Holographic timelike entanglement entropy and the black hole interior

Based on [hep-th: 2408.15752]
and work in progress, with M. P. Heller and A. Serantes

Fabio Ori

New insights in black hole physics from holography
Madrid, 17 June 2025

GHENT UNIVERSITY

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Introduction and motivation

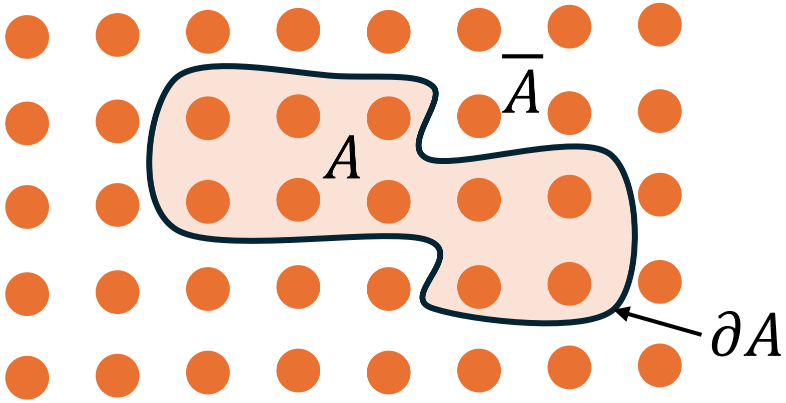
Entanglement entropy

From quantum mechanics to holography

- $|\Psi\rangle$ pure state. Select a region A, then the **reduced density matrix** $\rho_A = \mathrm{Tr}_{\bar{A}} |\Psi\rangle\langle\Psi|$. Its **von Neumann entropy** is the **entanglement entropy**:

Sorkin ’83

$$S_A := -\mathrm{Tr}_A(\rho_A \log \rho_A)$$



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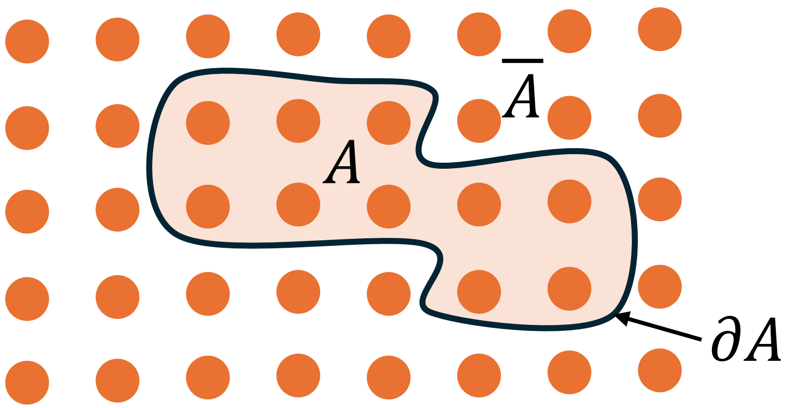
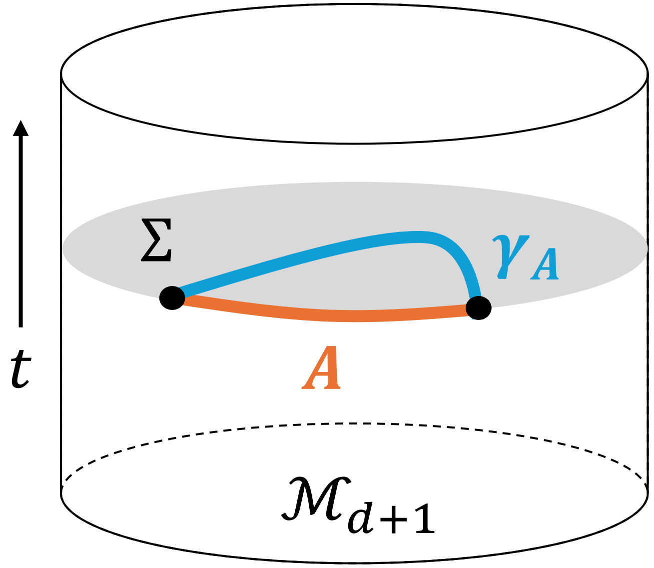




- Given a **$(d-1)$-dimensional region A**, find a **codimension-2 extremal surface γ_A** in the bulk spacetime, **anchored on ∂A**.

$$S_A := \min_{\mathrm{Area}[\gamma_A]} \frac{\mathrm{Area}[\gamma_A]}{4G_N^{(d+1)}}$$

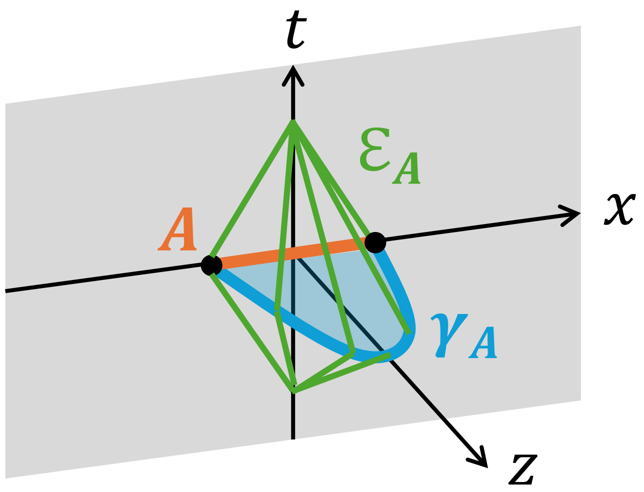
Ryu, Takayanagi ’06;
Hubeny, Rangamani, Takayanagi ’07

Motivation

Bulk reconstruction and new holographic probes

- Emergence of spacetime:
  entanglement wedge reconstruction.

Czech, Karczmarek, Nogueira, Van Raamsdonk ’12; Headrick, Hubeny, Lawrence, Rangamani ’14; Dong, Harlow, Wall ’16



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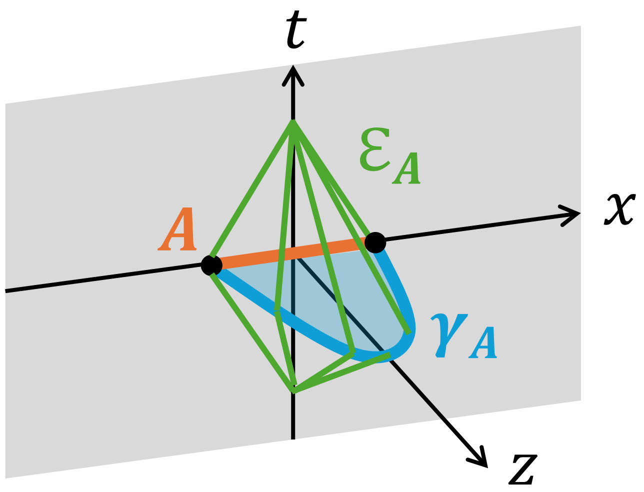

- **Field-theoretical quantity** related to the **emergence of the time direction**?

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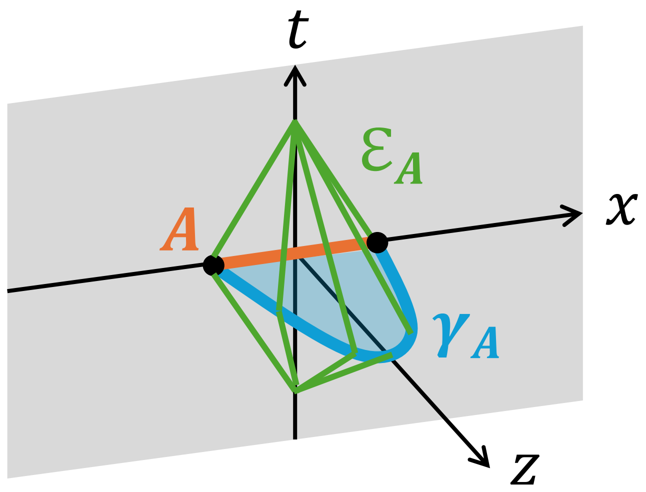
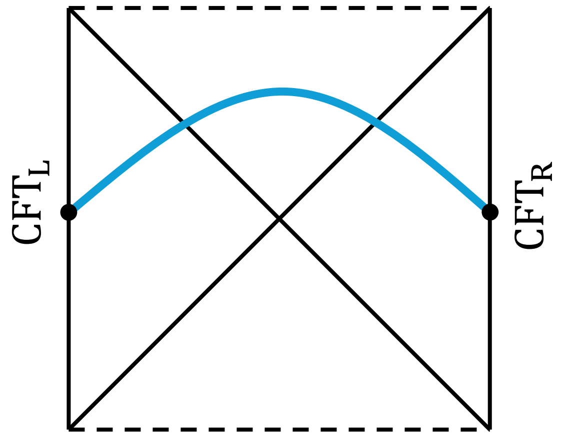


- **Field-theoretical quantity** related to the **emergence of the time direction**?

- Especially rare are observables that probe the **black hole interior**.



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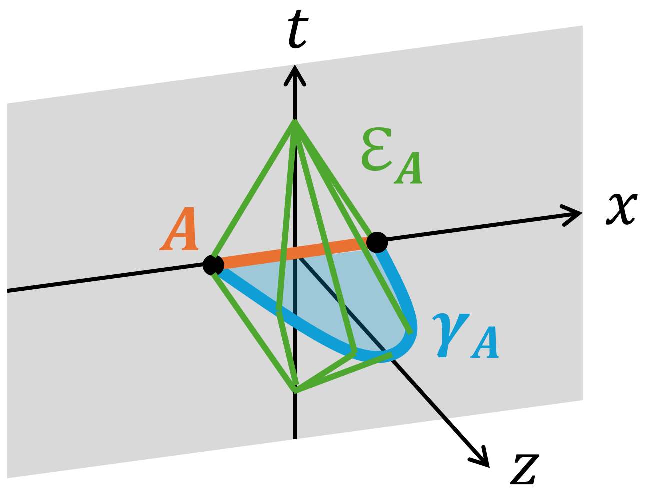

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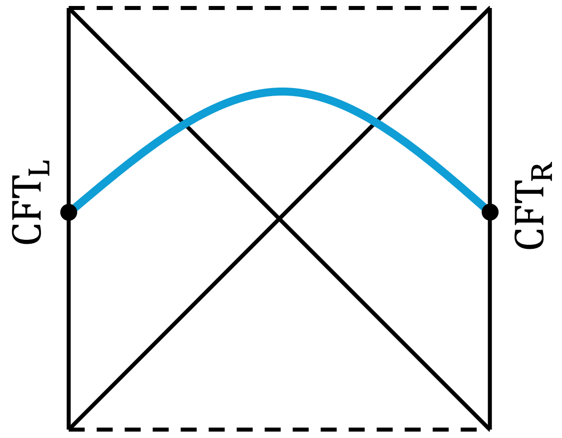


- What can we say on **one-sided probes of the black hole interior**?

The holographic dual to timelike entanglement

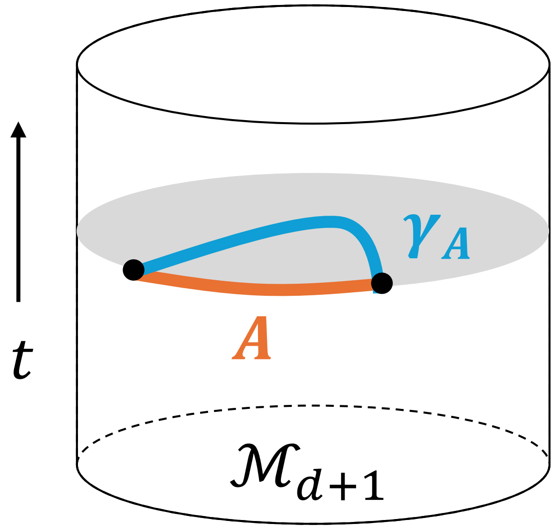
How do we define timelike entanglement?

Analytic continuation of the spacelike result

- CFT$_2$ result and its analytic continuation as $\Delta x^2 - \Delta t^2 < 0$:

Doi, Harper, Mollabashi, Takayanagi, Taki '23

$$S_A = \frac{c}{3}\log\left(\frac{\sqrt{\Delta x^2 - \Delta t^2}}{\varepsilon}\right) \xrightarrow{\text{Analytic continuation}} S_A^{(T)} = \frac{c}{3}\log\left(\frac{\sqrt{|\Delta x^2 - \Delta t^2|}}{\varepsilon}\right) + \frac{c\pi}{6}i$$



NB: in this talk, we will be agnostic on the general field theory definition of timelike entanglement.

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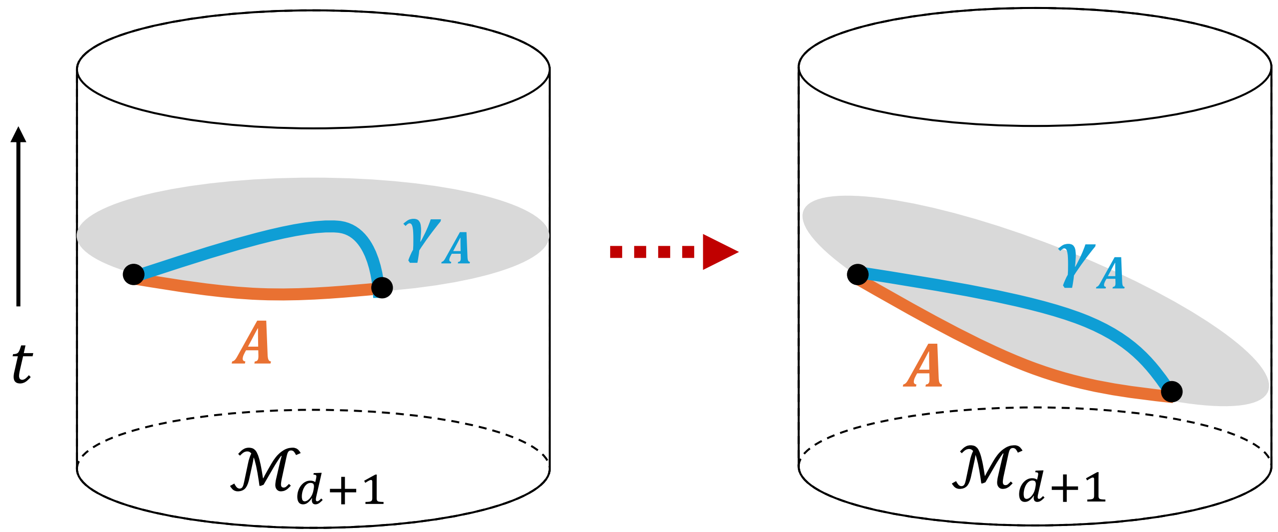

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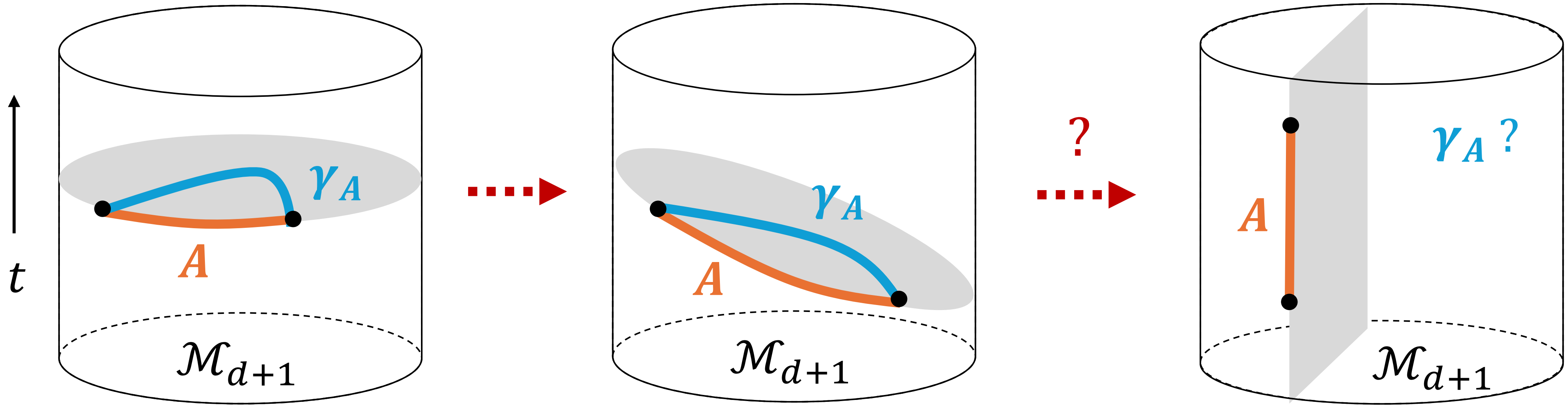


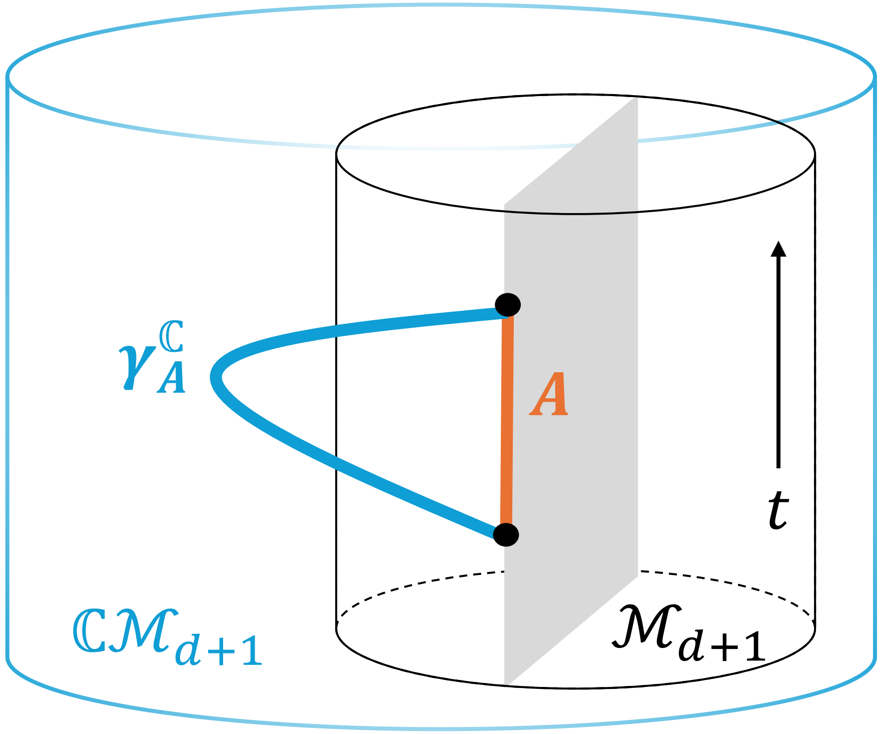
Geometric interpretation of timelike entropy

Complex bulk extremal surfaces

- Consider a $(d-1)$-dim. **timelike region A**.
- Find a **complex codimension-2 extremal surface γ_A** in a **complexified bulk**, **anchored on ∂A (<u>real boundary</u>)**. Then:

$$S_A^{(T)} \sim \frac{\mathrm{Area}[\gamma_A^{\mathbb{C}}]}{4G_N^{(d+1)}}$$

Heller, FO, Serantes '24



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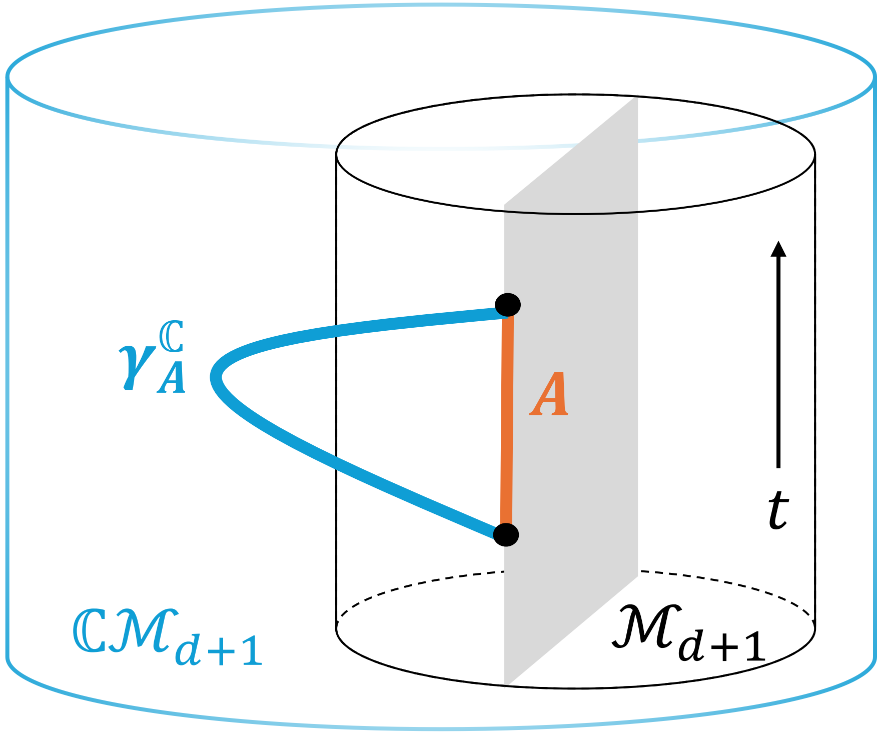
$$S_A^{(T)} \sim \frac{\text{Area}\left[\gamma_A^{\mathbb{C}}\right]}{4G_N^{(d+1)}}$$

\+ Matches with **analytic continuation** of all known closed form expressions.

\+ Can be applied to **any spacetime**.

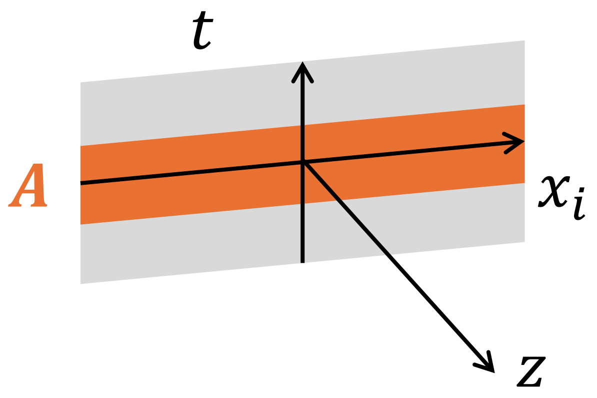
Heller, FO, Serantes '24



Timelike entanglement
and the black hole interior

Probes of the black hole interior?

Timelike entanglement in $SAdS_4$



- Strip subsystem A. Metric:

$$ds^2 = \frac{1}{z^2}\left(-f(z)\,dt^2 + \frac{dx_\parallel^2}{f(z)} + d\mathbf{x}_\perp^2\right)$$

$$f(z) = 1 - \left(\frac{z}{z_H}\right)^d$$

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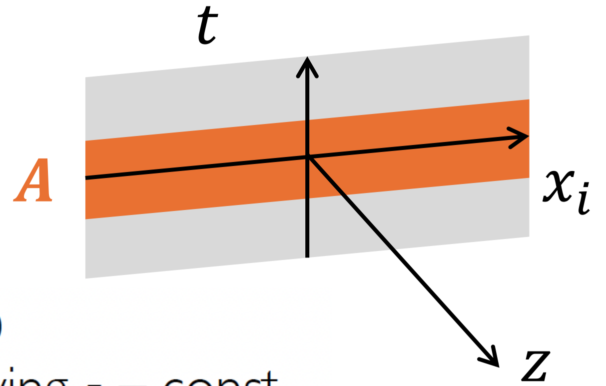

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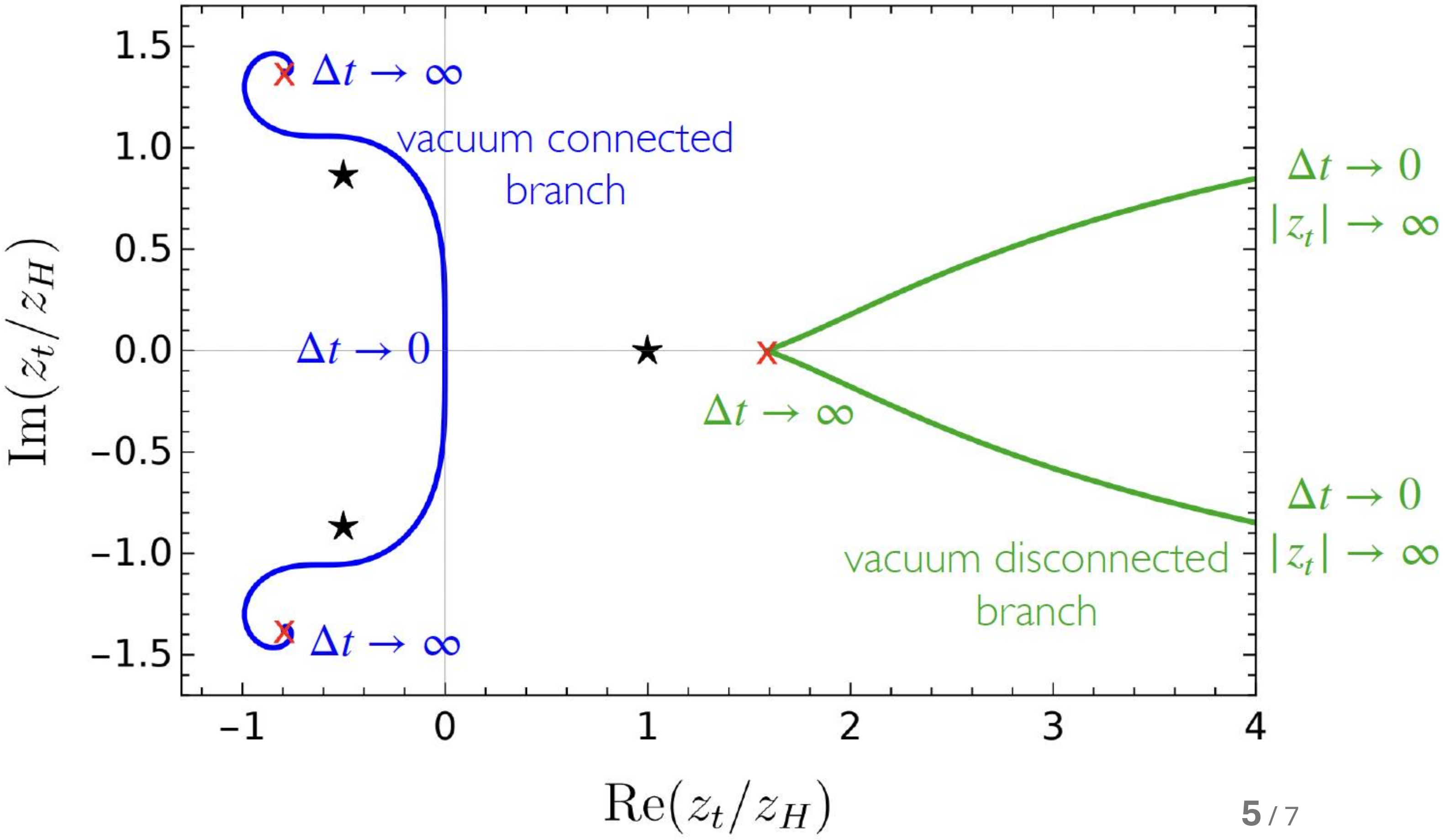
- New: **multiple surfaces**!
- **Vacuum-disconnected** ones probe a **complexified black hole interior**.

Heller, FO, Serantes ’24

★ : solutions of $f(z) = 0$

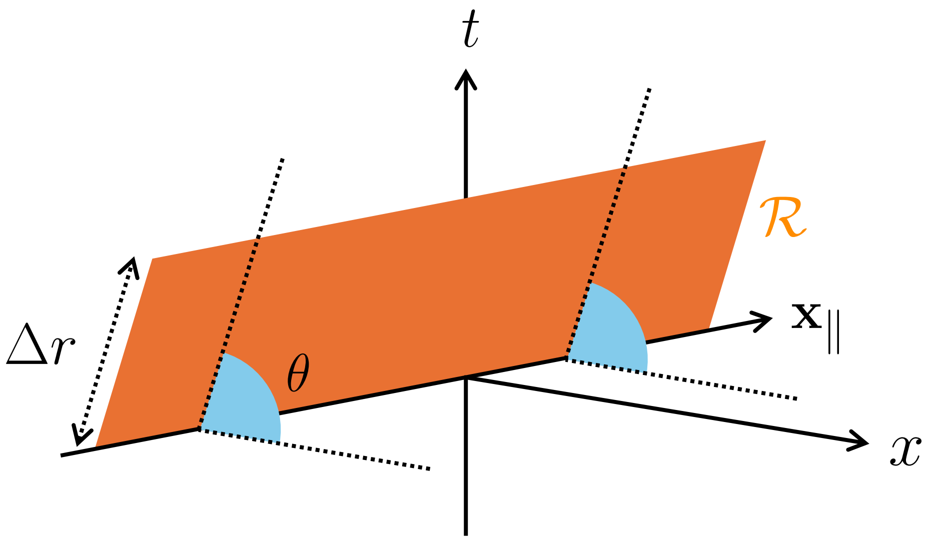
× : critical solutions having $z =$ const

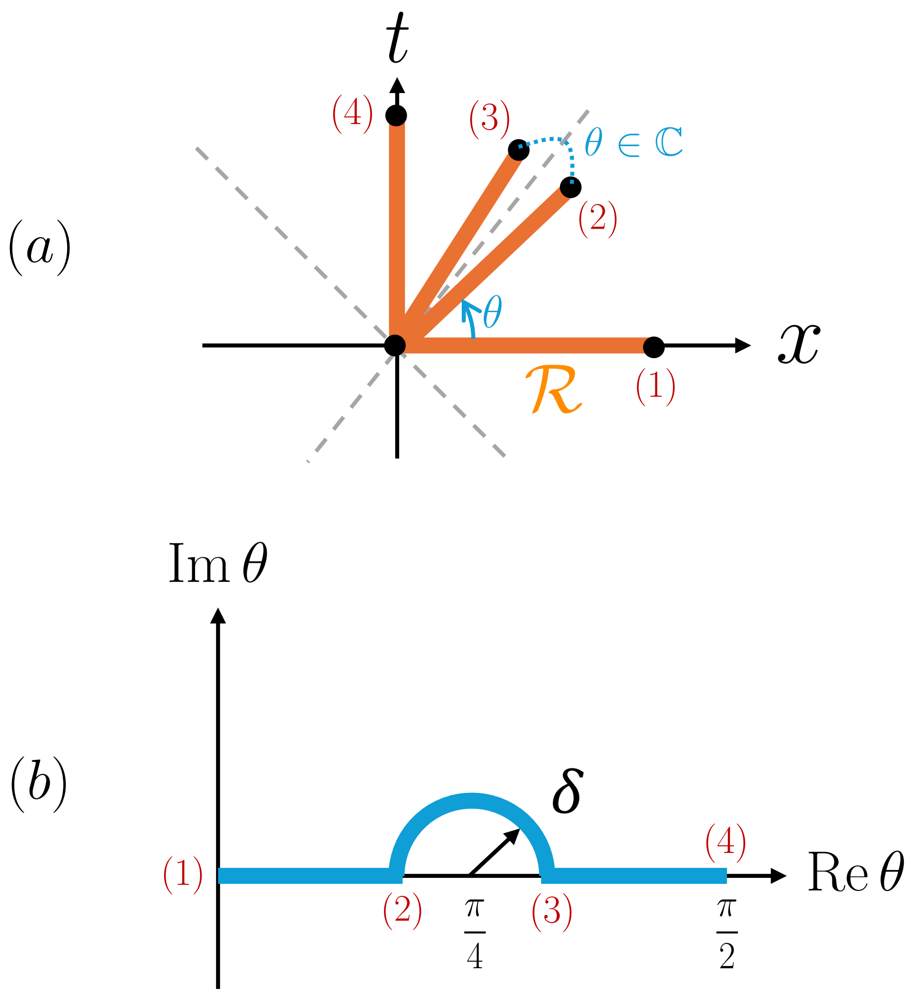


Probes of the black hole interior?

Interior from analytic continuation

Heller, FO, Serantes ’25
(to appear)





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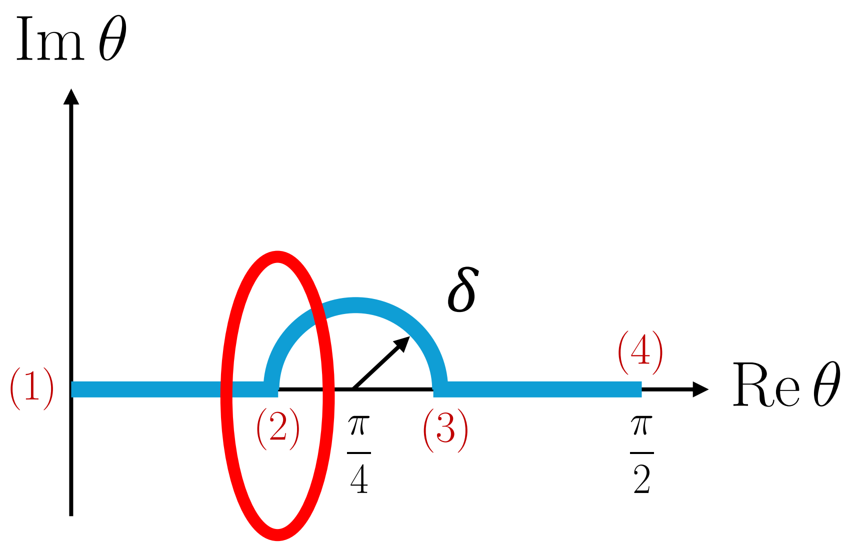
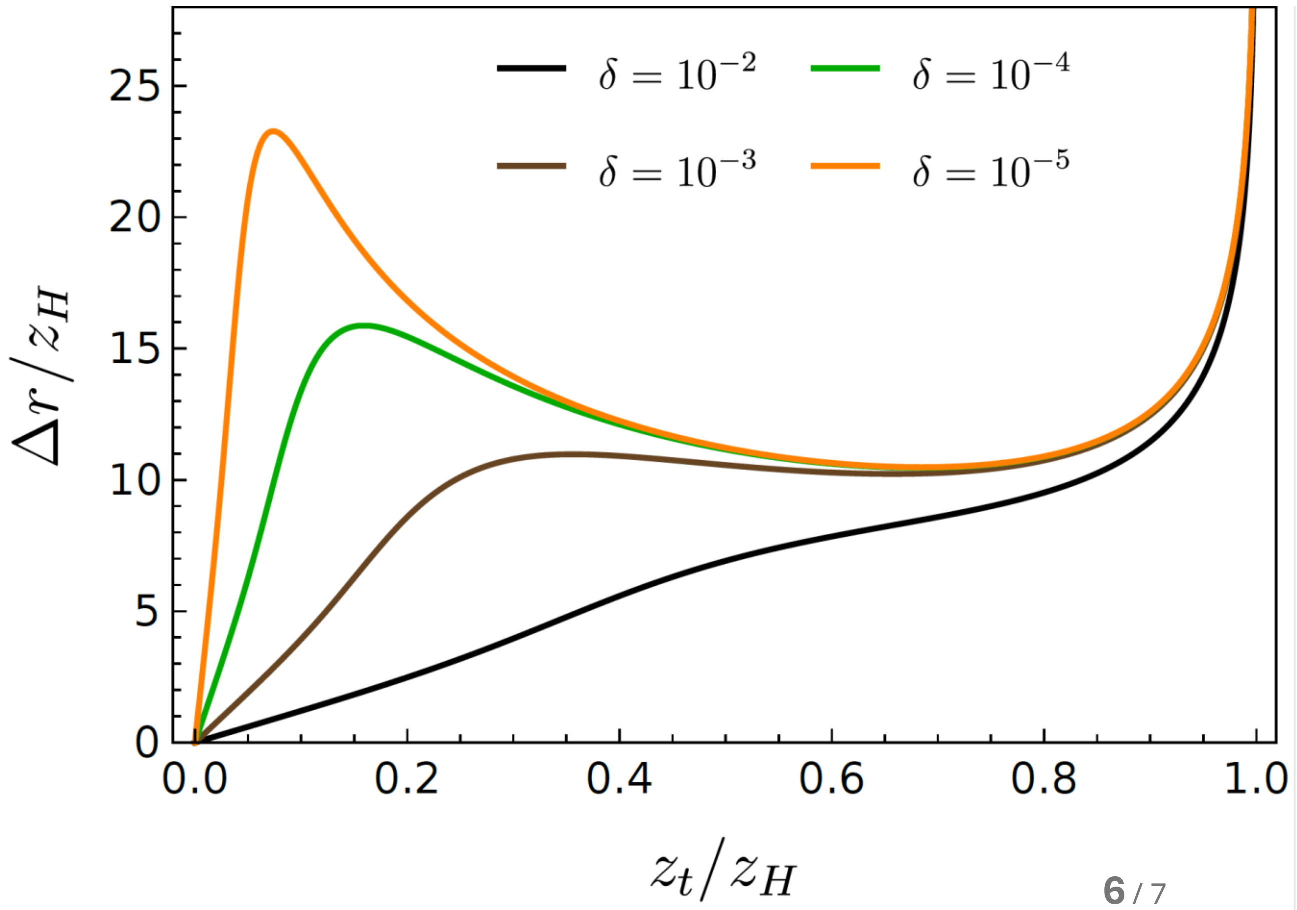


- New **phase transition** in ***spacelike*** **entanglement!**

Slightly (δ) before crossing the lightcone:



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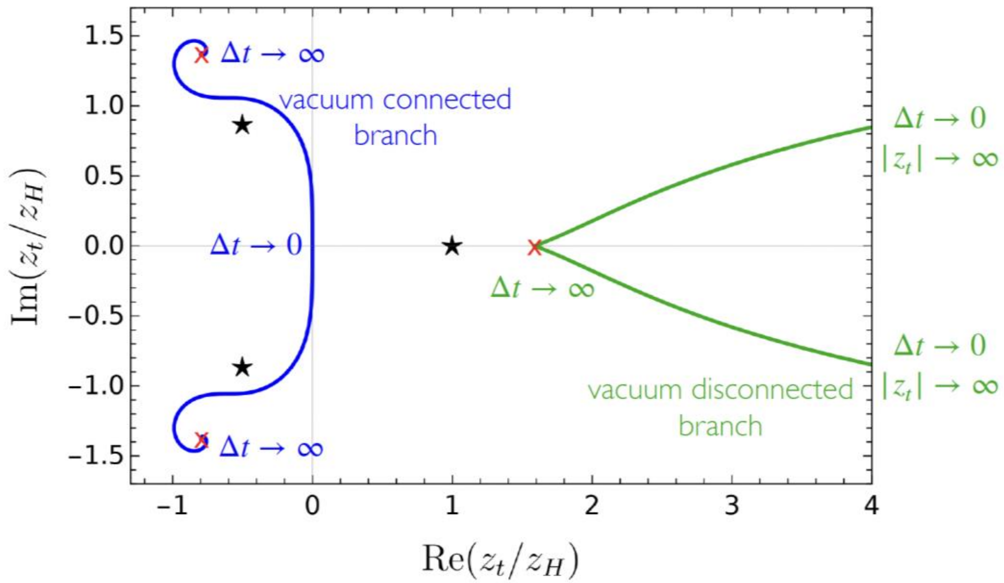
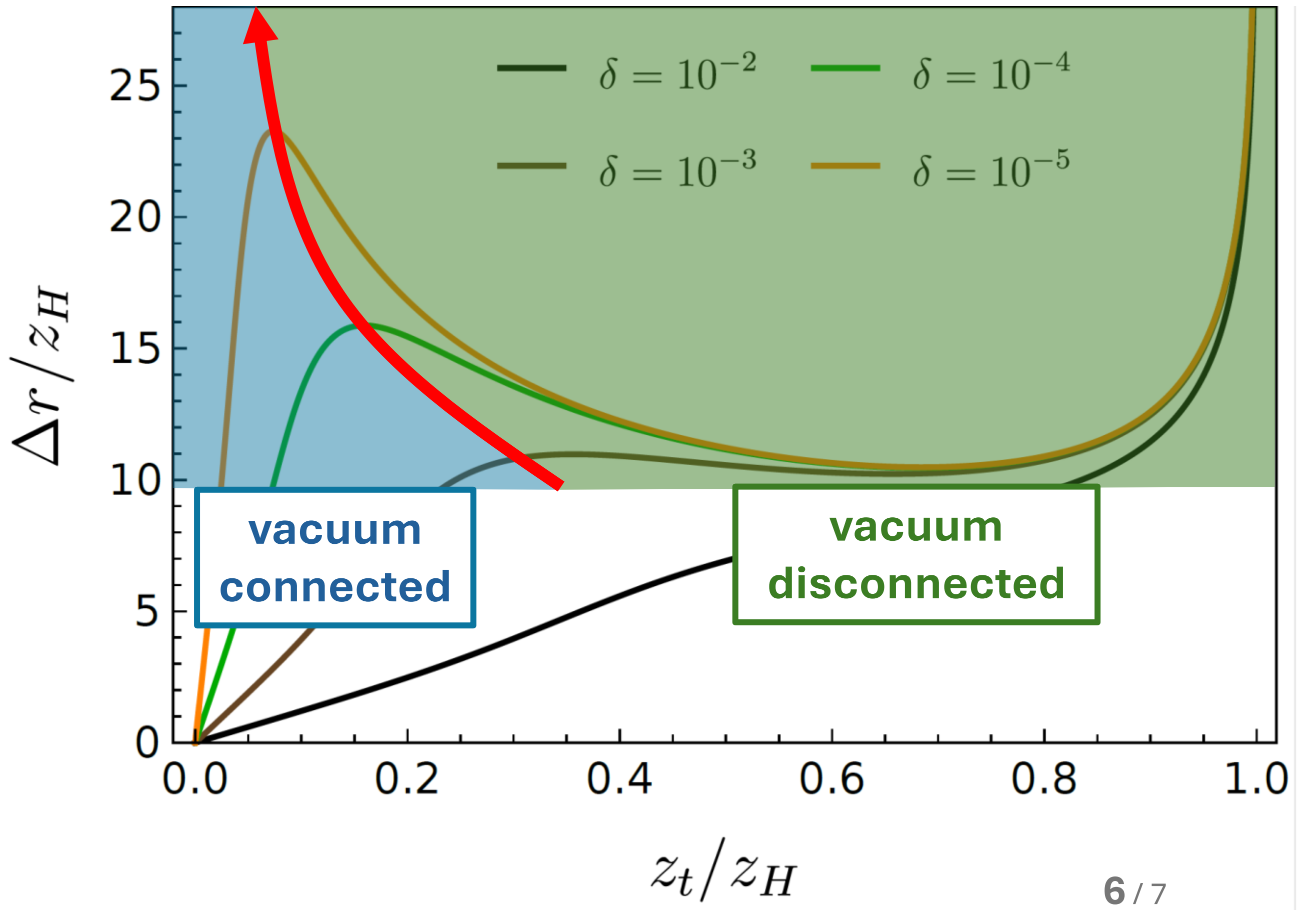


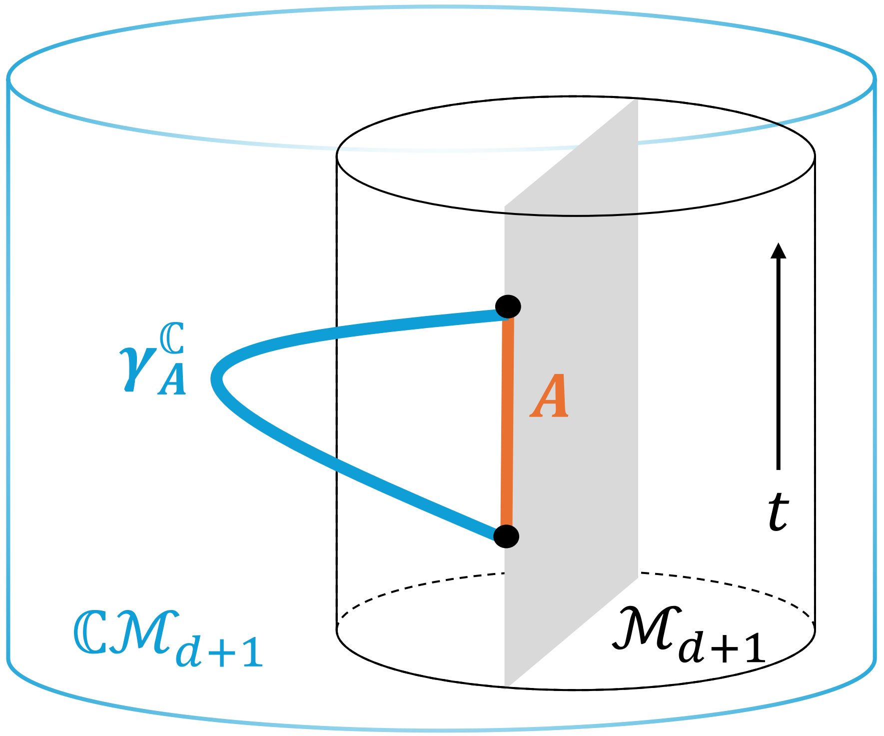
- **Dominant contributions** continue to timelike entropy.
- **Subdominant contributions** continue to the « **BH interior** ».

Slightly (δ) before crossing the lightcone:



Summary and outlook

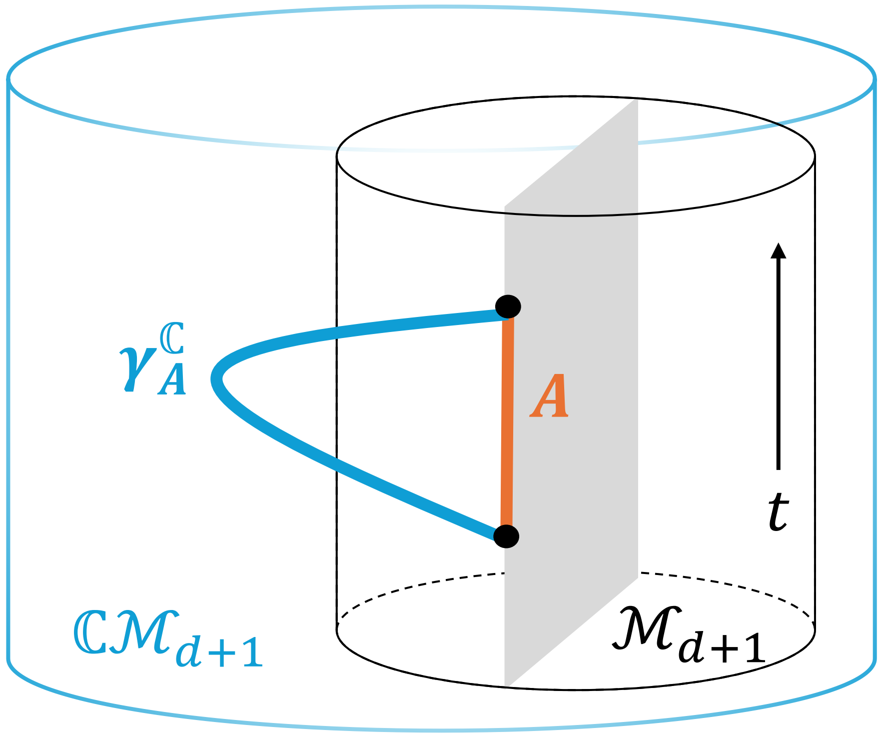
- **Timelike entanglement entropy** is understood geometrically only in a **complexified bulk spacetime**.
- **One-sided** holographic probes that arise from a **phase transition in spacelike entanglement** can reach a **complexified black hole interior**.



Summary and outlook

- **Timelike entanglement entropy** is understood geometrically only in a **complexified bulk spacetime**.
- **One-sided** holographic probes that arise from a **phase transition in spacelike entanglement** can reach a **complexified black hole interior**.

- *Relation to* ***two-sided settings****?*
- ***Complex holographic probes****:* *an (almost) unexplored landscape.*
- ***Field theoretical*** *realisations?* Tensor networks, Gaussian fields.

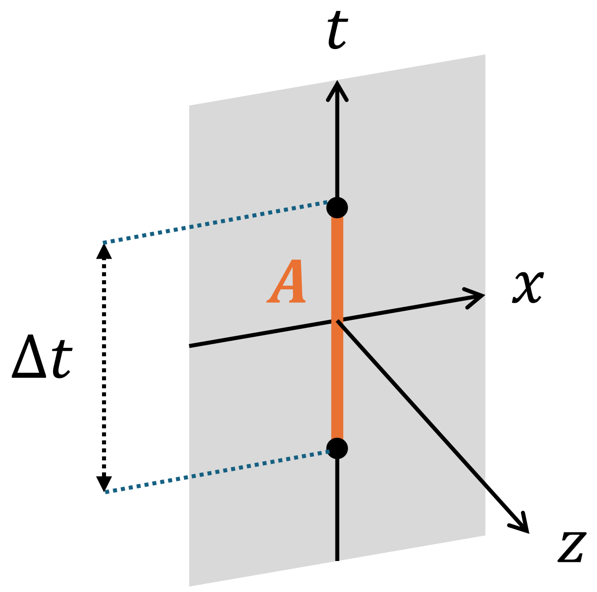


Carignano, Marimón, Tagliacozzo ’23, …

Thank you!

Supplemental slides

Simplest check of the prescription: AdS_3

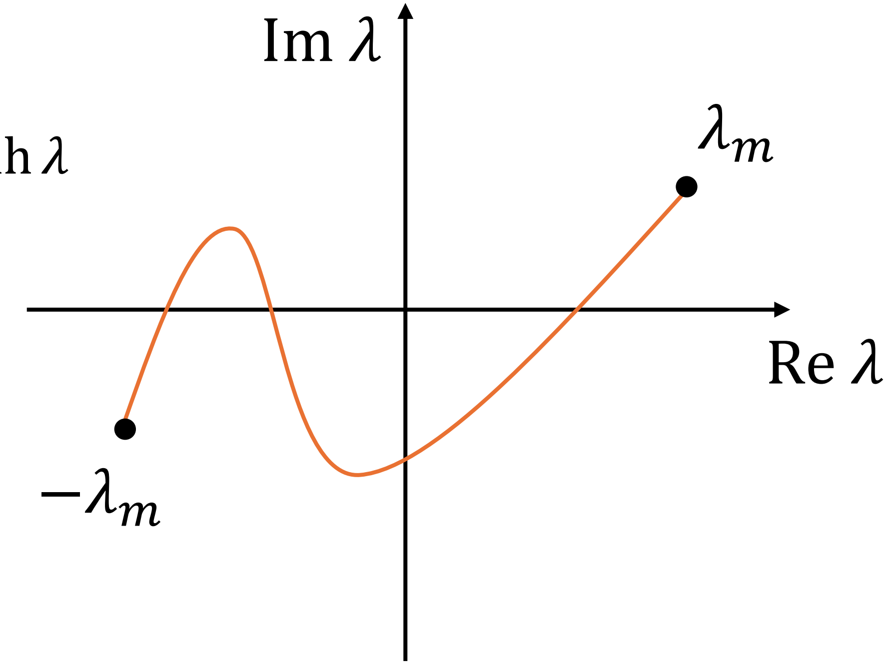


- Consider $A = \left[-\frac{1}{2}\Delta t, \frac{1}{2}\Delta t\right]$, at constant $x = 0$.
- We look for a geodesic joining two timelike-separated points at the boundary $z = \varepsilon \to 0$:

$$z(\lambda) = i\sqrt{\frac{1}{4}\Delta t^2 - \varepsilon^2}\,\operatorname{csch}\lambda\,, \qquad t(\lambda) = \sqrt{\frac{1}{4}\Delta t^2 - \varepsilon^2}\,\tanh\lambda$$

- The boundary conditions are at:

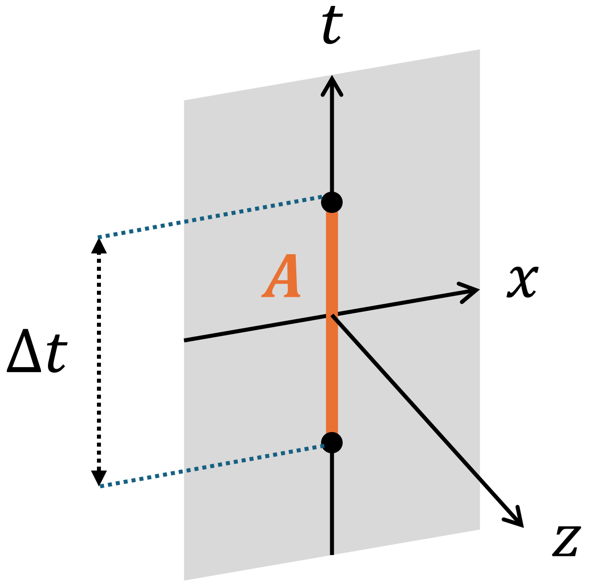
$$\lambda_m = \log\frac{\Delta t}{\varepsilon} + \frac{i\pi}{2} + O(\varepsilon)$$

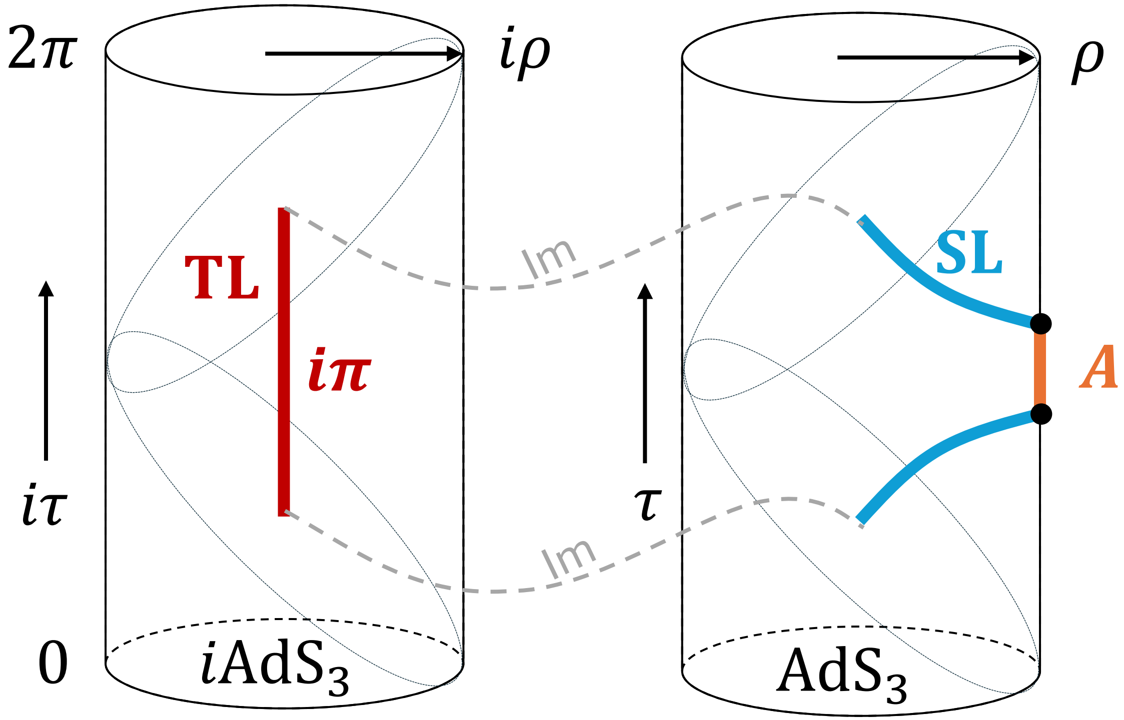


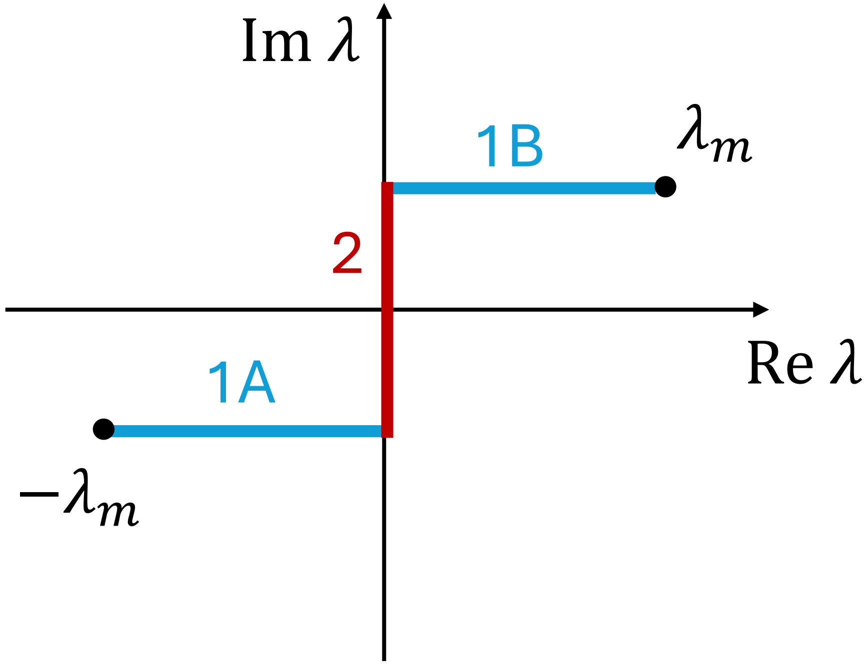
Supplemental slides

Simplest check of the prescription: AdS_3

- An interesting path:
  along paths 1A and 1B: $z, t \in \mathbb{R}$; along path 2: $z, t \in i\,\mathbb{R}$.



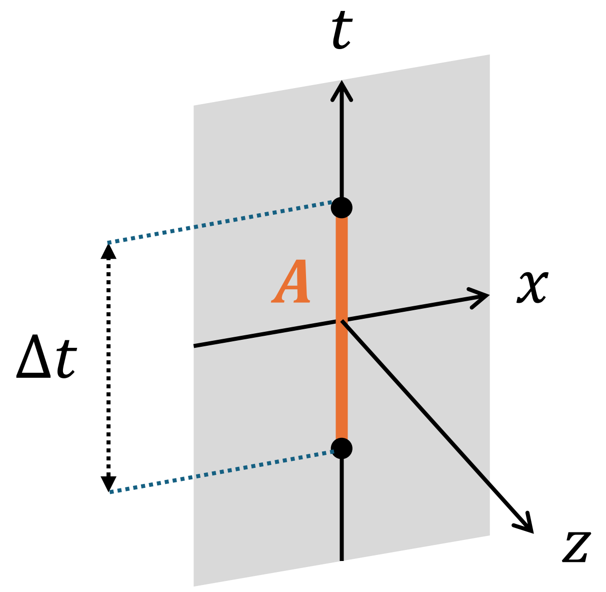


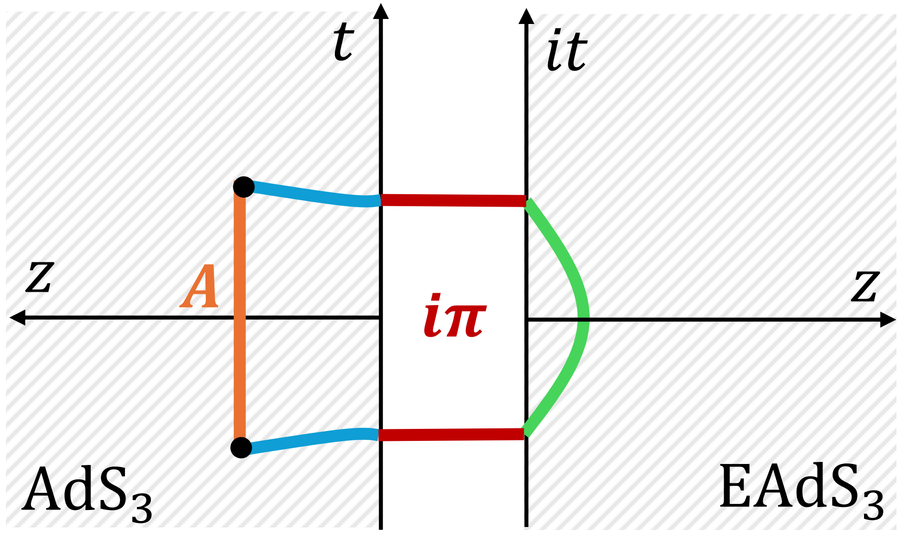


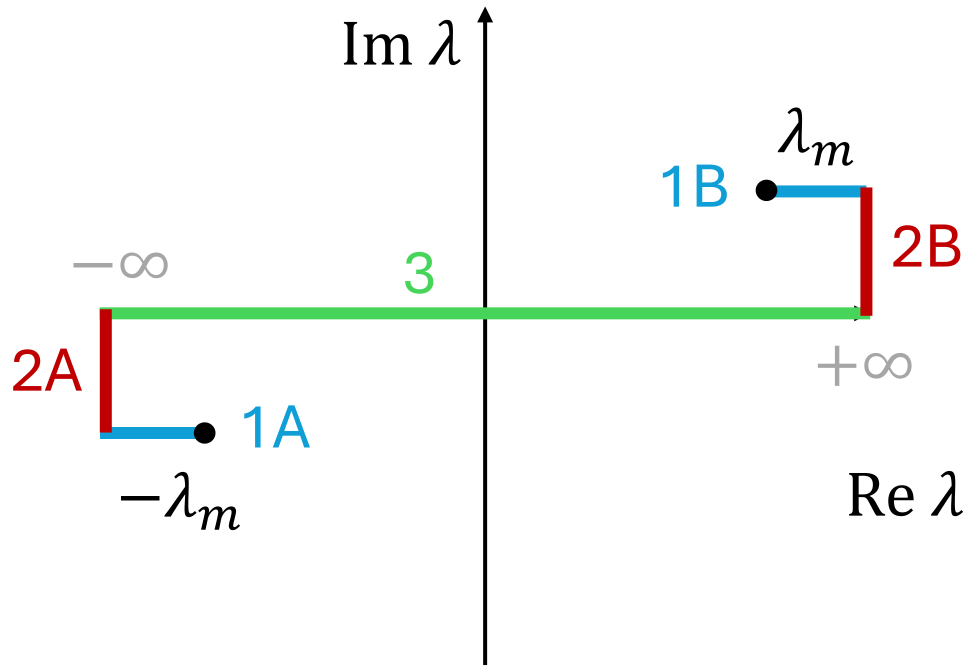
Supplemental slides

Simplest check of the prescription: AdS_3

- Another example:
  along paths 1A and 1B: $z, t \in \mathbb{R}$;
  along paths 2A and 2B: $z, t = \text{const}$;
  along path 3: $z, t \in \mathbb{R}$.







Supplemental slides

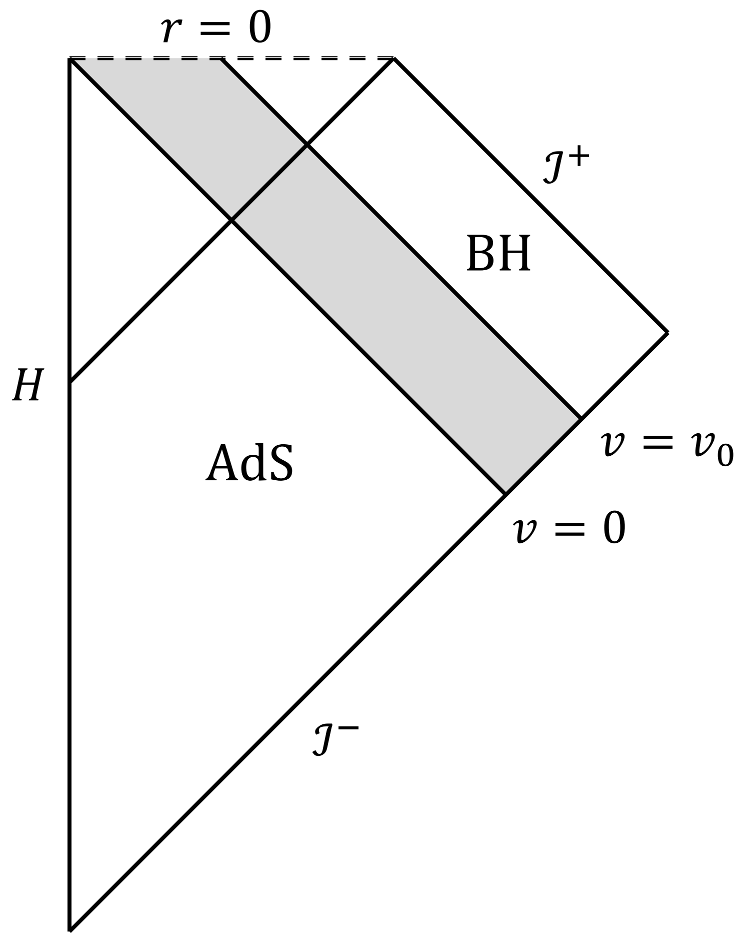
Vaidya spacetime in 2+1 dimensions

- Metric and mass-function:

$$\mathrm{d}s^2 = \frac{-(1 - m(v)\, z^2)\, \mathrm{d}v^2 - 2\mathrm{d}v\, \mathrm{d}z + \mathrm{d}x^2}{z^2}$$

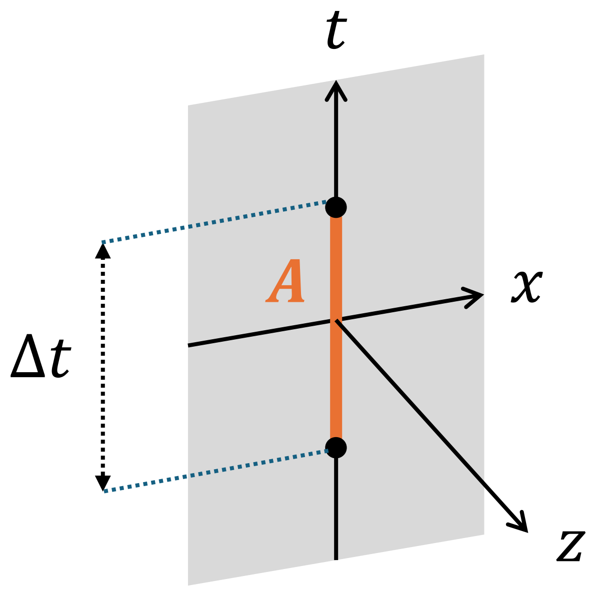
$$m(v) = \frac{\alpha}{2}(1 + \tanh \gamma v)$$

- It represents the formation of a black brane of temperature $T = \sqrt{\alpha}/2\pi$ on a timescale $1/\gamma$ by the **gravitational collapse** of a shell of **null dust**.



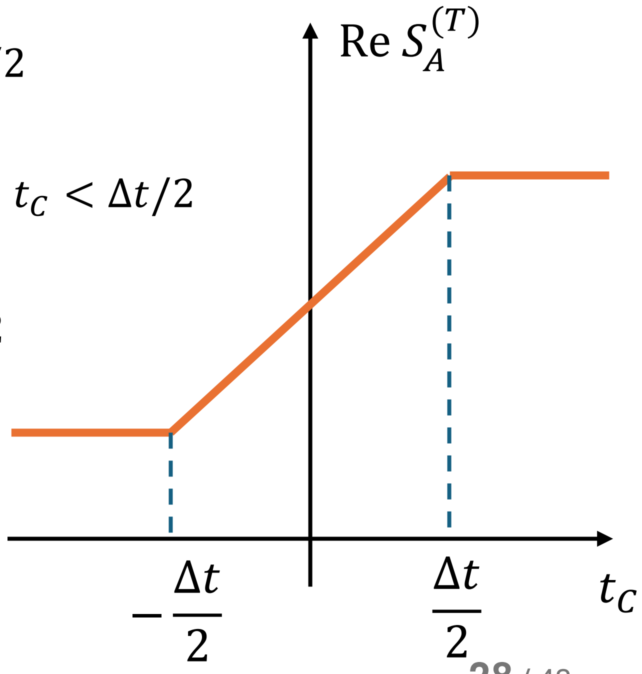
Supplemental slides

An exactly solvable timelike quench



- **Thin-shell** limit in AdS_3-Vaidya: analytically solvable.
- The shell is defined by $\boldsymbol{v = 0}$ (**matching at real $\boldsymbol{v}$**). For a region $A = [t_C - \Delta t/2, t_C + \Delta t/2]$ we have:

$$S_A^{(T)} = i\pi + \begin{cases} 2\log(\Delta t), & t_C < -\Delta t/2 \\ 2\log\left[\frac{2}{r_H}\sinh\left(\frac{r_H(2t_C+\Delta t)}{4}\right) - \cosh\left(\frac{r_H(2t_C+\Delta t)}{4}\right)\left(t_C - \frac{\Delta t}{2}\right)\right] & -\Delta t/2 < t_C < \Delta t/2 \\ 2\log\left[\frac{2}{r_H}\sinh\left(\frac{r_H}{\Delta t}\right)\right] & t_C > \Delta t/2 \end{cases}$$

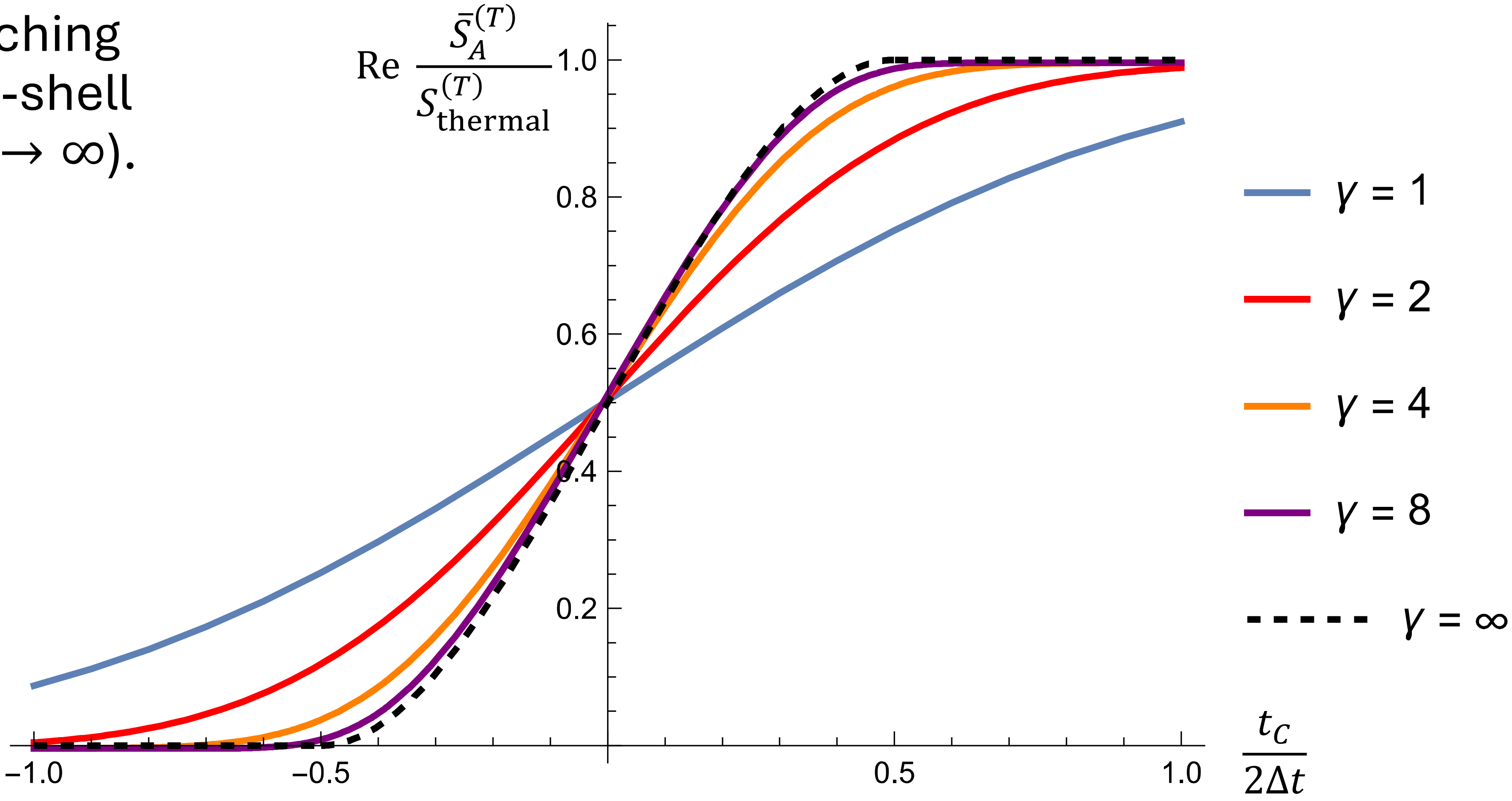


Balasubramanian, Bernamonti, Craps, Keränen, Keski-Vakkurif, Müller, Thorlacius, Vanhoofd ’12

Supplemental slides

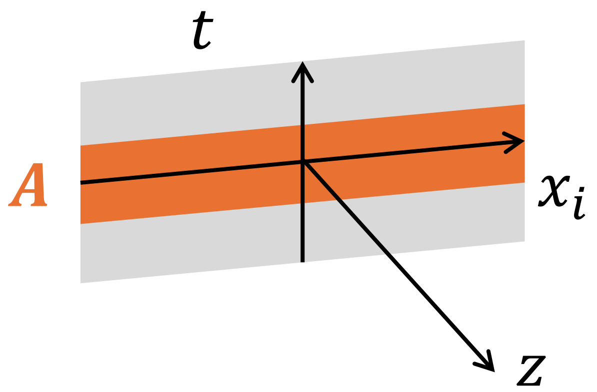
Crosscheck of complex geodesics

- Approaching the thin-shell limit ($\gamma \to \infty$).



Supplemental slides

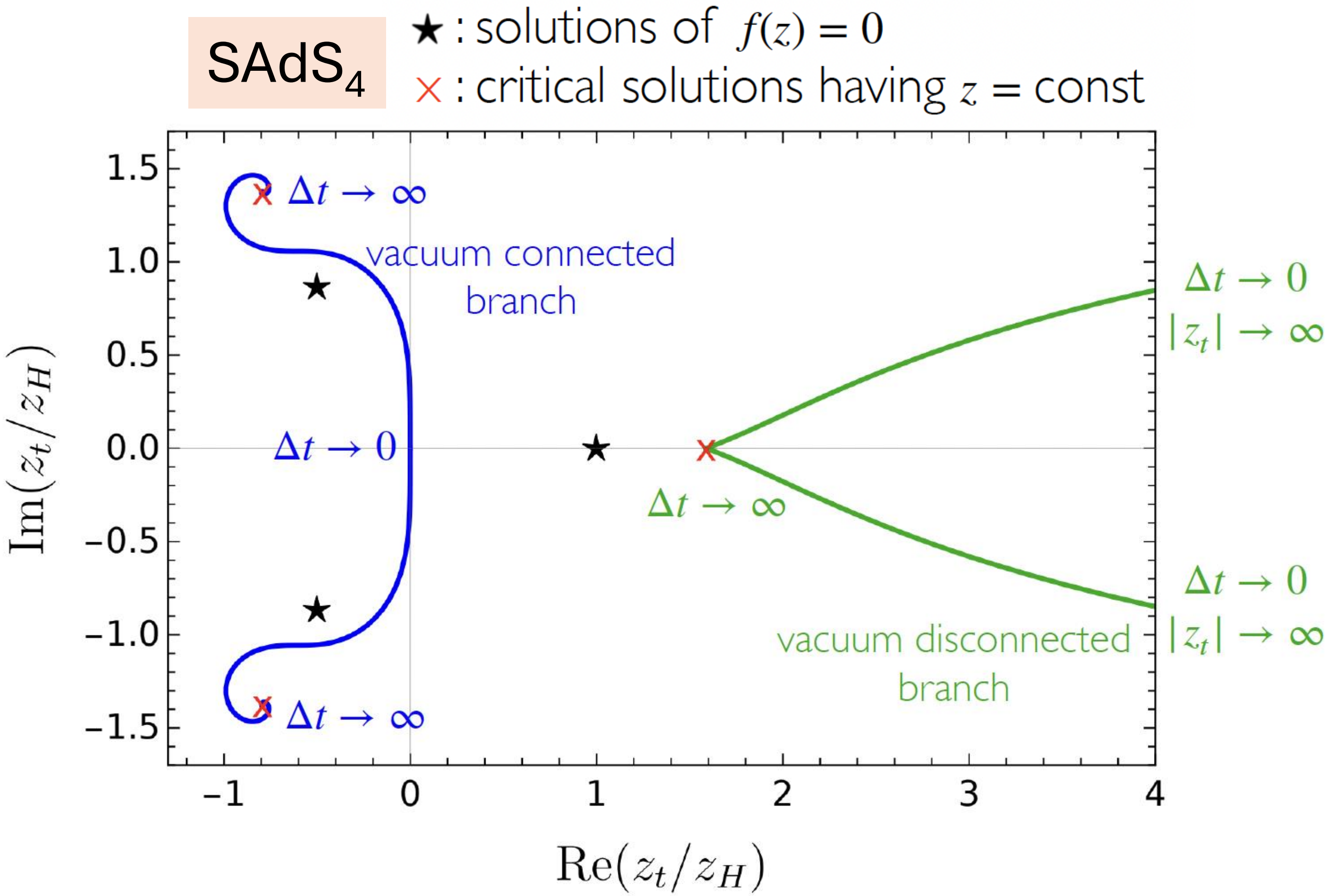
Timelike entanglement in $SAdS_4$



- Pick the surface with **minimal real part** of the **area**?

Prescription P1

$$S_A^{(T)} = \min_{\mathbf{Re}\,\mathrm{Area}[\gamma_A^{\mathbb{C}}]} \frac{\mathrm{Area}[\gamma_A^{\mathbb{C}}]}{4G_N^{(d+1)}}$$



Supplemental slides

Timelike entanglement in $SAdS_4$

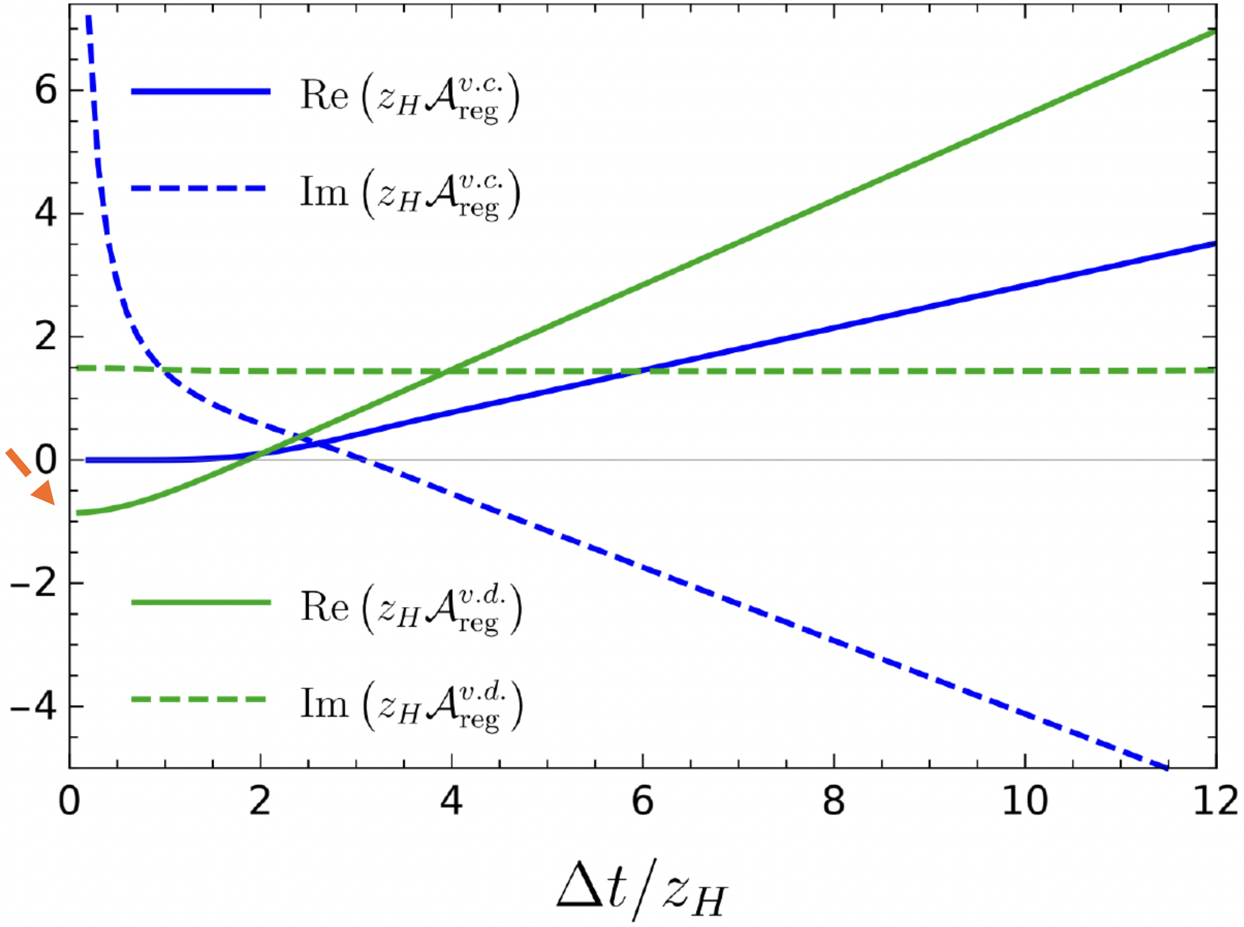
$$\mathcal{A}_{\text{reg}} = \lim_{\epsilon\to 0}\left(\mathcal{A} - \frac{2}{\epsilon}\right)$$

- Problem: the **UV/IR correspondence is violated**!
- Possible resolution:

Prescription P2

$$S_A^{(T)} = \frac{\text{Area}[\gamma_A^{\mathbb{C}}]}{4G_N^{(d+1)}}$$

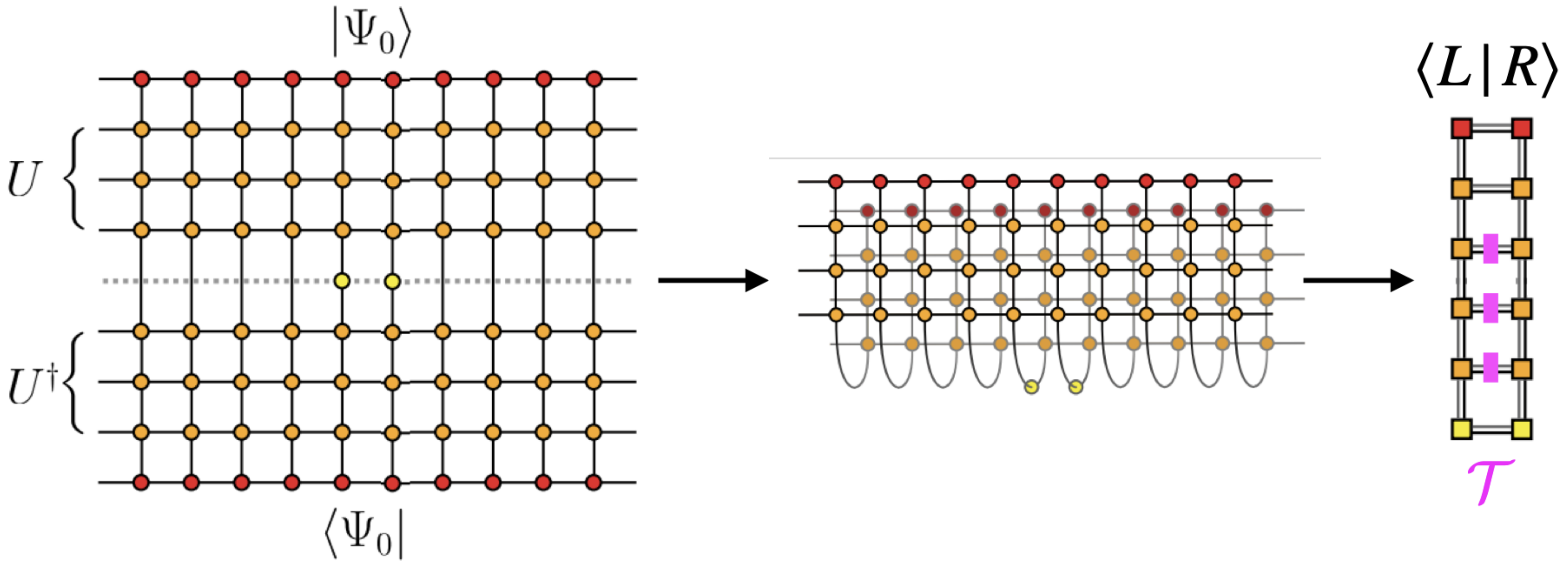
for a $\gamma_A^{\mathbb{C}}$ such that **upon analytic continuation $S_A^{(T)}$ reduces to standard entanglement entropy.**



Supplemental slides

Towards a field theory definition

- Connection with « **temporal entanglement** » in **tensor networks**:



Bañuls, Hastings, Verstraete '09; Hastings, Mahajan '23; Carignano, Marimón, Tagliacozzo '23

Entropies

$$\rho_L = \frac{\mathrm{Tr}_{\bar{\mathcal{T}}} |L\rangle\langle L|}{\langle L|L\rangle}$$

$$\rho_R = \frac{\mathrm{Tr}_{\bar{\mathcal{T}}} |R\rangle\langle R|}{\langle R|R\rangle}$$

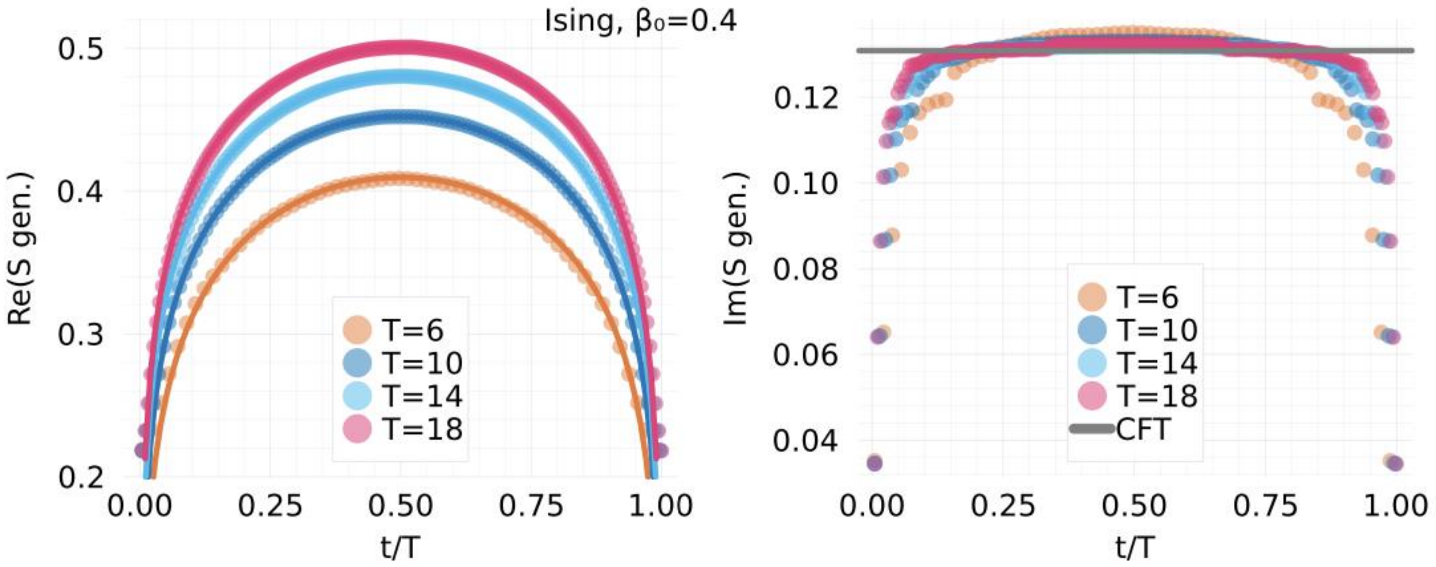
Pseudoentropy

$$\tilde{\rho}_{LR} = \frac{\mathrm{Tr}_{\bar{\mathcal{T}}} |R\rangle\langle L|}{\langle L|R\rangle}$$

Supplemental slides

Towards a field theory definition

- Tensor network evaluation of timelike entanglement entropy:



Carignano, Marimón, Tagliacozzo ’23

- Can we do more? **Quenches** and **time-dependent backgrounds** can be explored with tensor networks.

Foligno, Zhou, Bertini ’23