DSSYK and dilaton gravity models

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Based on arXiv:2404.03535, arXiv:2411.16922, arXiv:2501.17091 with A. Blommaert, A. Levine, J. Papalini and K. Parmentier
and arXiv:2503.20691 with A. Belaey, T. Tappeiner

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Outline

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$\rightarrow$ middle ground: **DSSYK**: bounded, continuous spectrum
$\rightarrow$ Possibility to learn deep lessons on holography for more microscopic systems

The SYK-model: introduction

SYK model: N 0+1 dimensional Majorana fermions $\psi_i(t)$, satisfying $\{\psi_i, \psi_j\} = \delta_{ij}$ with all-to-all random interactions of p fermions Sachdev-Ye '92, Kitaev '15:

$$H = \sum_{i_1 < \ldots < i_p} J_{i_1 \ldots i_p} \psi_{i_1} \ldots \psi_{i_p}$$

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Dynamics in the large N IR regime is captured by a Schwarzian effective action Kitaev '15, Maldacena-Stanford '16 . . ., which is in turn the boundary description of 2d JT gravity!
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But what is the gravitational dual of SYK beyond this regime? Largely unresolved question, with competing proposals in literature

e.g. Gross-Rosenhaus '17, Das-Jevicki-Suzuki '17, Das-Ghosh-Jevicki-Suzuki '17, Goel-Verlinde '21

The SYK-model: double-scaling limit

A tractable limit of SYK exists that is both analytically solvable and interesting: we double-scale $p \to \infty$ and $N \to \infty$ keeping ratio $\lambda \equiv p^2/N$ fixed
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Transfer matrix: $\sqrt{2|\log q|}\,\hat{T} = \hat{\alpha}^\dagger + \hat{\alpha}\frac{1-q^{2\hat{n}}}{1-q^2}$, $\quad n =$ chord number

where $\hat{\alpha}^\dagger\,|n\rangle = |n+1\rangle$ and $\hat{\alpha}\,|n\rangle = |n-1\rangle$

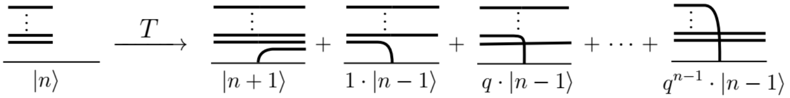

Figure taken from Berkooz-Mamroud '24 ...

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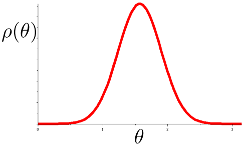

with $\rho(\theta) = (e^{\pm 2i\theta}; q^2)_\infty$

q-Pochhammer: $(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$

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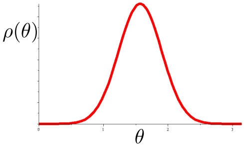

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$\rightarrow$ q-Hermite polynomials $\langle n|\theta\rangle \sim H_n(\cos(\theta)|q^2)$ diagonalize $\hat{T}$ with energy $-\cos\theta$, two-bdy wavefunctions when slicing the disk:

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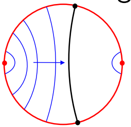

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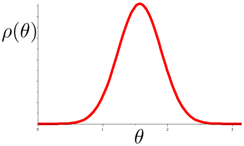

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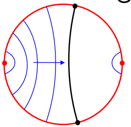

$\rightarrow$ DOS $\rho(\theta)$ from orthogonality relation of q-Hermite polynomials:

$$\sum_{n=0}^{+\infty} \frac{1}{(q^2;q^2)_n} H_n(\cos(\theta_1)|q^2) \, H_n(\cos(\theta_2)|q^2) = \frac{\delta(\theta_1 - \theta_2)}{(e^{\pm 2i\theta}; q^2)_\infty} \, .$$

DSSYK: other correlators

$\rightarrow$ Correlation functions in general mimic structure of the JT correlators replacing $\mathrm{SL}(2,\mathbb{R})$ by its q-deformation $\mathrm{SU}_q(1,1)$ with $q = e^{-\lambda}$ $(\lambda = p^2/N)$ Berkooz-Isachenkov-Narovlansky-Torrents '18 ...

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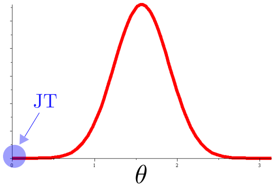
→ JT limit: $q \to 1$ undeformed limit (or $\lambda \to 0$) and $\theta = \lambda k$ (small energies)



DSSYK: from transfer matrix to q-Liouville

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$\hat{n} = \frac{\mathbf{L}}{2|\log q|}$:

$$\Rightarrow \quad \hat{T} \sim \mathbf{H} = -\cos(\mathbf{P}) + \frac{1}{2}e^{i\mathbf{P}}e^{-\mathbf{L}}, \qquad [\mathbf{L}, \mathbf{P}] = 2|\log q| i\hbar$$

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$\rightarrow$ To describe DSSYK, we require that $\mathbf{L}$ is positive and discrete

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$\Rightarrow$ same symplectic structure $(\mathbf{L}, \mathbf{P})$ and Hamiltonian $\mathbf{H}(\mathbf{L}, \mathbf{P})$

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First-order formulation of 2d dilaton gravity:

$$S = \frac{1}{2}\int d^2x\sqrt{-g}\,(\Phi R + V(\Phi)) =$$

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The general Poisson Sigma (PS) model is of the form: Ikeda '93, Schaller-Strobl '94 $S = \int_{\mathcal{M}} \left(A_i \wedge dJ^i - \frac{1}{2}A_i \wedge A_j P^{ji}(J)\right)$

where A_i is gauge connection, and J^i, $i = 1...m$ coordinatize a m-dimensional Poisson manifold target space:

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Identifying $A_i = (e_+, e_-, \omega)$ and $J^i = (E^+, E^-, \Phi)$

$\rightarrow$ 2d dilaton gravity is a special case of the PS model:

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This Poisson algebra is a classical non-linear symmetry algebra of the PS model ($\sim$ classical counterpart of a quantum algebra)

$\rightarrow$ For $V(\Phi)$ a sine function, this corresponds to the quantum algebra $U_q(\mathfrak{su}(1,1))$ which governed DSSYK correlators

First-order formulation of sine dilaton gravity (2)

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$\rightarrow$ Define PS model on a 2d strip $\mathcal{M}$, and add a boundary term as $\oint_{\partial\mathcal{M}} \mathbf{C}\, dt$ where $\mathbf{C}$ is a Casimir function on the Poisson algebra
$+$ boundary condition , $\quad A^i_\tau = \partial_{J^i}\mathbf{C}$
For the Poisson algebra $\{E^+, E^-\}_{\text{PB}} = \frac{V(\Phi)}{2}, \quad \{E^\pm, \Phi\}_{\text{PB}} = \pm E^\pm$
One shows most general Casimir function is: Klösch-Strobl '96
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$\mathbf{C} = \frac{1}{2}(E^+E^- + E^-E^+) + \int^\Phi V(\Phi) d\Phi$
Note: this generalizes the boundary term in JT gravity in BF formulation: TM '18 $\int_\mathcal{M} \text{Tr}BF - \frac{1}{2}\oint_{\partial\mathcal{M}} \text{Tr}B^2, \quad A_\tau^i = B^i$

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From Poisson-sigma model to the boundary (cfr. BF to particle on group, or CS to WZW), Schematically Blommaert-TM-Yao '23:

$\rightarrow$ Define PS model on a 2d strip $\mathcal{M}$, and add a boundary term as $\oint_{\partial\mathcal{M}} \mathbf{C}\, dt$ where $\mathbf{C}$ is a Casimir function on the Poisson algebra
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$\rightarrow$ One can "diagonalize" this six-dimensional symplectic space by phase space variables $(\varphi, p_\varphi, \beta, p_\beta, \gamma, p_\gamma)$ describing coordinates and conj. momenta on the "quantum group manifold"

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$\Rightarrow$ Action for a particle on a quantum group manifold $\text{SU}_q(1,1)$

$$L = p_\varphi \varphi' + p_\beta \beta' + p_\gamma \gamma' + \frac{\cos(\log q p_\varphi)}{2(\log q)^2} - \frac{\mu_\beta \mu_\gamma}{(\log q)^2} e^{-2\varphi - i \log q p_\varphi}$$

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- Structure: left- and right symmetry algebra (cfr. WZW or particle on group)
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  ⇒ We have found a q-deformation of this → **q-Schwarzian**

First-order formulation of sine dilaton gravity (4)

$$L = p_\varphi \varphi' + p_\beta \beta' + p_\gamma \gamma' + \frac{\cos(\log q p_\varphi)}{2(\log q)^2} - \frac{\mu_\beta \mu_\gamma}{(\log q)^2} e^{-2\varphi - i \log q p_\varphi}$$

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Corresponding Hamiltonian for the particle on the quantum group $\mathrm{SU}_q(1,1)$: ($\mathbf{L} = 2\varphi$ and $\mathbf{P} = -\log q p_\varphi$)

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→ Benefit of first-order formulation: group-theoretic structure is visible, already at the classical (Poisson-Lie) level + natural guess of sine dilaton potential

Sine dilaton gravity as two Liouville CFTs

Sine dilaton gravity is classically equivalent to two Liouville actions

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With ρ the conformal factor of the metric: $g_{\mu\nu} = e^{2\rho}\delta_{\mu\nu}$, perform the following field redefinition $(\rho, \Phi) \to (\phi, \chi)$:

$$b\phi = \rho + i|\log q|\Phi\,, \quad ib\chi = \rho - i|\log q|\Phi\,, \quad \pi b^2 = i|\log q|\,,$$

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The fields ϕ and χ have complex conjugate central charges

$$c_\phi = 13 + 6i\left(\frac{\pi}{|\log q|} - \frac{|\log q|}{\pi}\right)\,, \quad c_\chi = 13 - 6i\left(\frac{\pi}{|\log q|} - \frac{|\log q|}{\pi}\right)$$

See also Verlinde '19, '23, '24, Collier-Eberhardt-Mühlmann-Rodriguez '24, '25

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$$\boxed{\mathbf{H} = -\cos(\mathbf{P}) + \frac{1}{2} e^{i\mathbf{P}} e^{-\mathbf{L}}}$$

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$\rightarrow$ Diagonalizing $\mathbf{H}$ leads to difference equations, which have highly non-unique solutions $\sim\times$ periodic functions ("quasi-constants") unlike classical $q \rightarrow 1$ case governed by differential eqns

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Solution:

$$\psi^{\mathrm{L}}_\theta(\phi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dp\, e^{-ip\phi}\, S_b\left(-i\frac{bp}{2} \pm i\frac{b}{2}\frac{\theta}{|\log q|}\right) e^{-|\log q|\frac{p^2}{4} + \frac{\theta^2}{4|\log q|} - \frac{\pi p}{2}}$$

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Leads to orthogonality computation:

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$$Z(\beta) = \int_{-\infty}^{+\infty} d\theta \sin(\theta)\, \sinh\left(\frac{\pi\theta}{|\log q|}\right) e^{\beta\cos\theta}$$

→ Manifestly divergent energy density of states

$$\rho(E) = \sum_{m=-\infty}^{+\infty} \sinh\left(\frac{\pi(\arccos(E) + 2\pi m)}{|\log q|}\right)$$

→ Does not match with DSSYK, but is not completely wrong either: "fake" thermodynamics Stanford-Lin '23 or semiclassically only contains the black hole contribution and not the "observer" part

→ is formally related to the complex Liouville string Collier-Eberhardt-Mühlmann-Rodriguez '24 by analytic continuation (change of contour in $\theta \to i\theta$, and restriction of integration range)

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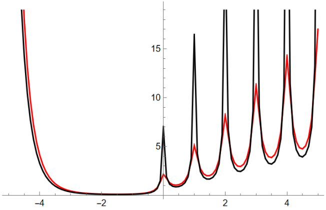
Indeed, from earlier explicit solution of $\psi^{\mathrm{R}} \rightarrow$ poles at positive n:

$\psi_\theta^{\mathrm{R}}(n) =$ 

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Using the technical identity:

$\sum_{m=-\infty}^{+\infty} 2\sin(\theta)\sinh\frac{\pi(\theta+2\pi m)}{|\log q|} e^{-\frac{(\theta+2\pi m)^2}{|\log q|}} = (e^{\pm 2i\theta};q^2)_\infty$

$$\Rightarrow \quad Z(\beta) = \int_0^{\pi} d\theta\, (e^{\pm 2i\theta};q^2)_\infty \exp\left(\beta\cos(\theta)\right)$$

$\rightarrow$ **matches DSSYK**

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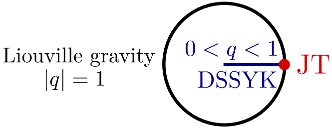
$\to$ No periodicity in $\mathbf{P}$ $\leftrightarrow$ Recursion relation has no singular points
$\Rightarrow$ No possibility for discrete quantization (or discrete bulk spacetime)



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$\rightarrow$ wavefunction only support on an integer lattice with $\hbar$ spacing
As $\hbar \rightarrow 0$, lattice disappears and effectively becomes continuous
$\Rightarrow$ No contradiction between classical continuum and quantum discretization of spacetime

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$\rightarrow$ Continuum path integral in phase space
$\rightarrow$ Is true for any Hamiltonian on S^1

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Overlap between conjugate states:

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$\rightarrow$ The DSSYK system is of this type but with $x \leftrightarrow p$

Towards a path integral perspective (3)

$\Rightarrow$ Euclidean DSSYK gravitational path integral:

$$\int_{L(0)=L(\beta)=0} \frac{\mathcal{D}L\mathcal{D}P}{V_{\text{gauge}}} \exp\left\{ \frac{1}{2|\log q|} \int_0^\beta d\tau \left(i\, P \frac{d}{d\tau} L + \cos(P) - \frac{1}{2} e^{iP} e^{-L} \right) \right\}$$

Towards a path integral perspective (3)

$\Rightarrow$ Euclidean DSSYK gravitational path integral:

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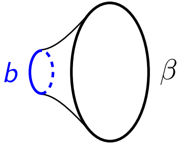
Bossi-Griguolo-Papalini-Russo-Seminara '24

Beyond the disk: single trumpet

Single trumpet amplitude was written down by Okuyama: Okuyama '23



$$= \frac{1}{\pi}\int_0^{\pi} d\theta\, e^{\beta\cos\theta}\cos(b\theta) = I_b(\beta)$$

where b is the discretized! geodesic length around the neck of the wormhole

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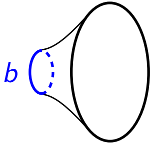

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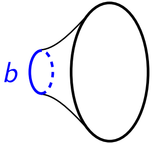

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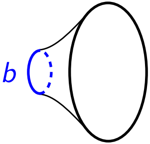

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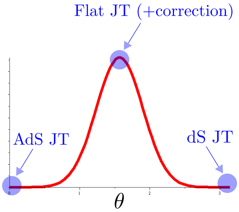
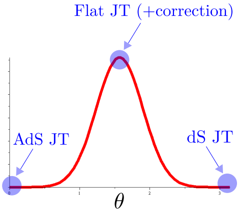
JT regimes in DSSYK



AdS JT gravity: $q \to 1$ and energies $\theta =$ (small)



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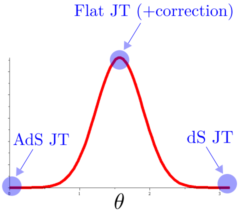

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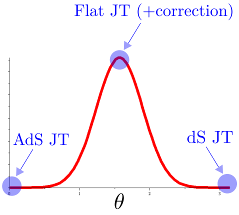

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dS JT gravity: $q \to 1$ and energies $\theta = \pi -$ (small)
Recently Okuyama '25: ETH matrix model Jafferis-Kolchmeyer-Mukhametzhanov-Sonner '22 yields dS matrix model Cotler-Jensen-Maloney '19

Towards more general dilaton gravity models (1)

Observation for disk partition function for various known models:

JT gravity:
$Z(\beta) = \int d\Phi\Phi \sinh(2\pi\Phi) e^{-\beta\Phi^2}$

Sinh dilaton gravity (= Liouville gravity):
$Z(\beta) = \int d\Phi \sinh\left(2\pi b^2\Phi\right) \sinh 2\pi\Phi e^{-\beta \cosh 2\pi b^2\Phi}$

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⇒ Suggests expression for generic dilaton gravity models:

$$Z(\beta) = \int_{-\infty}^{+\infty} d\Phi V(\Phi) \sinh(2\pi\Phi)\, e^{-\beta W(\Phi)}$$

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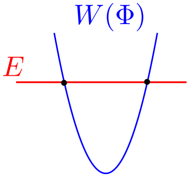
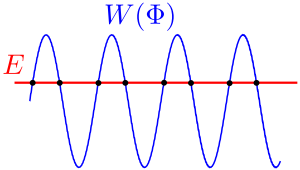


JT / Liouville gravity



Sine gravity

Compare to **Gas-of-defects approach:** Maxfield-Turiaci '20, Witten '20
Deforms JT potential $V(\Phi) = 2\Phi + \sum_n c_n e^{-\alpha_n \Phi}$ with sum of damped exponentials

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But: Subtlety with precise contour $\mathcal{C}$ and possible branch cut contribution $\rightarrow$ to be understood

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