Kasner interiors and complexity of analytic hairy black holes

Based on: "Kasner interiors from analytic hairy black holes" with D. Arean, H. S. Jeong, and J. F. Pedraza, [arXiv:2407.18430]

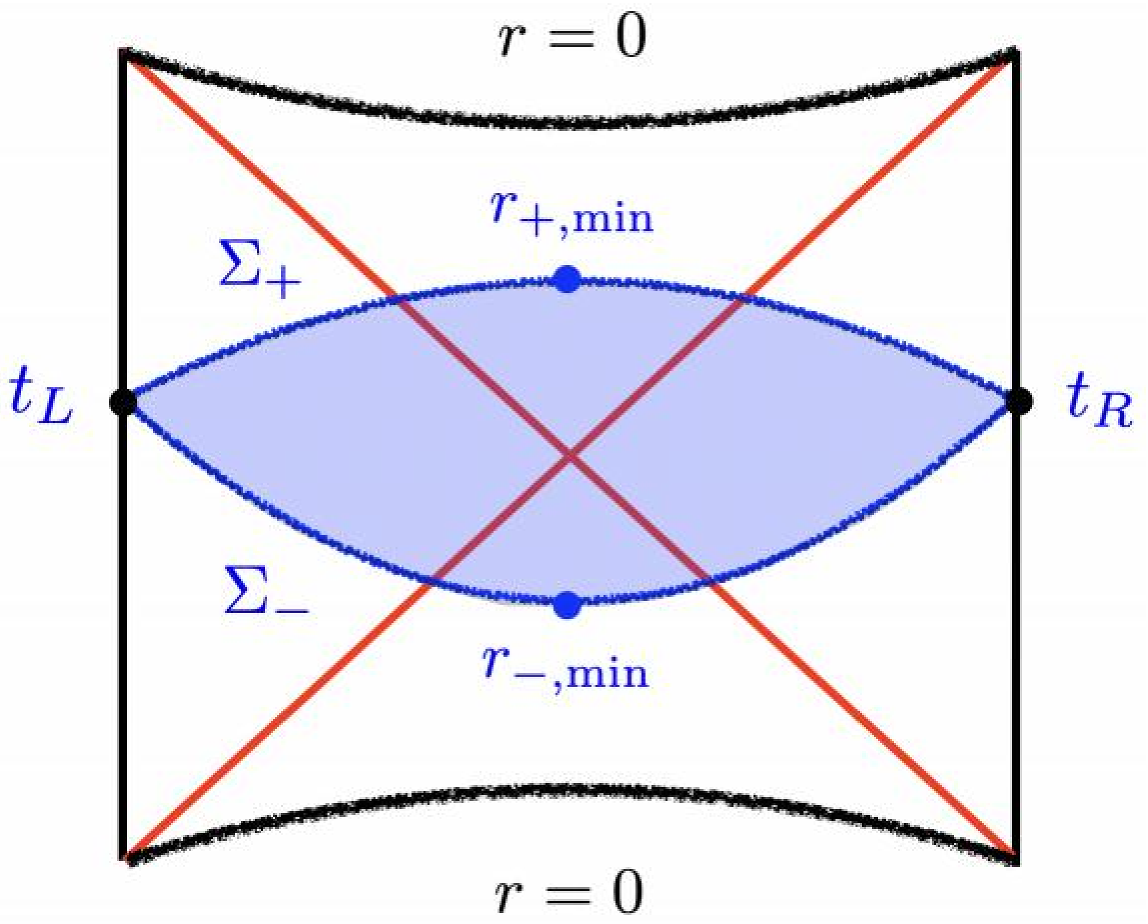
Le-Chen Qu

Introduction and motivation

In this work, we investigate Holographic Complexity to diagnose singularities. In the Kasner case, we propose a new variant of complexity that characterizes the late-time rate by the Kasner exponents, extending previous work by Jørstad, Myers and Ruan.

[Belin, Myers, Ruan, Sárosi, Speranza '21]

Complexity = Anything



The definition of complexity follows a two-step procedure. First, one needs to specify a bulk codimension-zero region M. To do so, one defines a functional of the form

$$W(\mathcal{M}) = \int_{\Sigma_+} \mathrm{d}^d\sigma \sqrt{h}\, F_+(g_{\mu\nu}, X_+^\mu) + \int_{\Sigma_-} \mathrm{d}^d\sigma \sqrt{h}\, F_-(g_{\mu\nu}, X_-^\mu) + \int_{\mathcal{M}} \mathrm{d}^{d+1}x \sqrt{g}\, G(g_{\mu\nu})$$

Complexity = Anything [Belin, Myers, Ruan, Sárosi, Speranza '21]

The region M can then be found by extremizing such a functional, such that

$$\delta[W(\mathcal{M})] = 0 \,.$$

To solve this extremization problem, one should impose boundary conditions such that M is anchored to the relevant boundary Cauchy slice. The second step amounts to evaluating a "complexity observable" on region M, so that

$$\mathcal{C}(\sigma_{\mathrm{CFT}}) = \int_{\Sigma_+} \mathrm{d}^d\sigma \sqrt{h}\, \mathcal{F}_+(g_{\mu\nu}, X_+^\mu) + \int_{\Sigma_-} \mathrm{d}^d\sigma \sqrt{h}\, \mathcal{F}_-(g_{\mu\nu}, X_-^\mu) + \int_{\mathcal{M}} \mathrm{d}^{d+1}x \sqrt{g}\, \mathcal{G}(g_{\mu\nu}) \,.$$

Complexity = Anything

[Jørstad, Myers, Ruan '21]

Since M includes regions outside and inside the event horizon, using Eddington Finkelstein coordinates is advantageous. The metric in these coordinates is given by:

$$\mathrm{d}s^2 = -D(r)\,\mathrm{d}v^2 + 2\sqrt{D(r)B(r)}\,\mathrm{d}v\,\mathrm{d}r + C(r)\mathrm{d}\vec{x}_i^2\,,$$

where the infalling coordinate v is defined as

$$v = t + r_*(r)\,, \qquad r_*(r) = \int_\infty^r \sqrt{\frac{B(\tilde{r})}{D(\tilde{r})}}\,\mathrm{d}\tilde{r}\,.$$

Observables on constant mean curvature slices

To define the gravitational observable, we first need to define the region of interest M. This is obtained by extremizing the functional

$$W_{\mathrm{CMC}} = \frac{1}{G_{\mathrm{N}} L}\left[\alpha_+ \int_{\Sigma_+} \mathrm{d}^d\sigma\sqrt{h} + \alpha_- \int_{\Sigma_-} \mathrm{d}^d\sigma\sqrt{h} + \frac{\alpha_{\mathrm{B}}}{L}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{g}\right]$$

By extremizing the above functional, the resulting hypersurfaces $\Sigma\pm$ are found to be Constant Mean Curvature (CMC) slices, with extrinsic curvatures given by

$$K_{\Sigma_+} = -\frac{\alpha_{\mathrm{B}}}{\alpha_+ L}, \qquad K_{\Sigma_-} = \frac{\alpha_{\mathrm{B}}}{\alpha_- L}.$$

To define the codimension-one observable, we then set G and either F+ or F− to zero. This results in the following complexity observable, evaluated on a single hypersurface:

$$\mathcal{C}^{\pm} = \frac{1}{G_N L}\int_{\Sigma_\pm} \mathrm{d}^d\sigma\sqrt{h}\,\mathcal{F}(\sigma)$$

[Jørstad, Myers, Ruan '21]

The maximization of CMC slices [Jørstad, Myers, Ruan '21]

Next, we provide an algorithm to obtain the CMC slices. We start by rewriting the bulk term as

$$\begin{aligned}\frac{\alpha_{\mathrm{B}}}{L}\int_{\mathcal{M}}\mathrm{d}^{d+1}x\sqrt{g} &= \frac{\alpha_{\mathrm{B}}}{L}\int\mathrm{d}^{d-1}x\int\mathrm{d}v\int_{r_+}^{r_-}\mathrm{d}r\sqrt{D(r)B(r)C(r)^{d-1}}\\ &= \alpha_{\mathrm{B}}V_{d-1}\left[\int_{\Sigma_-}\mathrm{d}\sigma\,\dot{v}_-(\sigma)\,b(r_-)\;-\int_{\Sigma_+}\mathrm{d}\sigma\,\dot{v}_+(\sigma)\,b(r_+)\right]\end{aligned}$$

where we defined a function b(r) and the spatial volume V such that

$$\frac{\partial b(r)}{\partial r}=\frac{\sqrt{D(r)B(r)C(r)^{d-1}}}{L}\,,\qquad V_{d-1}=\int\mathrm{d}^{d-1}x\,.$$

The maximization of CMC slices [Jørstad, Myers, Ruan '21]

Then, we obtain
$$W_{\mathrm{CMC}} = \frac{V_{d-1}}{G_{\mathrm{N}} L} \sum_{\varepsilon=+,-} \int_{\Sigma_\varepsilon} \mathrm{d}\sigma\, \mathcal{L}_\varepsilon \,,$$

where
$$\mathcal{L}_\varepsilon = \alpha_\varepsilon\, C(r_\pm)^{\frac{d-1}{2}} \sqrt{-D(r_\pm)\,\dot{v}_\pm^2 + 2\sqrt{D(r_\pm)B(r_\pm)}\,\dot{v}_\pm \dot{r}_\pm} - \varepsilon\, \alpha_{\mathrm{B}}\, \dot{v}_\pm\, b\,(r_\pm)\ .$$

Thus, finding the corresponding CMC slices (i.e., extremal surfaces of the functional W) is equivalent to solving the equations of motion derived from the "Lagrangian" L. To achieve this, it is useful to fix the gauge by choosing

$$\sqrt{-D(r_\pm)\dot{v}_\pm^2 + 2\sqrt{D(r_\pm)B(r_\pm)}\dot{v}_\pm \dot{r}_\pm} = C(r_\pm)^{\frac{d-1}{2}} \sqrt{B(r_\pm)D(r_\pm)}\,.$$

The maximization of CMC slices [Jørstad, Myers, Ruan '21]

By using gauge condition in the Lagrangian L, one can derive the equation of motion for the radial profile r±(σ) as: $\dot{r}_\pm^2 + \mathcal{U}\left(P_v^\pm, r_\pm\right) = 0\,,$

where the "potential" U is given by

$$\mathcal{U}\left(P_v^\pm, r_\pm\right) = -D(r_\pm)C(r_\pm)^{d-1} - \left(P_v^\pm \pm \frac{\alpha_\mathrm{B}}{\alpha_\pm} b(r_\pm)\right)^2$$

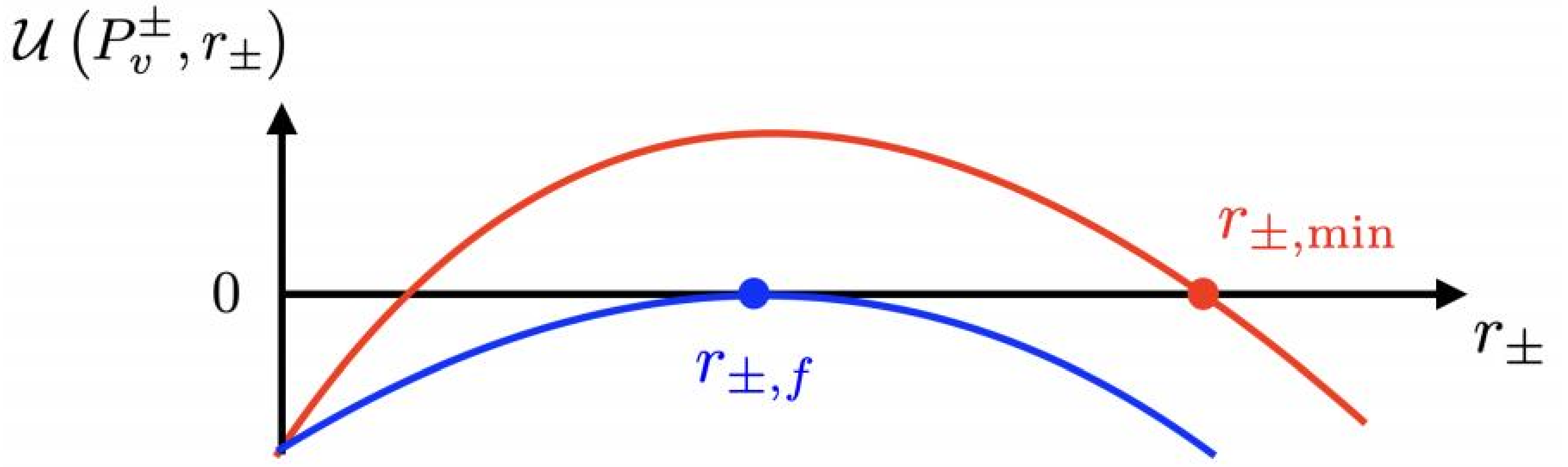
and the conjugate conserved momenta Pv are

$$P_v^\pm = \frac{\partial \mathcal{L}_\pm}{\partial \dot{v}_\pm} = \dot{r}_\pm - \sqrt{\frac{D(r_\pm)}{B(r_\pm)}}\dot{v}_\pm \mp \frac{\alpha_\mathrm{B}}{\alpha_\pm} b(r_\pm)\,.$$

Boundary time and conserved momentum

Note that the potential can exhibit zeros at a fixed Pv

$$\mathcal{U}\left(P_v^{\pm}, r_{\pm}\right) = 0\,.$$



the corresponding boundary time t

$$t_b = 2\int_{\infty}^{r_{\pm,\mathrm{min}}} \frac{\sqrt{B(r_{\pm})}\left(P_v^{\pm} \pm \frac{\alpha_B}{\alpha_{\pm}} b(r_{\pm})\right)}{\sqrt{D(r_{\pm})}\sqrt{-\mathcal{U}\left(P_v^{\pm}, r_{\pm}\right)}} \mathrm{d}r_{\pm}\,.$$

The late-time limit and large mean curvature

To determine the behavior at late times (t → ∞), it is essential to discuss the expansion of the potential near r±,min,

$$\mathcal{U}\left(P_v^{\pm}, r_{\pm}\right) = \partial_{r_{\pm}} \mathcal{U}\left(P_v^{\pm}, r_{\pm}\right)\Big|_{r_{\pm}=r_{\pm,\min}} \left(r_{\pm} - r_{\pm,\min}\right) + \frac{1}{2}\partial_{r_{\pm}}^{2} \mathcal{U}\left(P_v^{\pm}, r_{\pm}\right)\Big|_{r_{\pm}=r_{\pm,\min}} \left(r_{\pm} - r_{\pm,\min}\right)^{2} + \cdots$$

Next, we identify the 'final' zero, r±,f , such that

$$\mathcal{U}\left(P_v^{\pm}, r_{\pm,f}\right) = \partial_{r_{\pm}} \mathcal{U}\left(P_v^{\pm}, r_{\pm}\right)\Big|_{r_{\pm}=r_{\pm,f}} = 0\,.$$

Then, due to the quadratic term, the integral may become divergent, resulting in an infinite t.

The late-time limit and large mean curvature

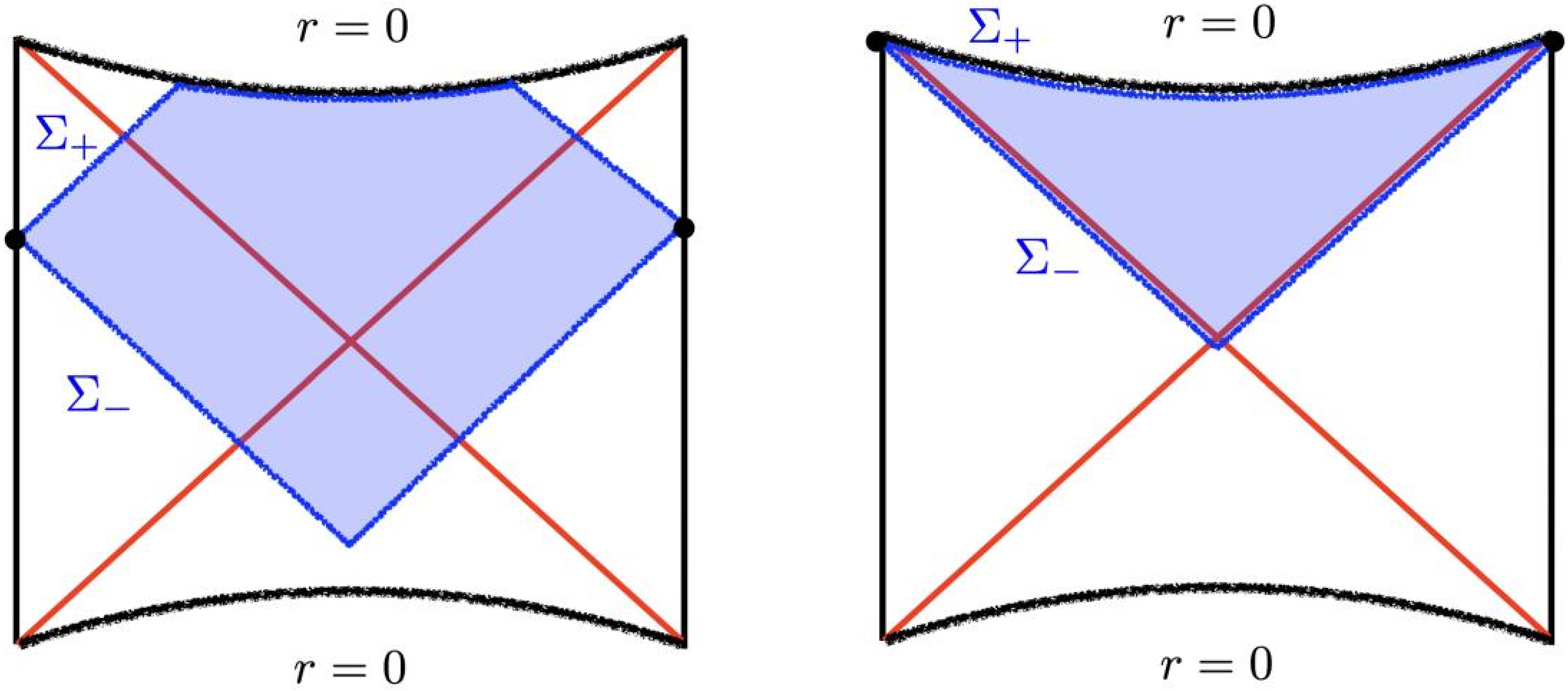
The equation to determine r±,f is given by

$$K_{\Sigma_\pm}^2 + \frac{\left[C(r_{\pm,f})D'(r_{\pm,f}) + (d-1)D(r_{\pm,f})C'(r_{\pm,f})\right]^2}{4B(r_{\pm,f})D(r_{\pm,f})^2C(r_{\pm,f})^2} = 0\,.$$

In particular, in the limit of large mean curvature, K → ± ∞ , one can show that

$$r_{\pm,f} \to \begin{cases} 0\,, & (r_{\pm,f} \text{ is the singularity})\,, \\ r_h\,, & (r_{\pm,f} \text{ is the horizon})\,. \end{cases}$$

The late-time limit and large mean curvature



(a) At a fixed finite time

(b) At the late-time limit

Complexity in the late-time limit

Building upon the previous discussion, we now investigate the late-time behavior of the complexity, evaluated on the previously extremized surface Σ±:

$$\begin{aligned}\mathcal{C}^{\pm} &= \frac{1}{G_N L}\int_{\Sigma_\pm} \mathrm{d}^d\sigma\,\sqrt{h}\,\mathcal{F}(\sigma)\\ &= \frac{V_{d-1}}{G_\mathrm{N} L}\int_{\Sigma_\pm} \mathrm{d}\sigma\, C(r_\pm)^{\frac{d-1}{2}}\;\mathcal{F}(r_\pm)\sqrt{-D(r_\pm)\dot{v}_\pm^2 + 2\sqrt{D(r_\pm)B(r_\pm)}\,\dot{v}_\pm\dot{r}_\pm}\end{aligned}$$

We consider the limit of large mean curvature

$$(K_{\Sigma_\pm} \to \pm\infty): \quad \lim_{t_b\to\infty}\frac{\mathrm{d}\mathcal{C}^{\pm}}{\mathrm{d}t_b} = \frac{V_{d-1}}{G_\mathrm{N} L}\,\mathcal{F}(r_{\pm,f})\,C(r_{\pm,f})^{\frac{d-1}{2}}\sqrt{-D(r_{\pm,f})}$$

Complexity with the past boundary

The past boundary Σ− approaches the event horizon

$$\mathcal{F} = 1 : \quad \lim_{t_b \to \infty} \frac{\mathrm{d}\mathcal{C}^-}{\mathrm{d}t_b} = \frac{V_{d-1}}{G_\mathrm{N} L} C(r_+)^{\frac{d-1}{2}} \sqrt{-D(r_+)} = 0$$

$$\mathcal{F} = |LK| : \quad \lim_{t_b \to \infty} \frac{\mathrm{d}\mathcal{C}^-}{\mathrm{d}t_b} = \frac{V_{d-1}}{G_\mathrm{N}} \left| \frac{C(r)D'(r) + (d-1)D(r)C'(r)}{2\sqrt{B(r)D(r)}} \right| C(r)^{\frac{d-3}{2}} \Bigg|_{r=r_+}$$

$$= 8\pi T_+ S_+ ,$$

Timelike singularity

For timelike singularities, the future boundary Σ+ approaches the inner horizon r− as the mean curvature becomes large,

$$\mathcal{F} = 1: \qquad \lim_{t_b \to \infty} \frac{\mathrm{d}\mathcal{C}^+}{\mathrm{d}t_b} = \frac{V_{d-1}}{G_\mathrm{N} L} C(r_-)^{\frac{d-1}{2}} \sqrt{-D(r_-)} = 0\,,$$

$$\mathcal{F} = |LK|: \quad \lim_{t_b \to \infty} \frac{\mathrm{d}\mathcal{C}^+}{\mathrm{d}t_b} = \frac{V_{d-1}}{G_\mathrm{N}} \left| \frac{C(r)D'(r) + (d-1)D(r)C'(r)}{2\sqrt{B(r)D(r)}} \right| C(r)^{\frac{d-3}{2}} \Bigg|_{r=r_-}$$

$$= 8\pi\, T_- S_-\,,$$

Kasner singularity

Finally, we consider C+ in the presence of Kasner singularities. In the absence of an inner horizon, the future boundary Σ+ approaches the spacelike singularity, r → 0, so in this case we expect to be able to diagnose the singularity. To be more specific, we consider the Kasner metric

$$\mathrm{d}s^2 = -c_r^2\,\mathrm{d}r^2 + c_t^2\,r^{2p_t}\,\mathrm{d}t^2 + r^{2p_x}\,\mathrm{d}\vec{x}_i^2\,,$$

Substituting this expression yields

$$\lim_{t_b\to\infty}\frac{\mathrm{d}\mathcal{C}^+}{\mathrm{d}t_b} = \frac{V_{d-1}}{G_\mathrm{N}L}\,\mathcal{F}(r)\,c_t\,r^{p_t+(d-1)p_x}\,.$$

Kasner singularity

Now, for the two choices of F we find:

$$\mathcal{F} = 1: \qquad \lim_{t_b \to \infty} \frac{\mathrm{d}\mathcal{C}^+}{\mathrm{d}t_b} = \frac{V_{d-1}}{G_\mathrm{N} L} c_t \, r^{p_t + (d-1)p_x} \to 0 \,,$$

$$\mathcal{F} = |LK|: \qquad \lim_{t_b \to \infty} \frac{\mathrm{d}\mathcal{C}^+}{\mathrm{d}t_b} = \frac{V_{d-1}}{G_\mathrm{N}} \frac{c_t}{c_r} (p_t + (d-1)p_x) \, r^{p_t + (d-1)p_x - 1} = \frac{V_{d-1}}{G_\mathrm{N}} \frac{c_t}{c_r}$$

Specifically, we find that

$$\mathcal{F} = L\sqrt{K_{\mu\nu}K^{\mu\nu}}: \qquad \lim_{t_b \to \infty} \frac{\mathrm{d}\mathcal{C}^+}{\mathrm{d}t_b} = \frac{V_{d-1}}{G_\mathrm{N}} \frac{c_t}{c_r} \sqrt{p_t^2 + (d-1)p_x^2} \, r^{p_t + (d-1)p_x - 1}$$
$$= \frac{V_{d-1}}{G_\mathrm{N}} \frac{c_t}{c_r} \sqrt{1 - p_\phi^2} \,,$$

Thank you!