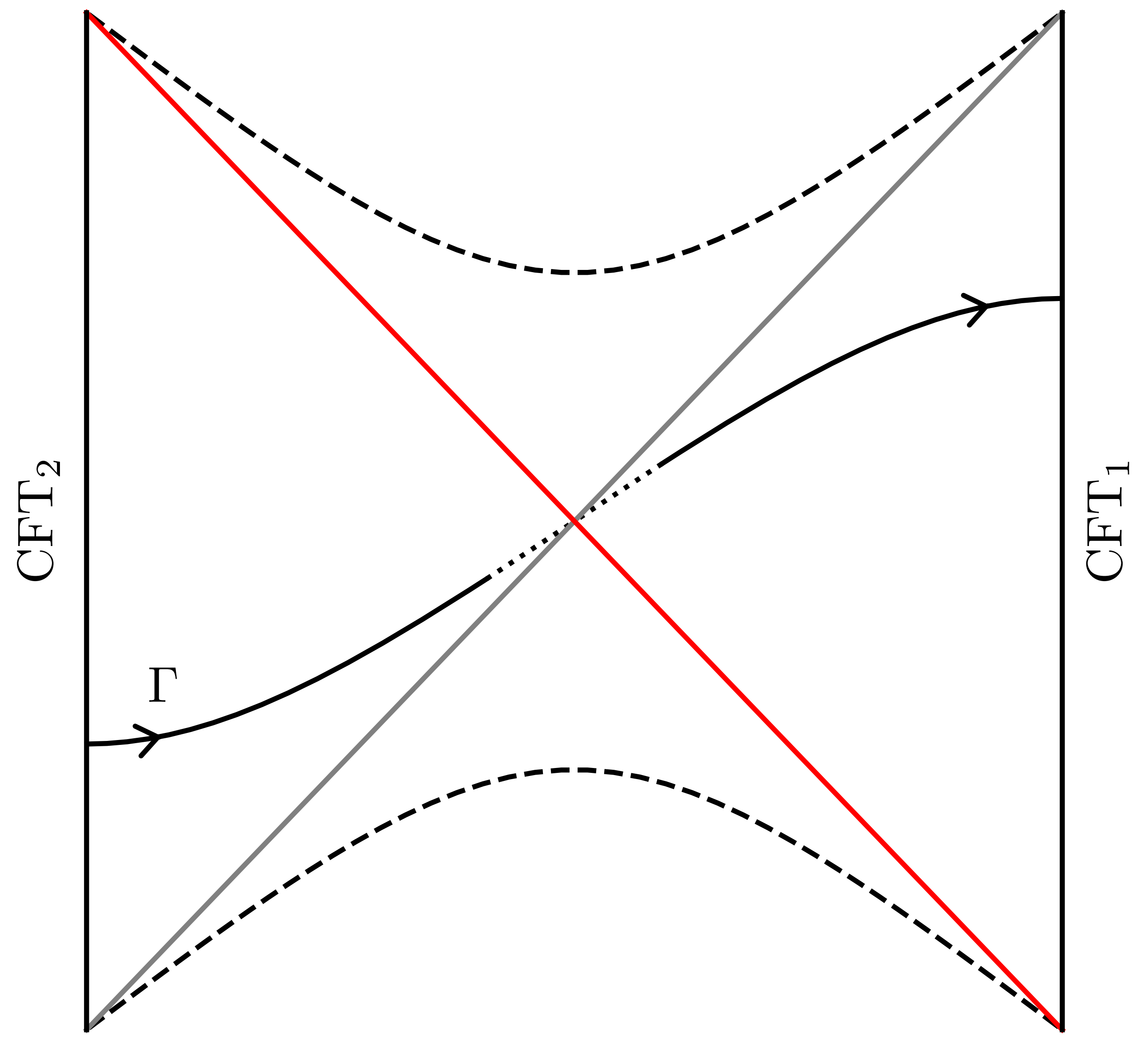

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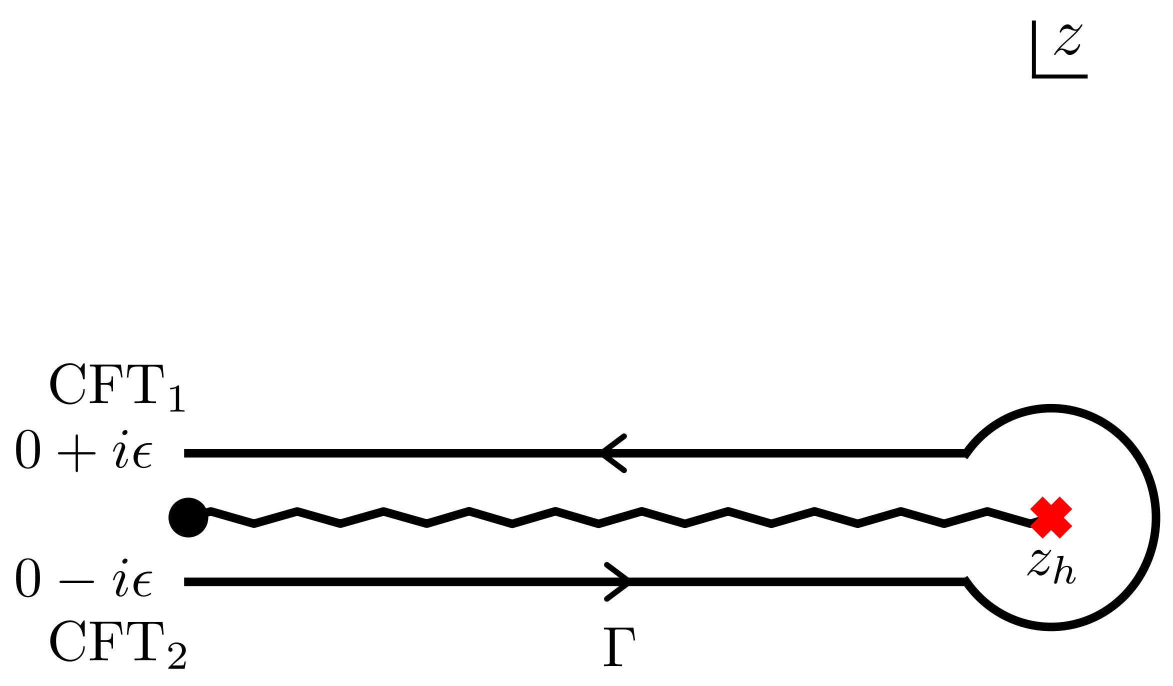

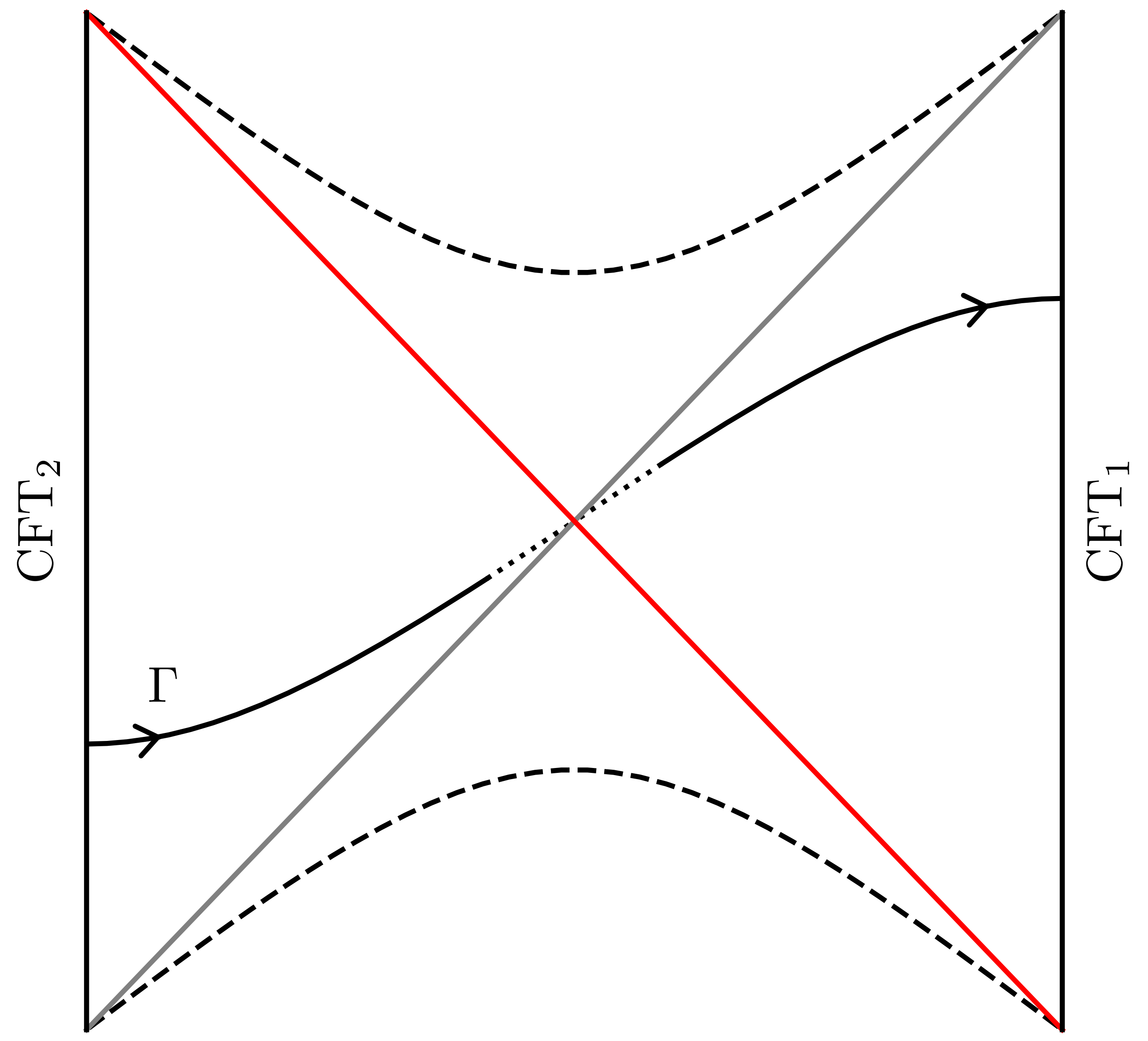

→ Avoid QNM branch point singularities

→ Schwinger-Keldysh contour in dual QFT (similar to [Glorioso, Crossley, Liu ’18])

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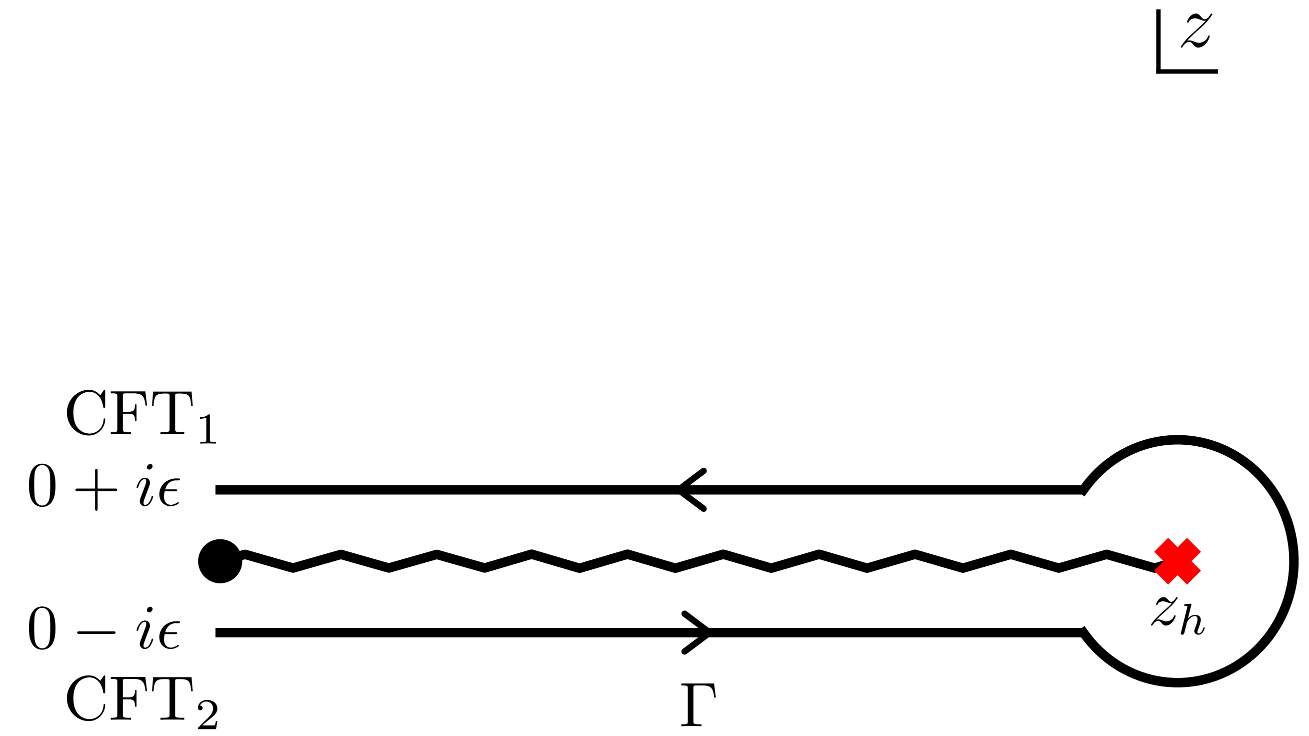

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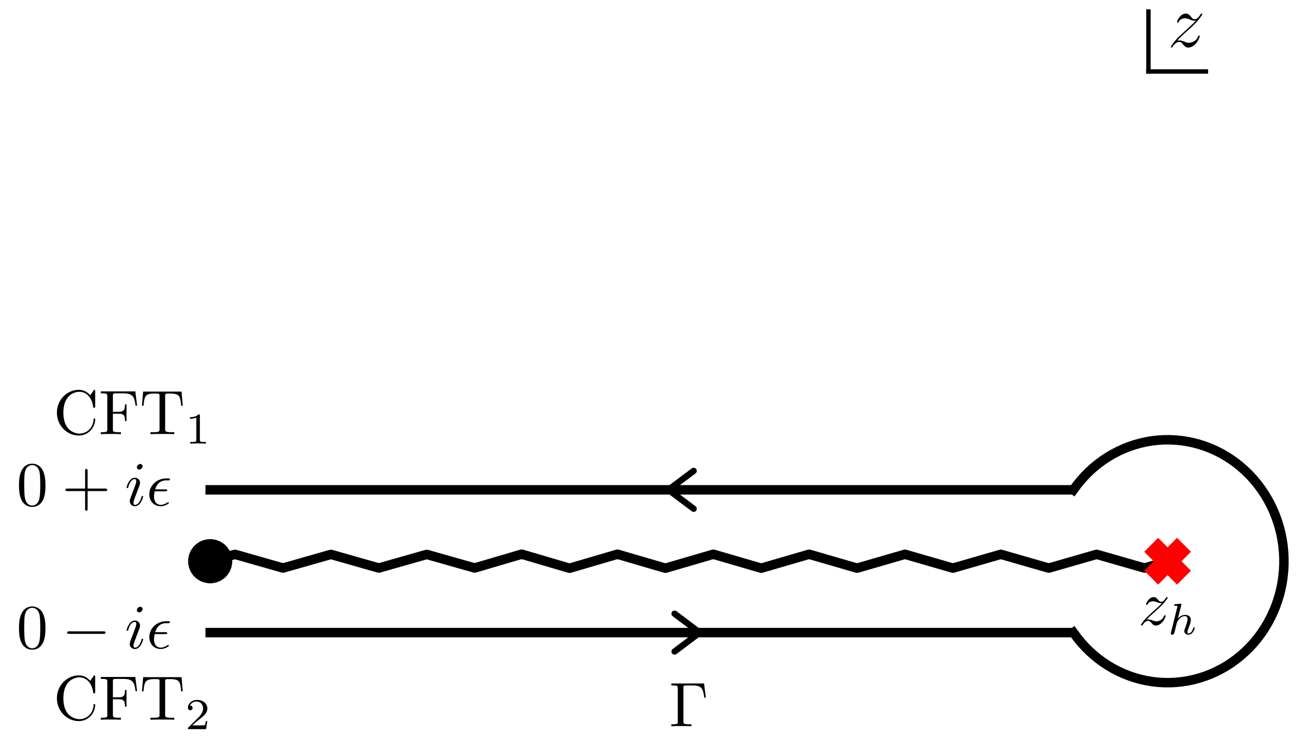

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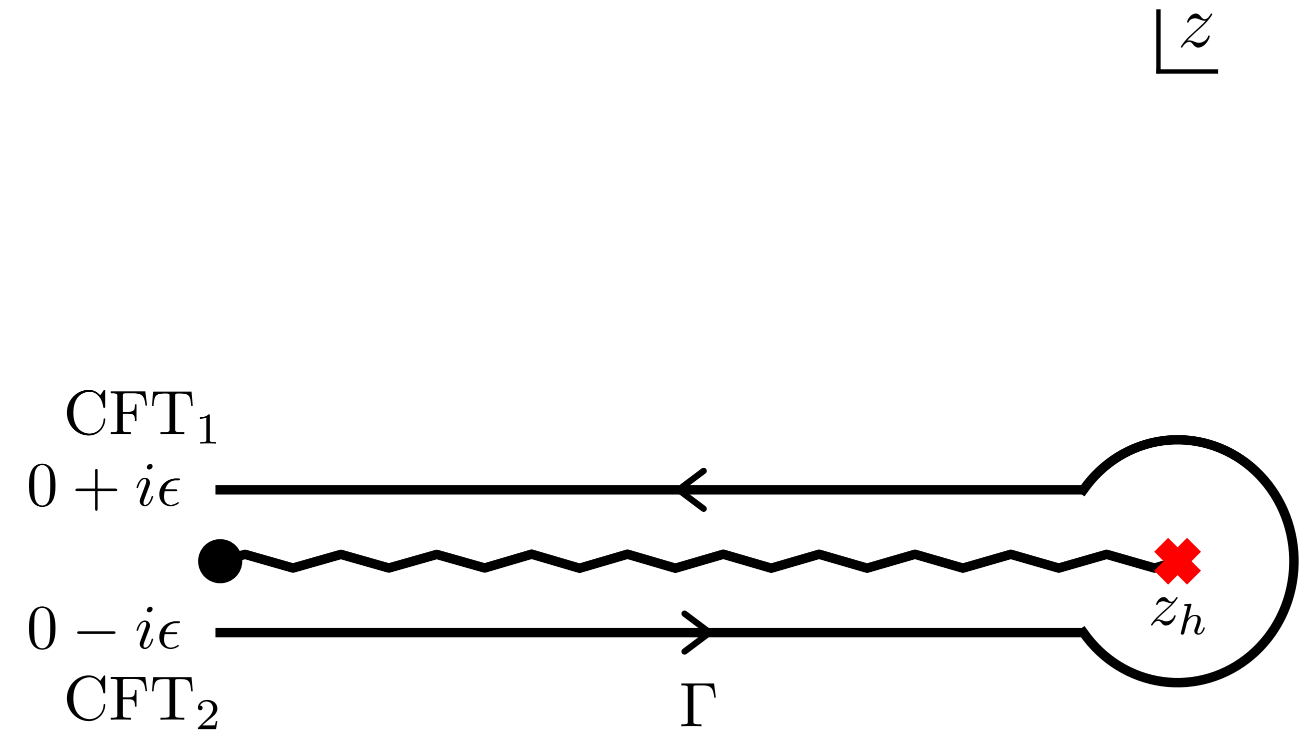

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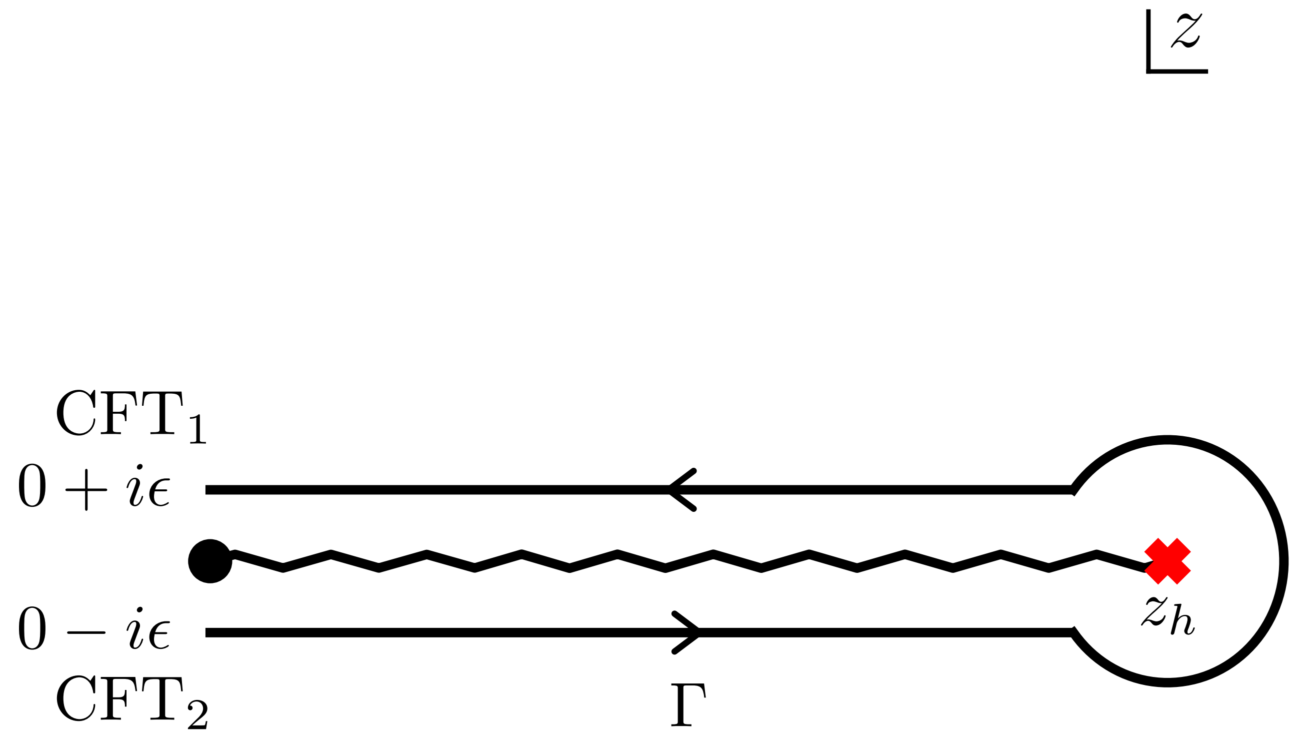

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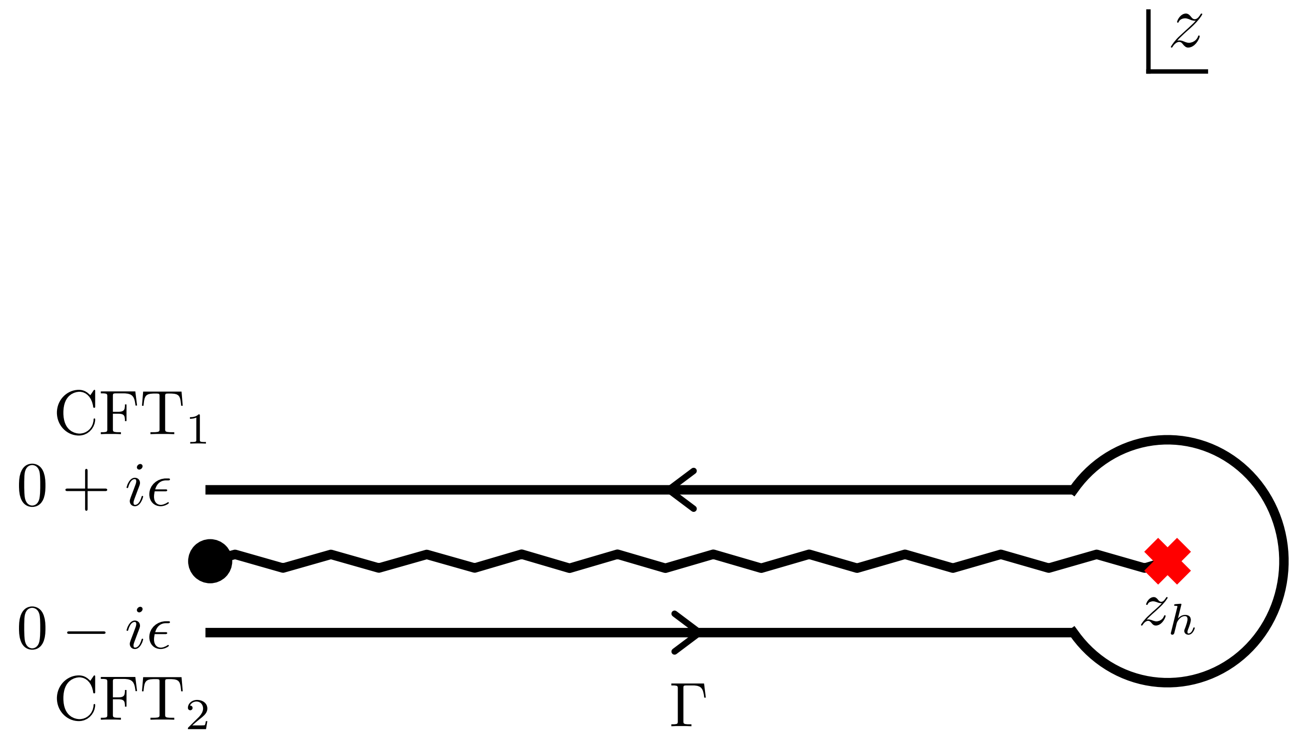

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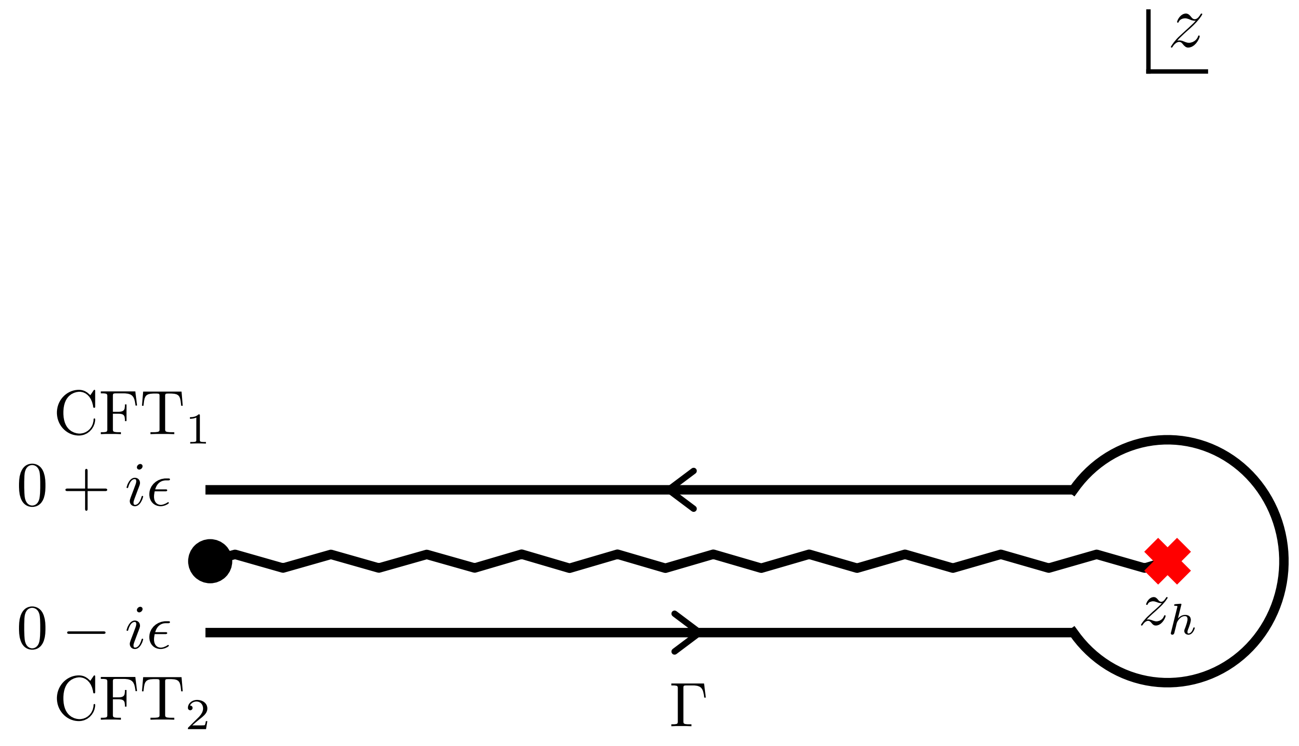

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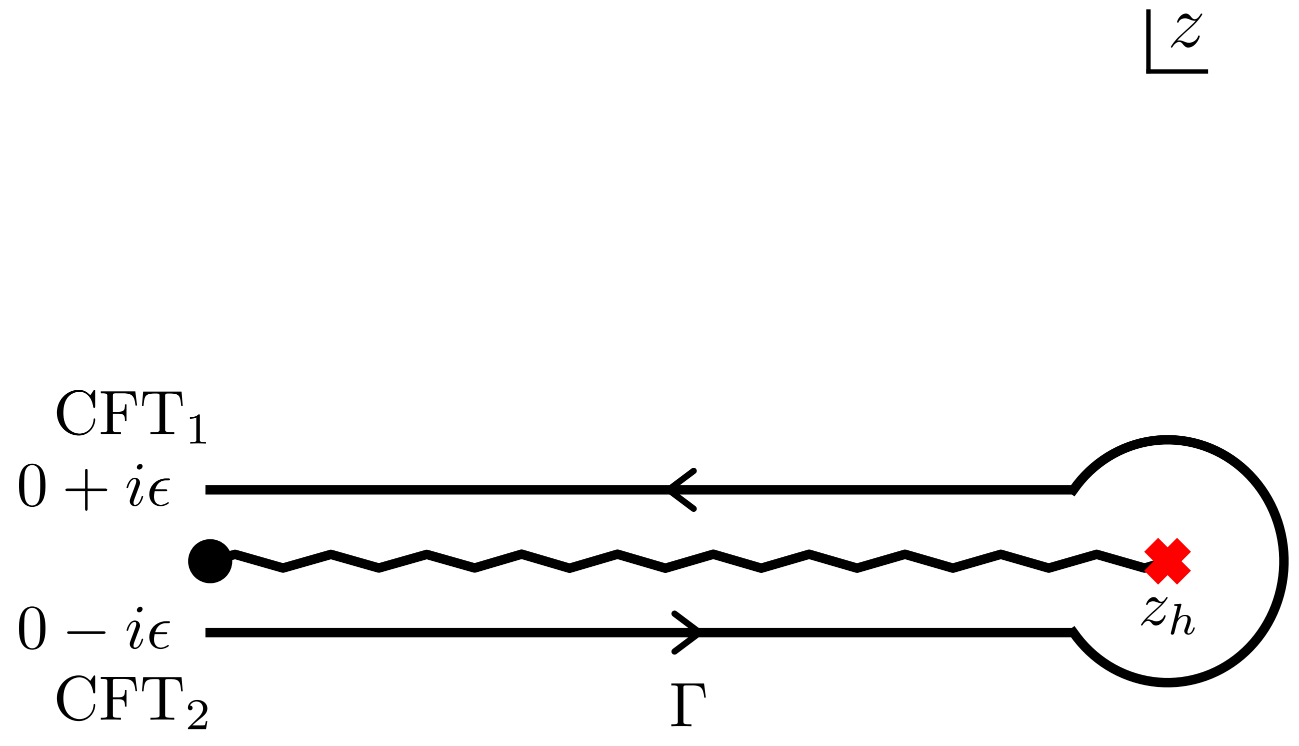

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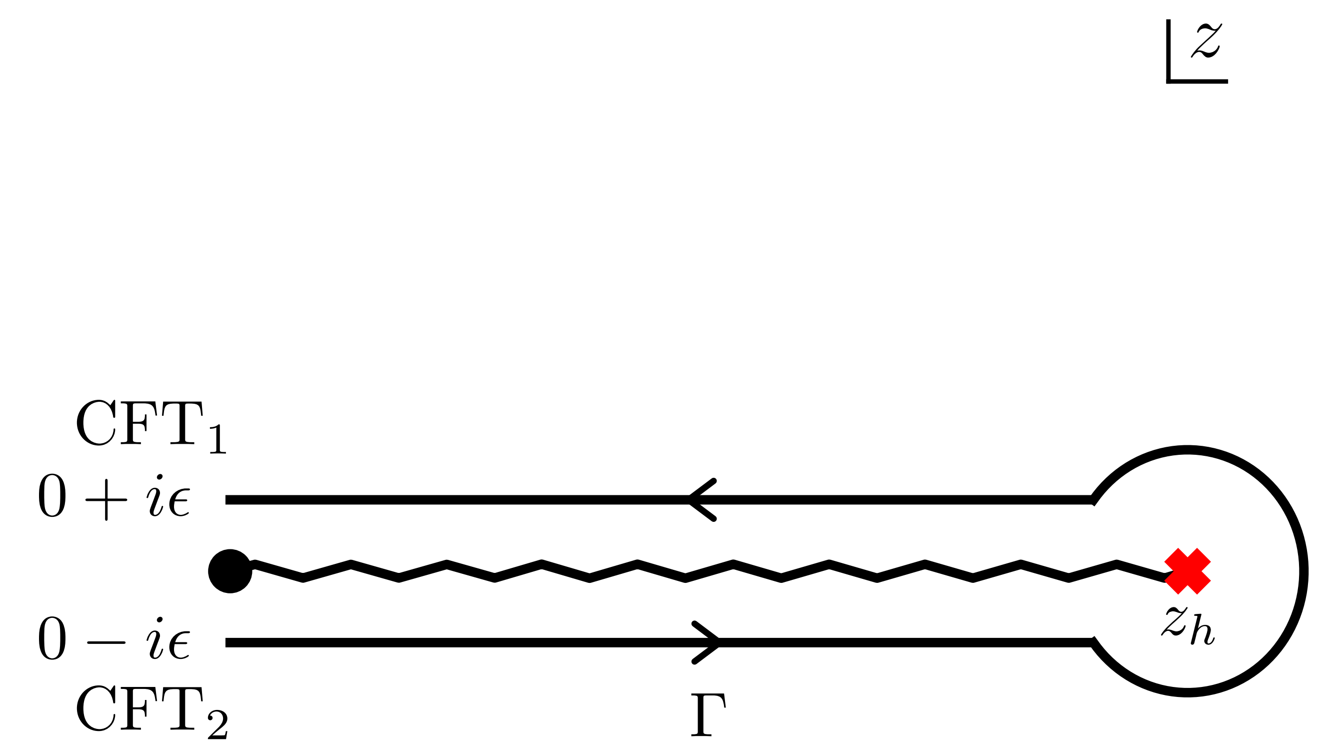

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- We prove that regular, normalisable eigenfunctions of $\mathscr{H}$ on Γ are **QNMs** and **anti-QNMs**

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→ Fundamental origin in right / left eigenfunction orthogonality for 'self-adjoint' $\mathscr{H}$

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  e.g. 'energy norms'

Thank you!

A. Generalisation

- Extended to wide class of AdS black holes

$$ds^2 = \frac{1}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{g(z)} + d\sigma_{d-1}^2\right)$$

- KG starting product:

$$\langle a, b\rangle_{KG} = i\int d\sigma_{d-1}\int_\Gamma dz\,\frac{1}{z^{d-1}\sqrt{f(z)}\sqrt{g(z)}}\,a^*\overleftrightarrow{\partial}_t b$$

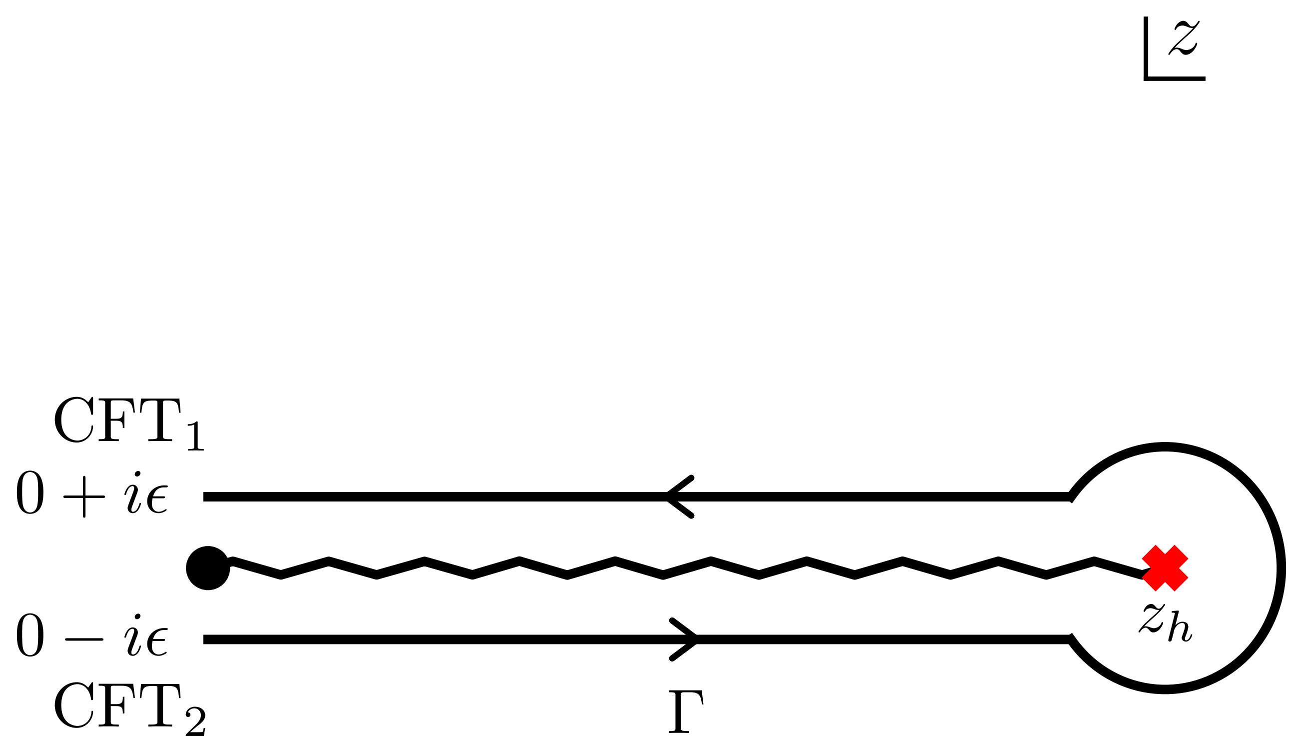
- Complex scalar Φ, $m_\Phi^2 = \Delta(\Delta - d)$ with Hamiltonian $\mathscr{H}$

$$\mathscr{H}^\dagger = \mathscr{H}$$

B. Eigenfunctions of $\mathscr{H}$ on Γ

- General solution on each copy

$$Z^{(i)} = c_A^{(i)} \widetilde{G}_A(\omega, z) + c_R^{(i)} \widetilde{G}_R(\omega, z)$$



- Normalisability + analytic continuation around $z = z_h$

$$Z^{(1)} = z^{\Delta} + \ldots$$

$$Z^{(2)} = -\frac{(2\Delta - d)}{G_{12}} z^{d-\Delta} + \ldots + \frac{G_{22}}{G_{12}} z^{\Delta} + \ldots$$

- Poles of $G_{12} = n e^{\frac{\beta\omega}{2}} (G_R - G_A)$ are QNMs and anti-QNMs