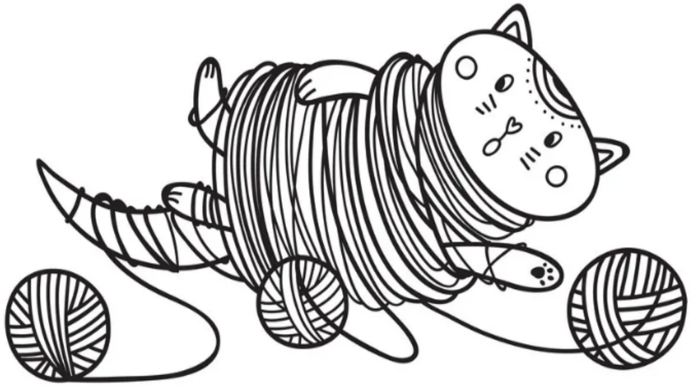

Covariant Quantum Bit Threads

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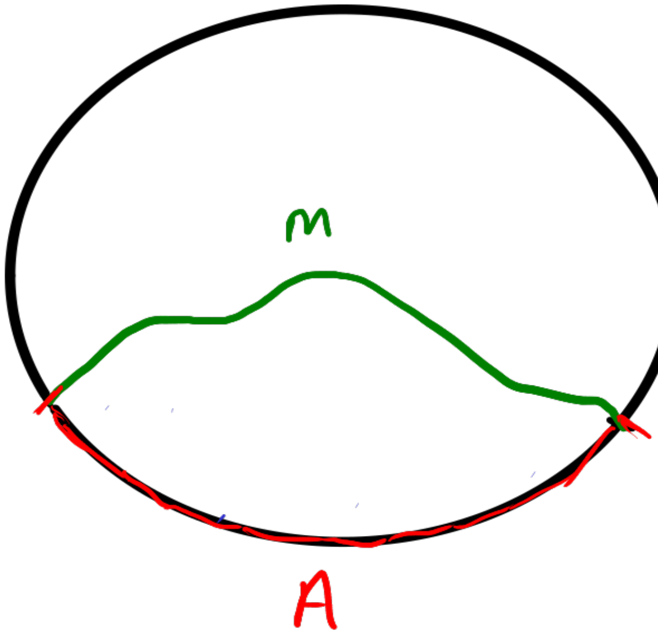
New Insights in Black Hole Physics from Holography
Madrid 2025

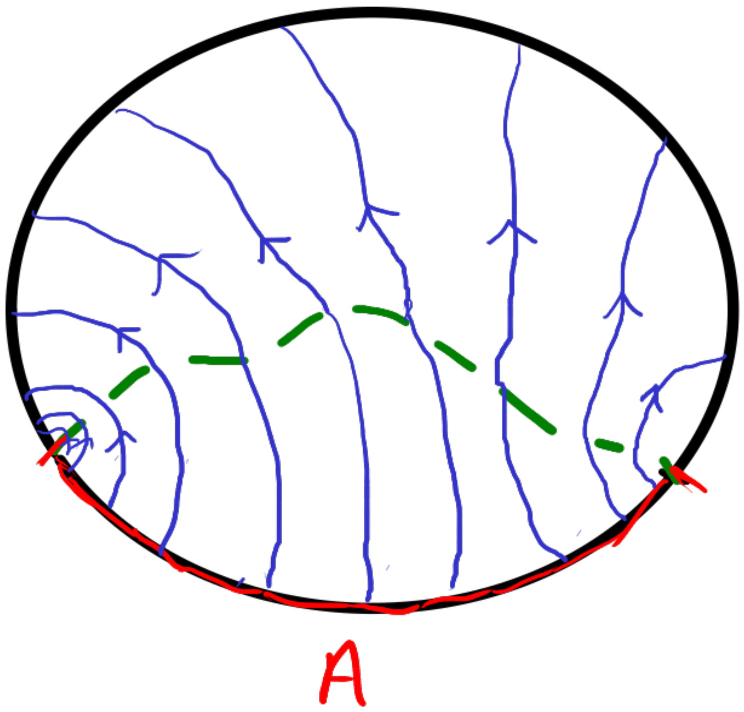
What is holographic entanglement entropy?

- A prescription for calculating entanglement entropy holographically.
- Can be surface-based. Entanglement entropy equals an extremised functional on a set of bulk surfaces.
- Can be flow-based: bit threads. Entanglement entropy equals max flux of a constrainted vector field.

$$S(A) = \min_{m \sim A} \text{Area}(m)$$



$$S(A) = \max_{v} \int_A v$$



Motivation: why is holographic entanglement entropy important?

It's hard to calculate entanglement entropies in QFTs any other way.

Useful applications of holographic entanglement entropy:

- Bulk reconstruction
- Black hole information problem
- Entropy inequalities
- Quantum chaos and scrambling
- Condensed matter applications: quenches and phase transitions

What are covariant quantum bit threads?

	Surface-based	Flow-based
Classical and time-reflection symmetric	Ryu-Takayanagi (RT) '06	**Classical bit threads** Freedman-Headrick '16
Not time-reflection symmetric	Hubeny-Rangamani-Takayanagi (HRT) '07	**Covariant bit threads** Headrick-Hubeny '22
Quantum corrections	Faulkner-Lewkowycz-Maldacena (FLM) '13 Engelhardt-Wall '14	**Quantum bit threads** AR '21 Agon-Pedraza '21 Headrick-Reddy-AR '25
Irrelevant terms in gravitational EFT	Dong '13 Camps '13	Harper-Headrick-AR '18
Not time-reflection symmetric + quantum corrections	Quantum extremal surfaces (QES)	**Covariant quantum bit threads** Headrick-Reddy-AR '25

Motivation: why consider flow-based prescriptions when we already have surface-based prescriptions?

The more prescriptions the better: different perspectives give new insights and different problems favour different approaches.

Advantanges of bit threads over surface-based prescriptions:

- Technical (numerics): Bit thread prescriptions are convex optimisation problems. So, in numerical optimisation, we never get stuck in local optima.
- Conceptual:
    - Physical meaning of entropy inequalities is manifest.
    - Surfaces jump (e.g. mutual information phase transitions), flow configurations do not.

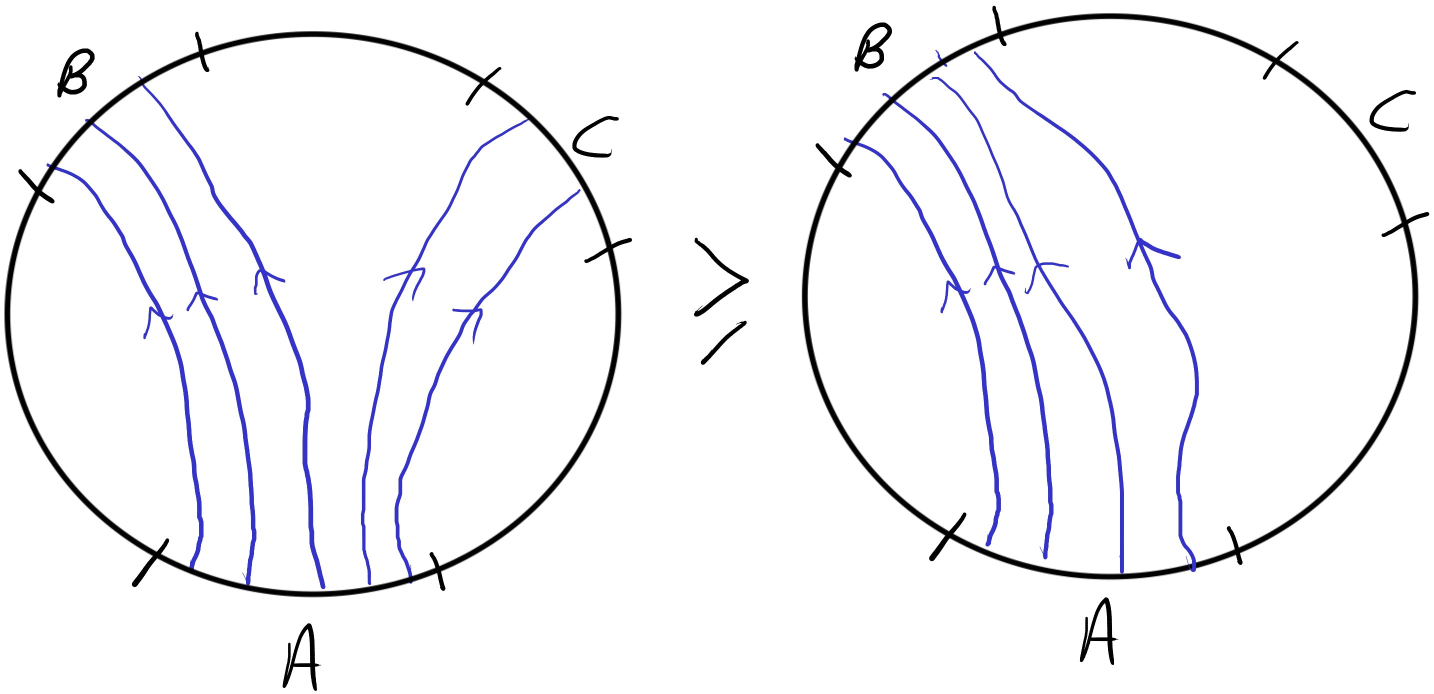
Entropy proofs with bit threads

Strong subadditivity:

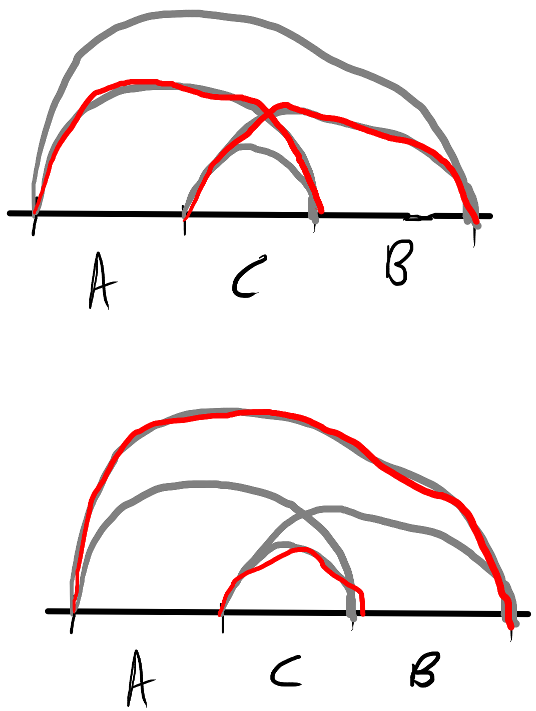
$$I(A:BC) \geq I(A:B)$$

Correlation can only increase under inclusion

Flow proof:



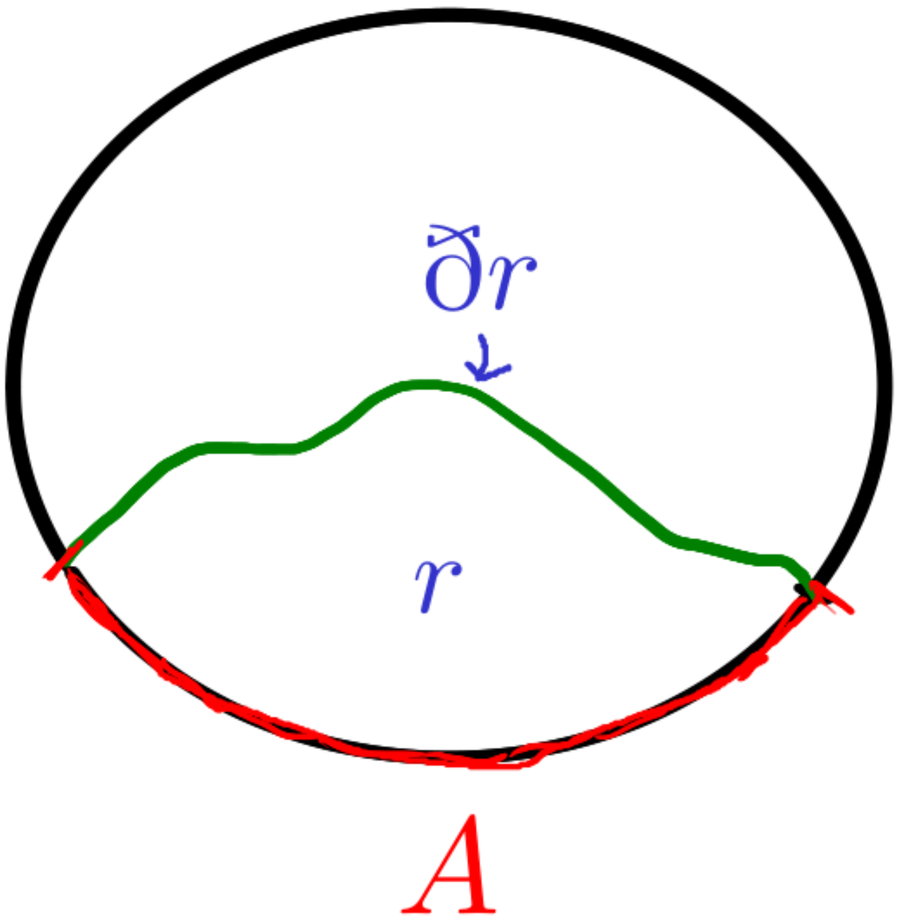
Surface proof:



Bit threads make physical QI meaning manifest

Quantum minimal surface (QMS) prescription

Assume the state is time-reflection symmetric



QMS prescription:

$$S(A) = \min_r S_{gen}(r)$$

Generalised entropy:

$$S_{gen}(r) := \frac{|\eth r|}{4G_N} + S_{bulk}(r)$$

Ryu-Takayanagi but accurate to all orders in 1/N.

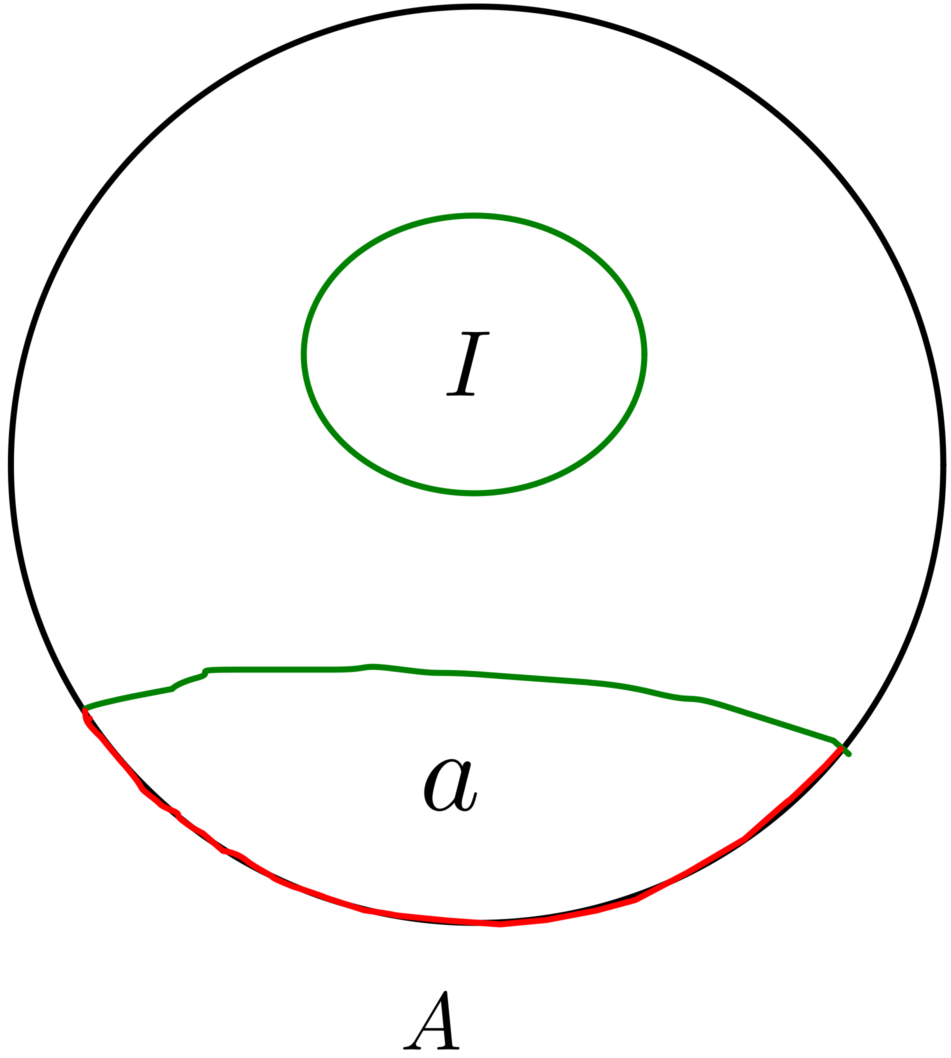
Quantum bit threads (non-covariant)

Prescription:

$$\max_{v} \int_A v \qquad \text{subject to} \quad |v| \leq \frac{1}{4G_N} \text{ and } \forall r : \left| \int_r \nabla \cdot v \right| \leq S_{bulk}(r)$$

- Quantum bit threads can start and end at points in the bulk.
- Provably equivalent to the quantum minimal surface prescription.
- Island boundaries are boundary-disconnected bottlenecks for quantum bit threads.

Islands and quantum bit threads



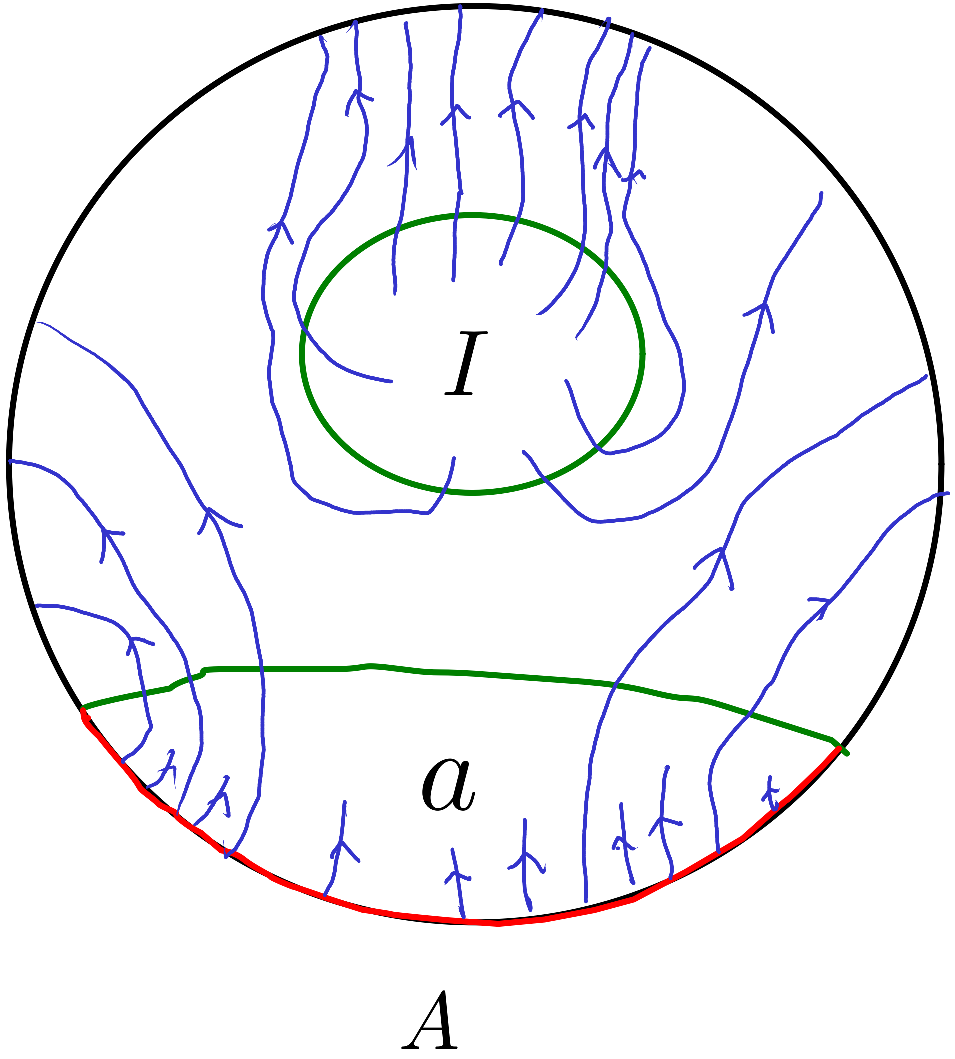
Constraint: $\left|\int_r \nabla \cdot v\right| \leq S_{bulk}(r)$

$S_{bulk}(a)$ is large: many bit threads can end in a.

$S_{bulk}(a \cup I)$ is small: threads that disappear in a must reappear in I.

(Next slide shows max-flow configuration)

Islands and quantum bit threads



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Further developments of (non-covariant) quantum bit threads

(Soon-to-be-published with M. Headrick and S.R. Kasireddy)

- Cutoff-independent prescriptions: constraints in terms of generalised entropy.
- Prescriptions involving measures on oriented and unoriented bulk curves (called thread distributions).
- Stricter and looser versions of the constraints.
- New proofs of equivalence to the QES prescription.
- What we learn about the bulk entanglement structure from the quantum bit thread configurations.

Covariant quantum V-threads

We need a covariant prescription to tackle problems like evaporating black holes.

Quantum extremal surface prescription:

$$S(A) = \min \operatorname{ext} S_{gen}(\eth r)$$

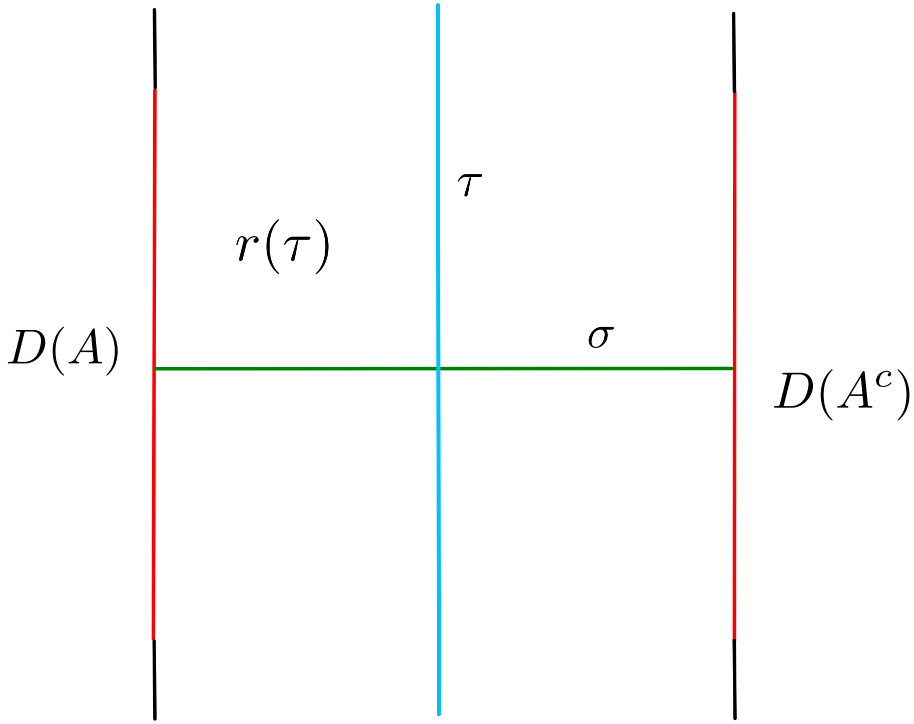
Covariant quantum V-threads

Covariant quantum bit thread prescription (V-flow):

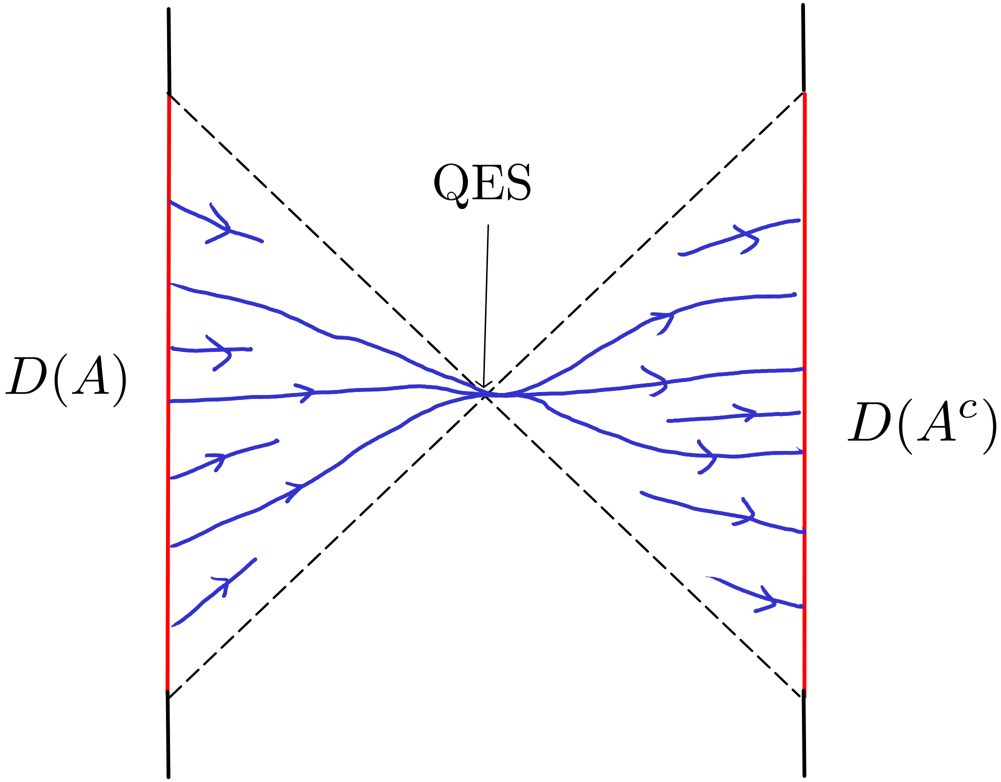
$$S(A) = \max_V \int_{D(A)} *V, \quad \text{subject to} \quad \exists\sigma\forall\tau : -\int_{r(\tau)} d * V \leq S_{bulk}(\sigma \cap \tau)$$

and a norm bound.

Time sheets and Cauchy slices:



Max V-flow:



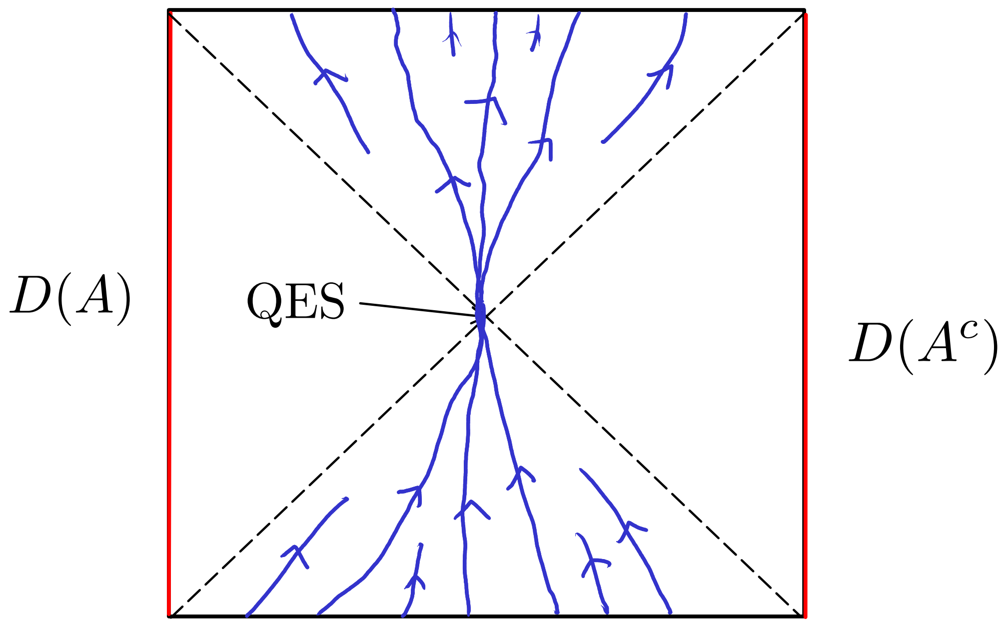
Covariant quantum V-threads and evaporating black holes

Evaporating AdS black holes with a flat space bath:

Other aspects of covariant quantum bit threads

- Proofs of equivalence to the QES prescription. Using Lagrange dualisation.
- U-flow prescription (roughly speaking, timelike vector field, with flow lines from past to future infinity).
- Thread distribution formulations (measures on bulk curves).
- Quantum maximin and minimax (relaxed in various ways).

U-flows:



Messages to take away

1) Quantum bit thread prescriptions:
    - New: covariant, cutoff-independent, and thread distributions.

2) Quantum bit threads jump across the bottleneck (QES).

3) Boundaries of islands are emergent, boundary-disconnected bottlenecks for quantum bit threads.

Future directions

- Numerics: use these prescriptions to calculate boundary entropies.

- Algebraic quantum bit threads: apply what we've learned about generalised entropy and von Neumann algebras.

- Non-AdS holography: there are surface-based proposals for entanglement entropy in non-AdS holography, what are the flow-based equivalent prescriptions?

- Flow prescriptions for Renyi entropies and other QI quantities.

Bonus slides

Convex optimisation

Lagrangian duality

1) Start with "primal" constrained optimisation problem

2) Add Langrange multipliers

3) Optimise over original variables to get "dual" problem:

Basic example

1) $\min\limits_{x}(x^2)$ subject to $x = 0$ Primal

2) = $\min\limits_{x}\max\limits_{\lambda}(x^2 + \lambda x)$ + Lagrange multipliers

3) = $\max\limits_{\lambda}(-\lambda^2/4)$ Dual

Convex optimisation

Primal:

$$\max_{v} \int_{A} n_{\mu} v^{\mu}$$

subject to $\quad |v| \leq \frac{1}{4G_N}$ and $\forall(\sigma \in \Omega_A) : \left(-\int_{\sigma} \nabla_{\mu} v^{\mu}(x) \leq S_{bulk}(\sigma) \right)$

After adding Lagrange multipliers:

$$\sup_{v} \inf_{\mu,\phi} \left[\int_{A} v + \int_{\Sigma} \phi(x) \left(\frac{1}{4G_N} - |v| \right) + \int_{\Omega_A} d\mu(\sigma) \left(\left(\int_{\Sigma} \chi(\sigma, x) \nabla_{\mu} v^{\mu} \right) + S_{\text{bulk}}(\sigma) \right) \right]$$

Resultant dual:

$$\min_{m \sim A} \left(\frac{\text{Area}(m)}{4G_N} + S_{\text{bulk}}(\sigma(m)) \right)$$