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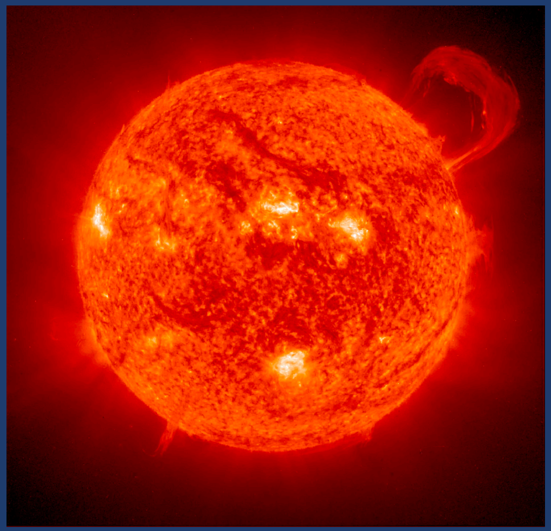
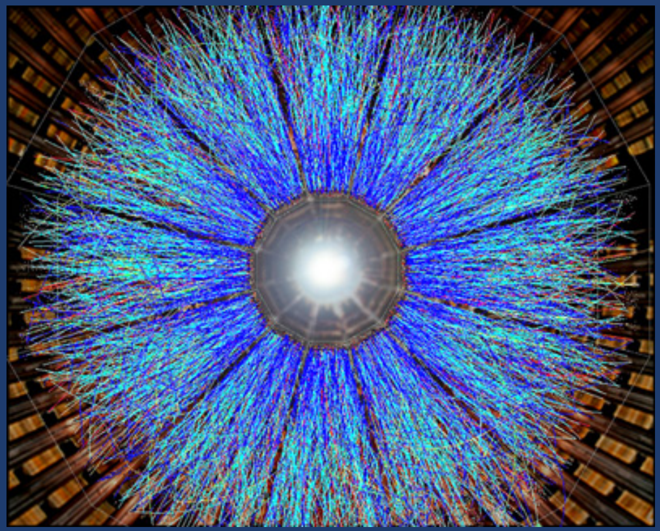
THE SPECTRA OF HOLOGRAPHIC THERMAL FIELD THEORIES, BLACK HOLES AND THE IMPLICATIONS OF THE SPECTRAL DUALITY RELATION

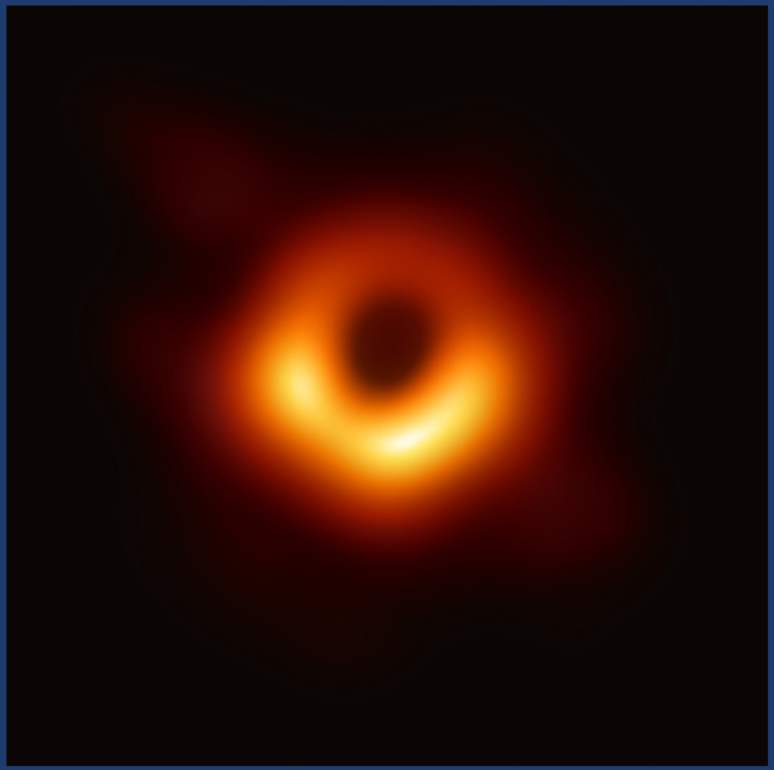
MADRID, 18.6.2025

THERMAL FIELD THEORY AND BLACK HOLES



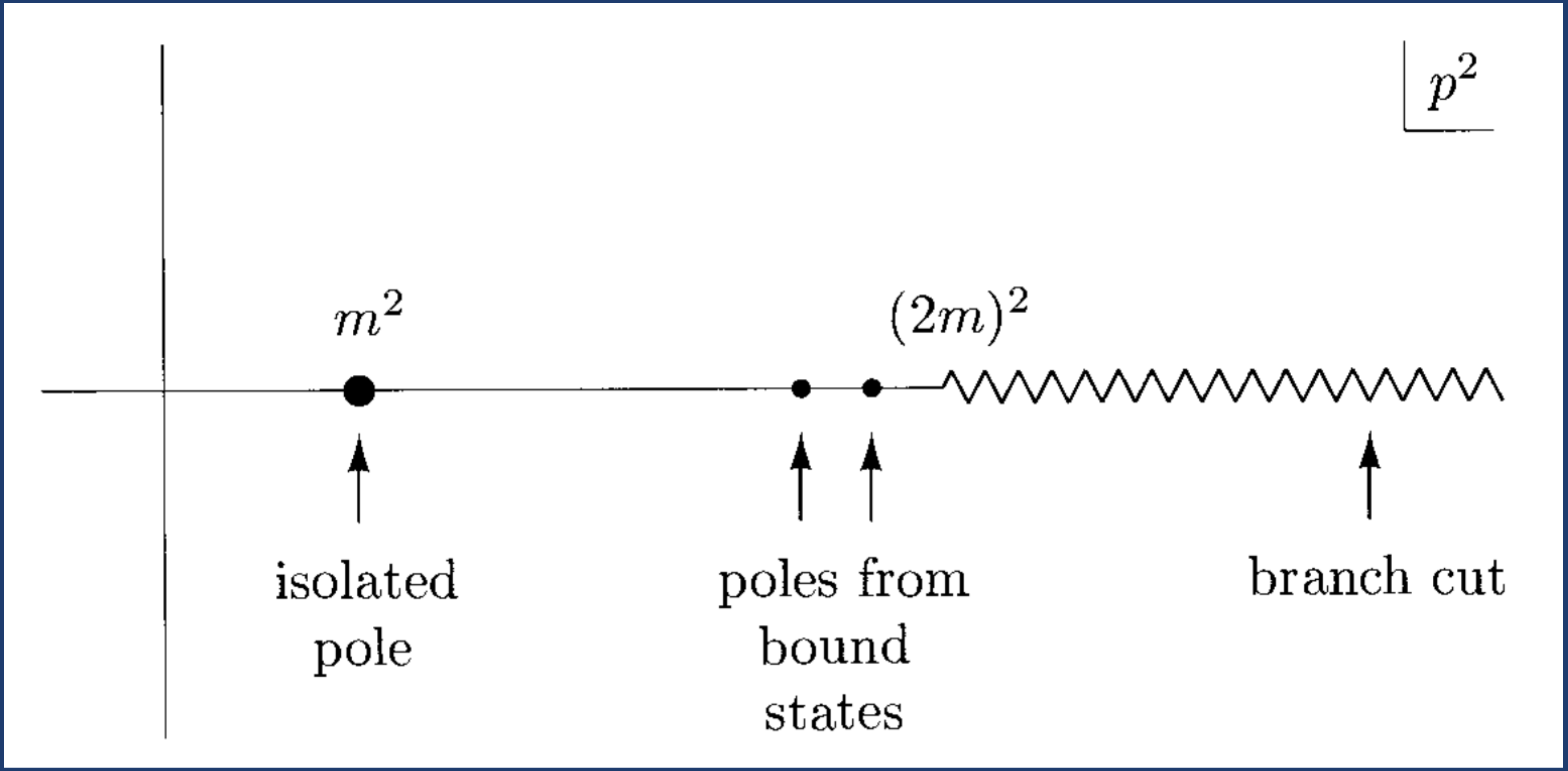


$$Z[\beta = 1/T] = \int \mathcal{D}\Phi \, e^{-\beta H} e^{\frac{i}{\hbar}\int d^d x \, \mathcal{L}(\Phi,\lambda)}$$



SPECTRUM OF A SIMPLE $T = 0$ CORRELATOR

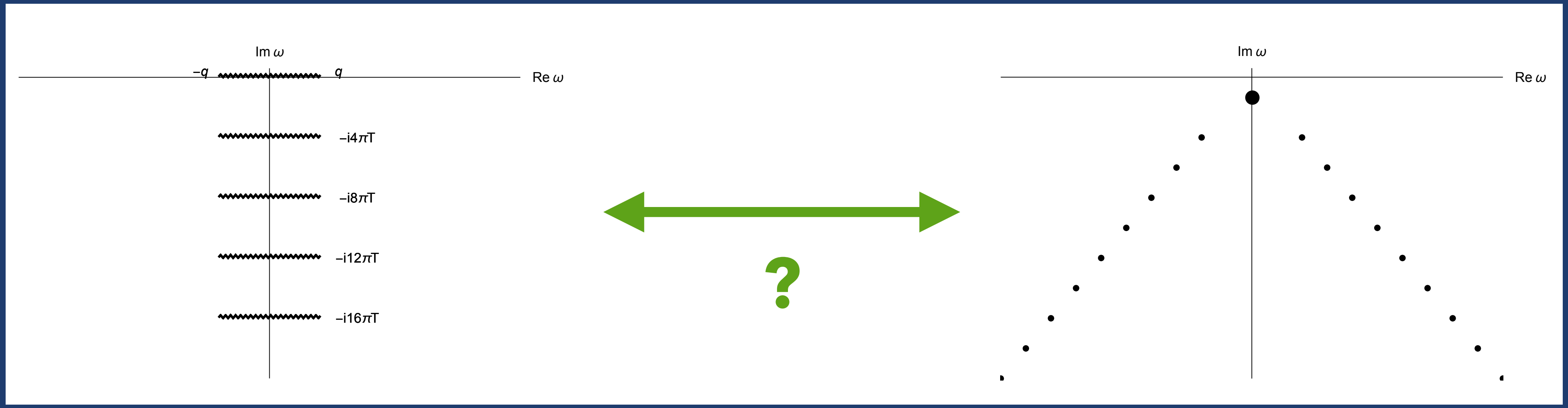
$$\langle \phi(p)\phi(-p) \rangle = \frac{Z(p^2)}{p^2 - m^2 + \Sigma(p^2)}$$



[from Peskin and Schroeder]

ANALYTIC STRUCTURE OF THERMAL CORRELATORS

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}$$



$\lambda \to 0$

[Hartnoll, Kumar, (2005)]

$\lambda \to \infty$

holography (*N=4 SYM-type theories*)

meromorphic momentum space correlator

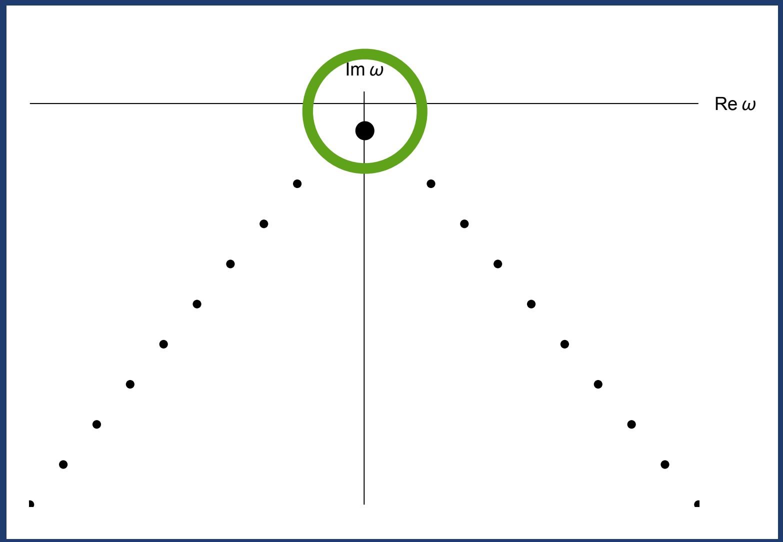
what is the structure of thermal correlators and black hole QNMs?
what is the minimal information necessary to determine them completely?

methods:

- kinetic theory: solving the collision integral
- perturbative QFT: calculation of Feynman diagrams
- holography: solving differential equations in black hole backgrounds

LOW-ENERGY SPECTRUM AND HYDRODYNAMICS

- low-energy limit of (some) thermal QFTs is described hydrodynamics



- conservation laws and global conserved operators

$$\nabla_\mu T^{\mu\nu} = 0$$

- tensor structures (symmetries, gradient expansions) and transport coefficients (QFT)

$$T^{\mu\nu} = \sum_{n=0}^{\infty} \left[\sum_i^N \lambda_i^{(n)} \mathcal{T}^{\mu\nu}_{(n)} \right] \xrightarrow[u^\mu \sim T \sim e^{-i\omega t + iqz}]{\nabla_\mu T^{\mu\nu} = 0} \omega(q) = \sum_{n=1}^{\infty} \alpha_n q^n$$

$$\partial u^\mu \sim \partial T \ll 1 \longrightarrow \omega / T \sim q/T \ll 1$$

equilibrium temperature

$q = \sqrt{\mathbf{q}^2}$

- dispersion relations are poles:

diffusion $\omega = -iDq^2$

sound $\omega = \pm v_s q - i\Gamma q^2$

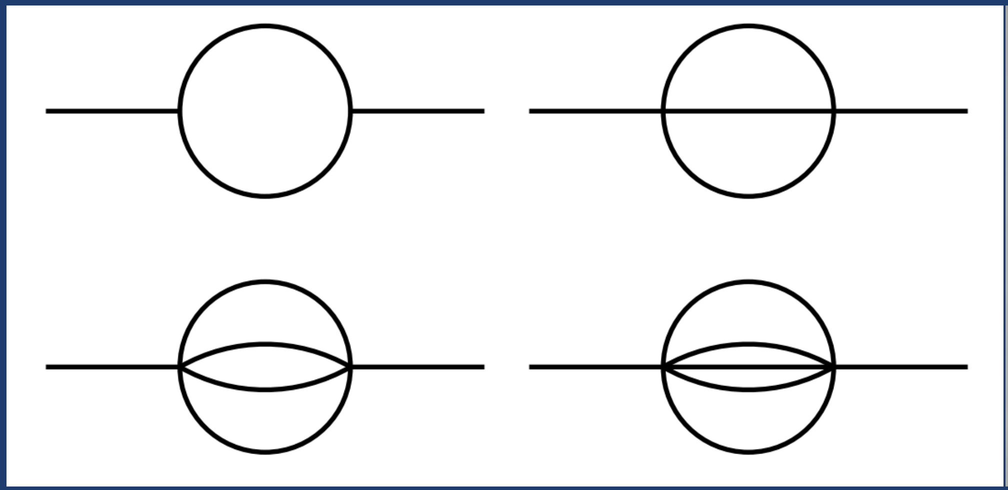
LOW-ENERGY SPECTRUM AND HYDRODYNAMICS

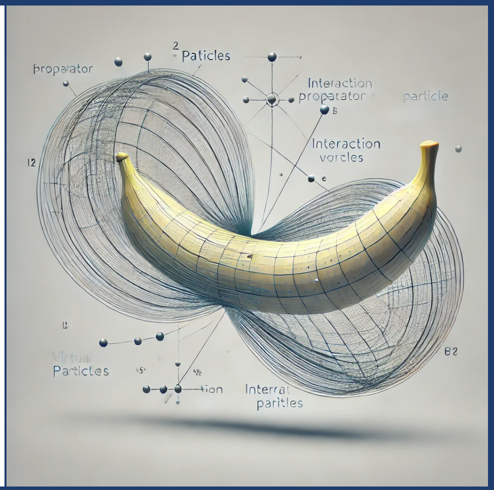
- Schwinger-Keldysh effective field theory of diffusion to all loops: long time tails…
  [Chen-Lin, Delacretaz, Hartnoll (2019) ; Delacretaz (2020); SG, Lemut, Pelaič, Soloviev, PRD (2024)]
- tree-level result (classical hydrodynamics) has one diffusive pole at $\omega = -iDq^2$
- non-analytic dispersion relations:

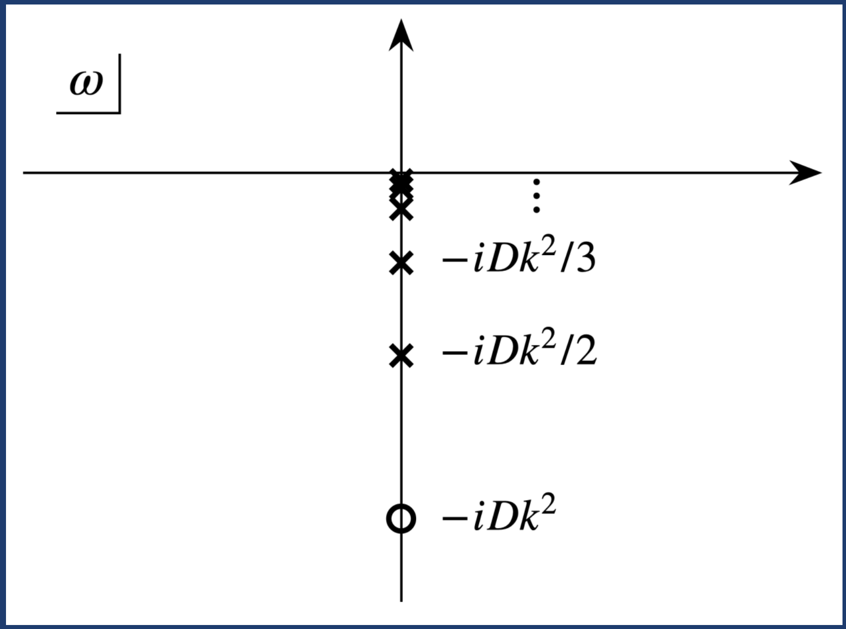
$$\omega = -iDq^2 \pm \delta\omega,$$

$$\delta\omega = \sum_{n=1}^{\infty} c_n \gamma_n^{\frac{nd}{2}-1} q^{2+nd} \begin{cases} 1, & nd \text{ odd}, \\ \ln(\gamma_n q^2), & nd \text{ even} \end{cases}$$

- n-loop result has a (pair) of diffusive pole(s) and branch point at $\omega = -\dfrac{iDq^2}{n+1}$
- go to *all* orders with ‘bananas’:





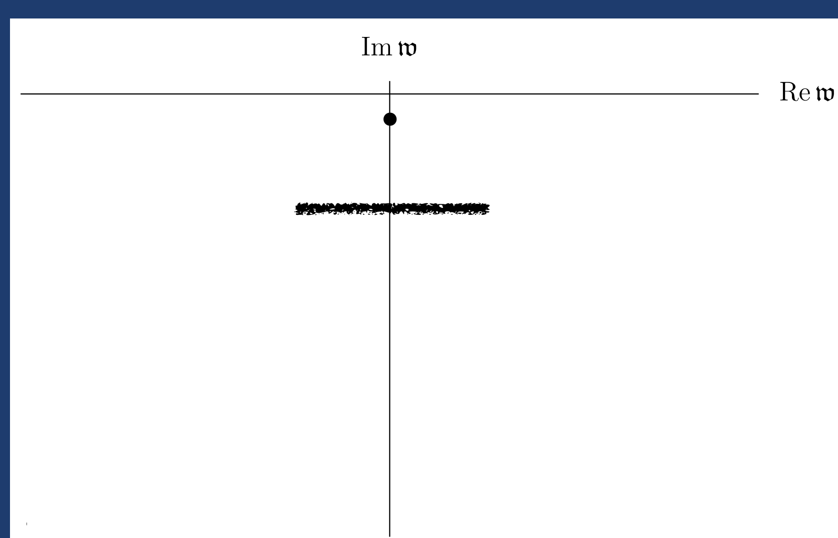


KINETIC THEORY AND THERMAL SPECTRUM

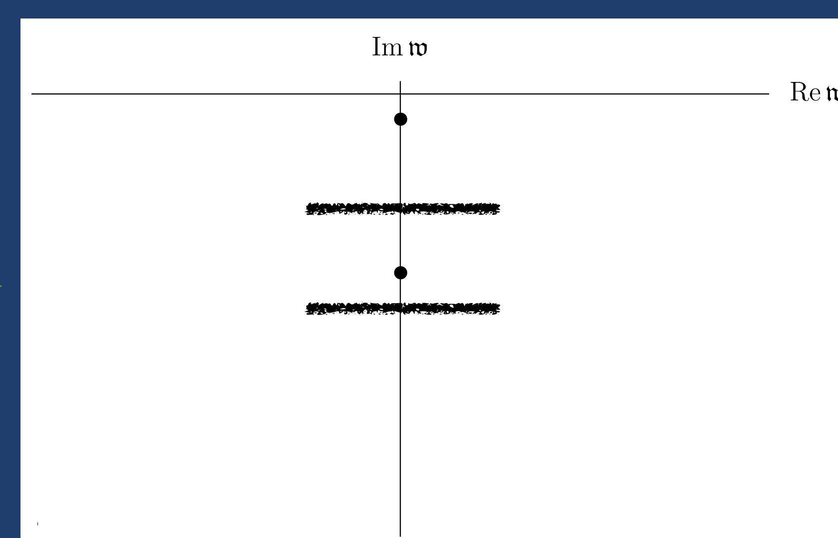
$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}$$

$\lambda \to 0$ $\lambda \to \infty$

poles from a cut?



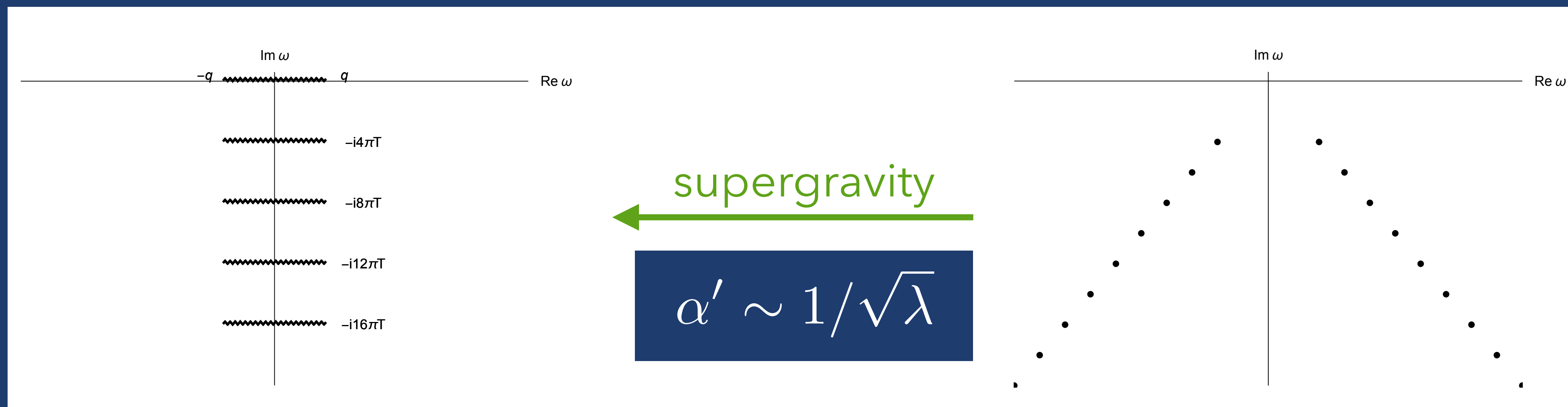
Boltzmann equation – RTA [Romatschke, (2016)]



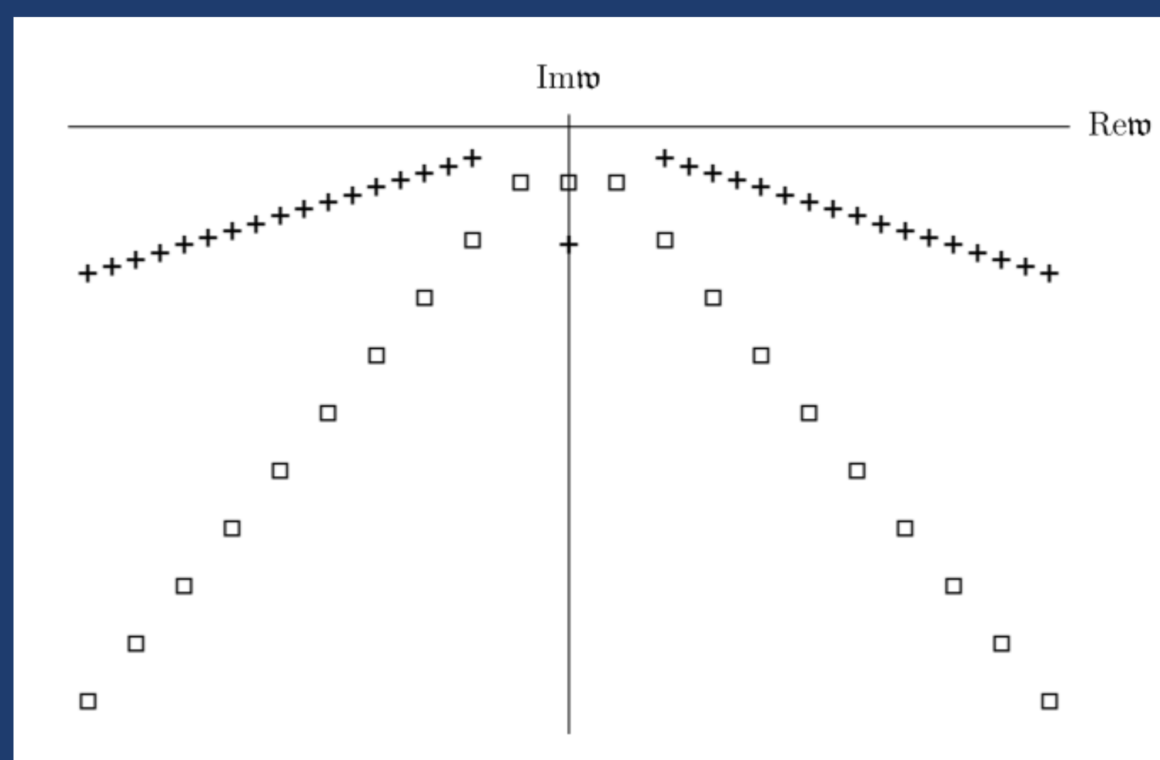
BBGKY hierarchy – RTA-like truncations [SG, Soloviev (2025)]

HOLOGRAPHY AND THERMAL SPECTRUM

$$\langle T_{\mu\nu}(-\omega,-q),T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)} = \frac{B(\omega,q)}{\prod_{i=0}^{\infty}(\omega-\omega_i(q))}$$

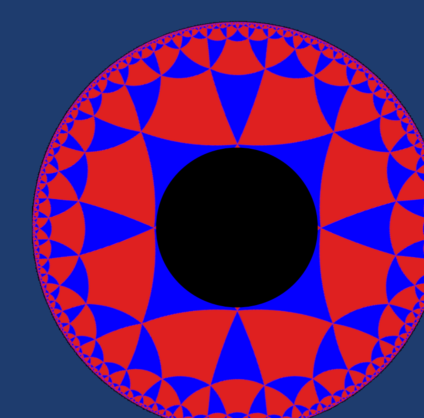


$\lambda \to 0$ | cut from poles | $\lambda \to \infty$



correlators remain meromorphic

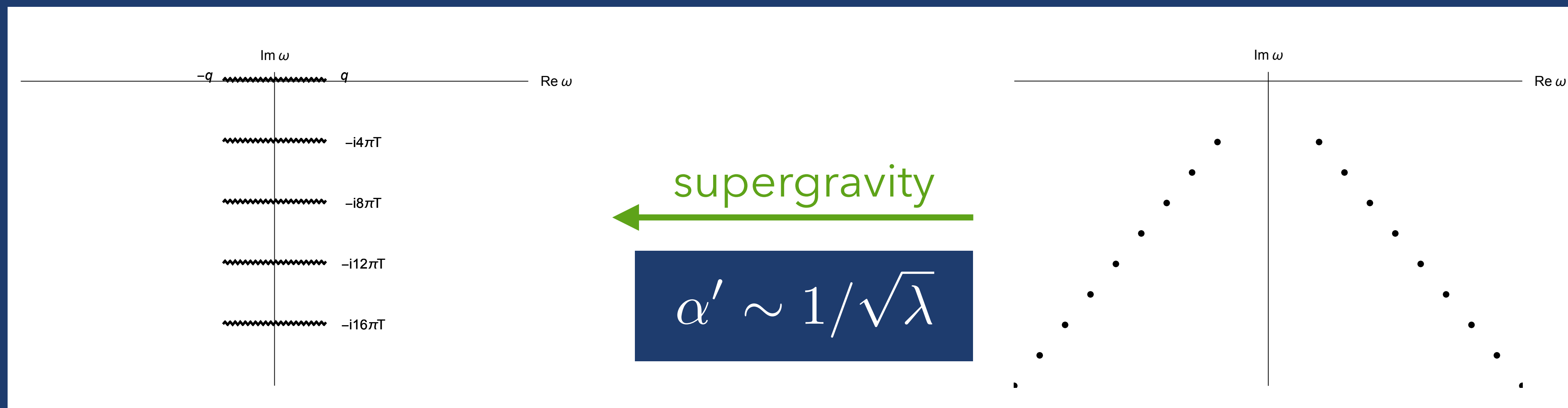
quasinormal modes of black holes



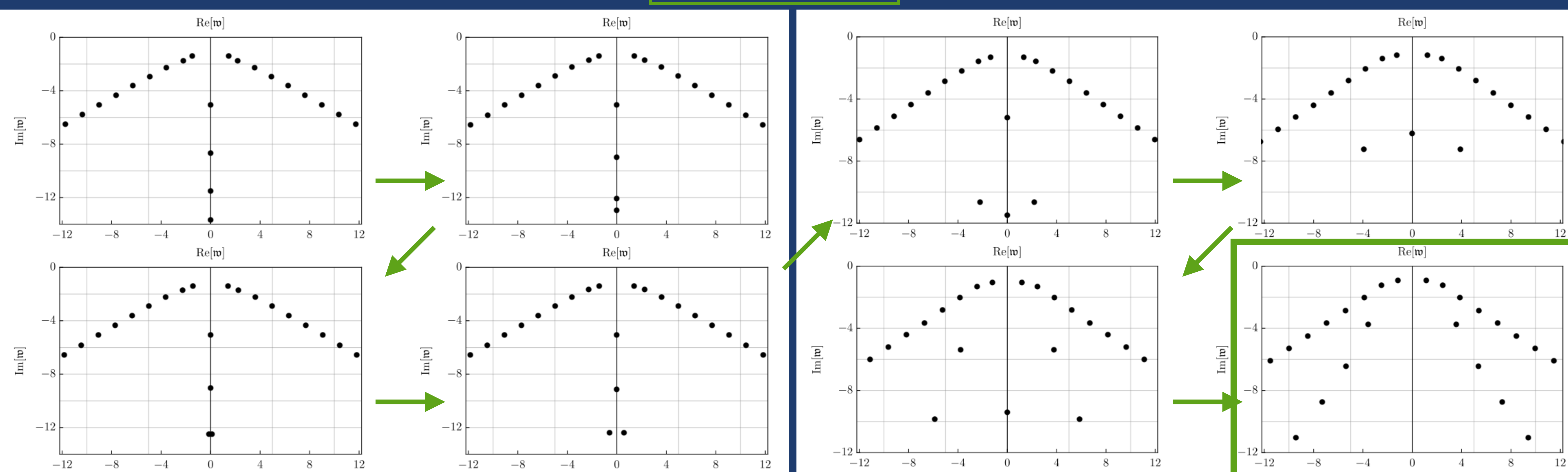
[SG, Starinets …, several papers]

HOLOGRAPHY AND THERMAL SPECTRUM

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)} = \frac{B(\omega,q)}{\prod_{i=0}^{\infty}(\omega-\omega_i(q))}$$



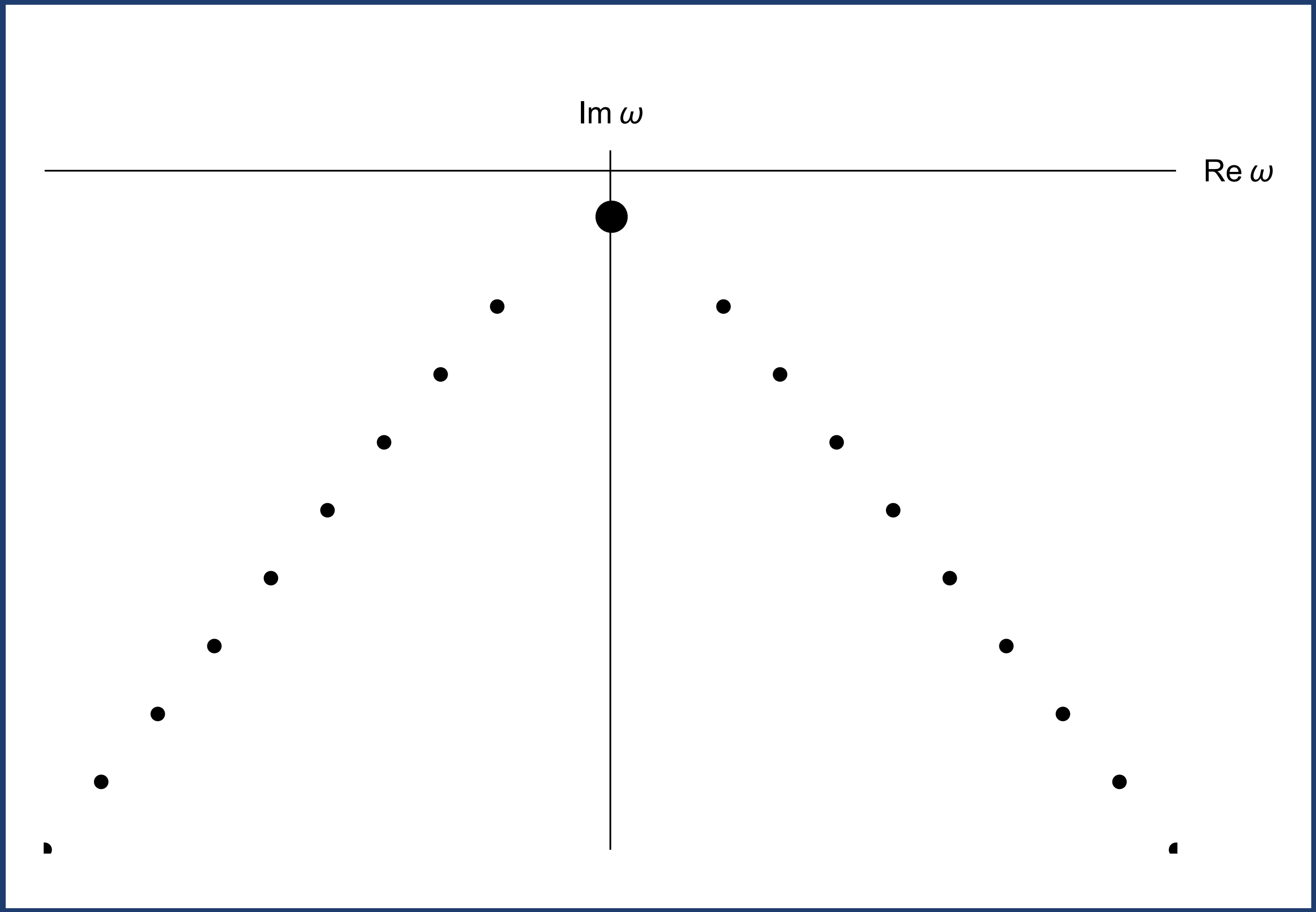
$\lambda \to 0$ cut from poles $\lambda \to \infty$



[SG, Starinets …, several papers; see also Dodelson on SYK (2025)]

HOLOGRAPHY AND THERMAL SPECTRUM

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)} = \frac{B(\omega,q)}{\prod_{i=0}^{\infty}(\omega-\omega_i(q))}$$



for the rest of the talk:

correlators are meromorphic in $\omega \in \mathbb{C}$

OUTLINE

I. spectrum can be reconstructed from a single QNM

II. spectrum can be reconstructed from pole skipping and hydrodynamics

III. spectral duality relation

I. SPECTRUM CAN BE RECONSTRUCTED FROM A SINGLE QNM

[SG, Lemut, JHEP (2022)]

I. SPECTRAL RECONSTRUCTION FROM ONE QNM

- hydrodynamic series are convergent Puiseux series (shear *p=1*, sound *p=2*)
  [SG, Kovtun, Starinets, Tadić, PRL (2019); … ; Withers; JHEP (2018); Heller, et.al. (2020, …)]

$$\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \ldots$$

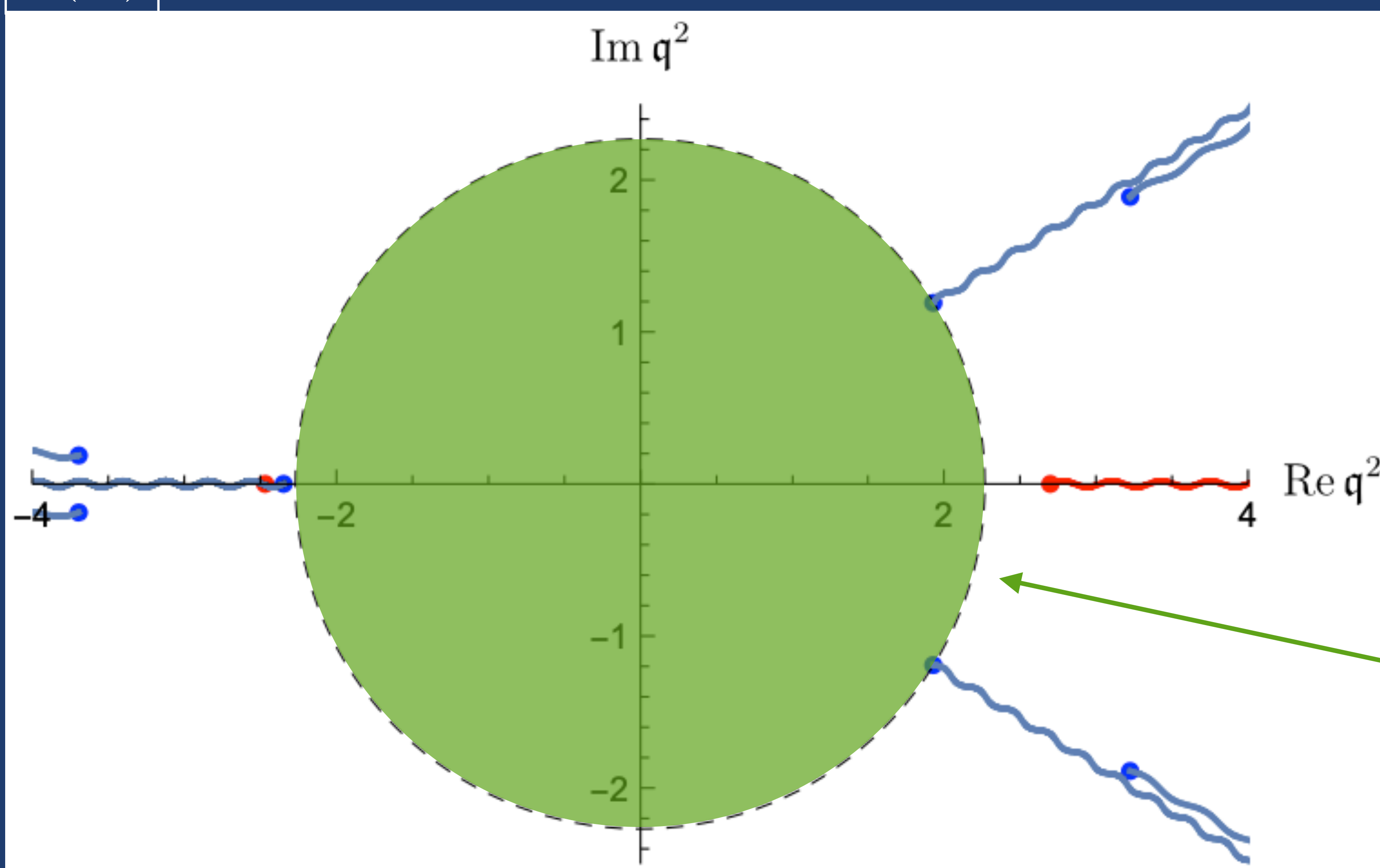
$$\mathfrak{w}_{\text{sound}} = -i \sum_{n=1}^{\infty} a_n e^{\pm \frac{i\pi n}{2}} \left(\mathfrak{q}^2\right)^{n/2} = \pm v_s \mathfrak{q} - \frac{i}{2}\mathfrak{G}\mathfrak{q}^2 + \ldots$$

$$R_{\text{shear}}(\lambda) = 2.22\left(1 + 674.15\,\lambda^{-3/2} + \cdots\right)$$

$$R_{\text{sound}}(\lambda) = 2\left(1 + 481.68\,\lambda^{-3/2} + \cdots\right)$$

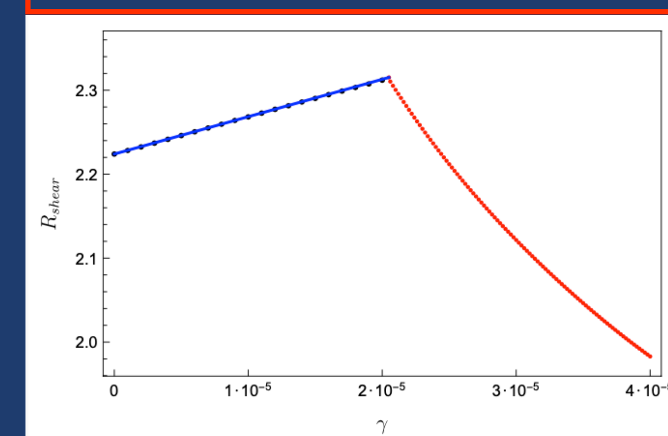
- dispersion relations are holomorphic in a disk

$\omega(q^2)$



holomorphic disk

N=4 SYM radius convergence
[SG, Starinets, Tadić, JHEP (2021)]



I. SPECTRAL RECONSTRUCTION FROM ONE QNM

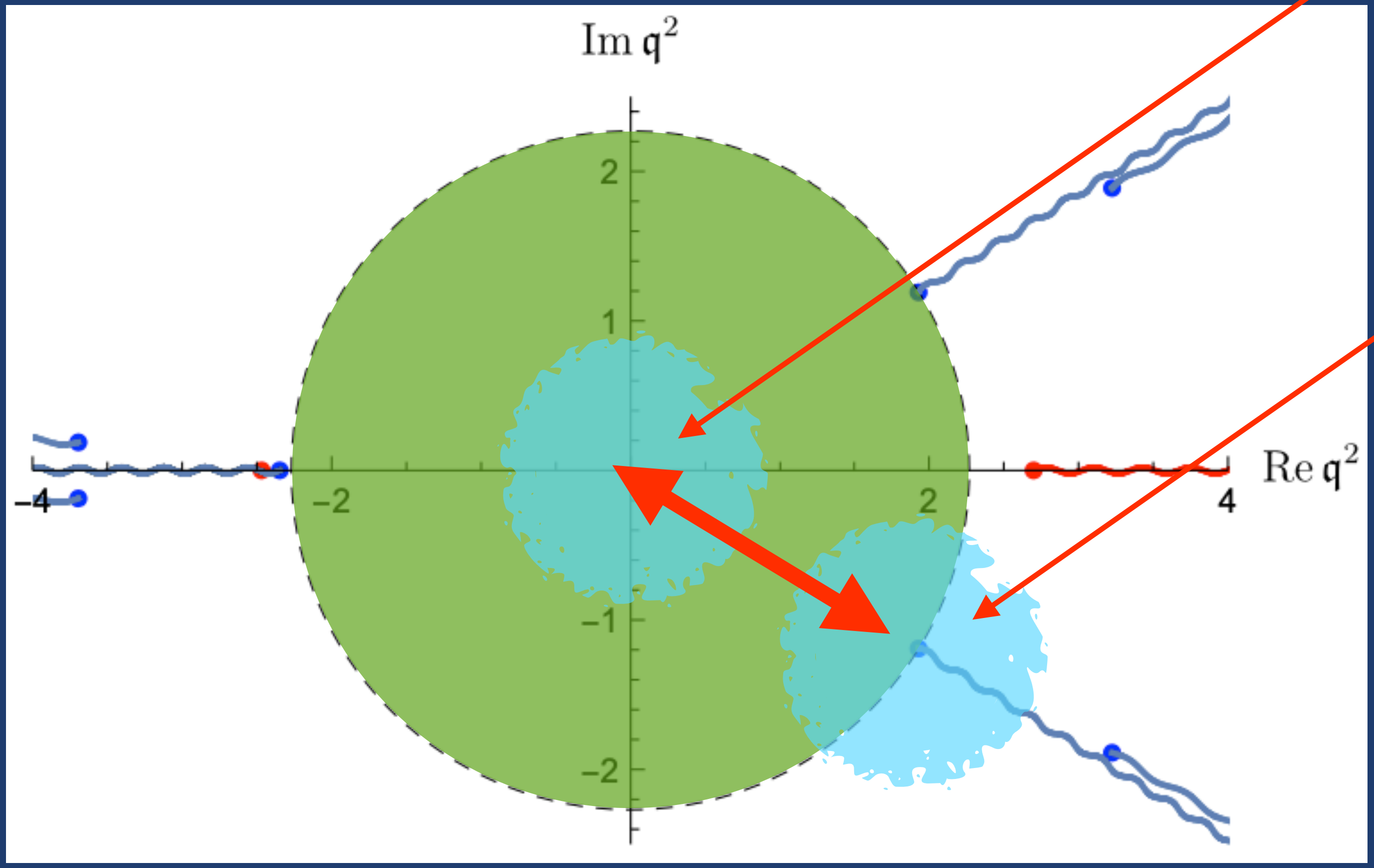
claim: systematic reconstruction of *all* modes connected via *level-crossing* is possible by exploration (analytic continuations) of the Riemann surface connecting physical modes

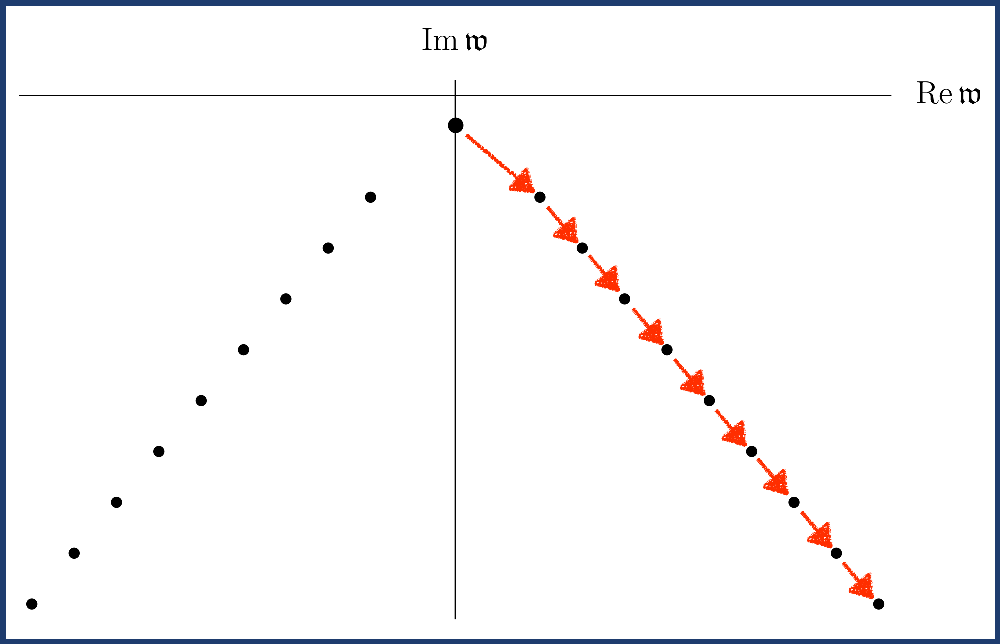
- algorithm combining a theorem by Puiseux and a theorem by Darboux
- statement should hold for spectra that are 'sufficiently complicated' like the Heun function

$$\omega_0(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$\omega_0(z) = -i \sum_{n=0}^{\infty} e^{\frac{i\pi n}{2}} b_n (z - z_1)^{n/2}$$

$$\omega_1(z) = -i \sum_{n=0}^{\infty} e^{-\frac{i\pi n}{2}} b_n (z - z_1)^{n/2}$$





all UV modes from one IR mode

EXAMPLE: DIFFUSION OF M2 BRANES (ADS4/CFT3)

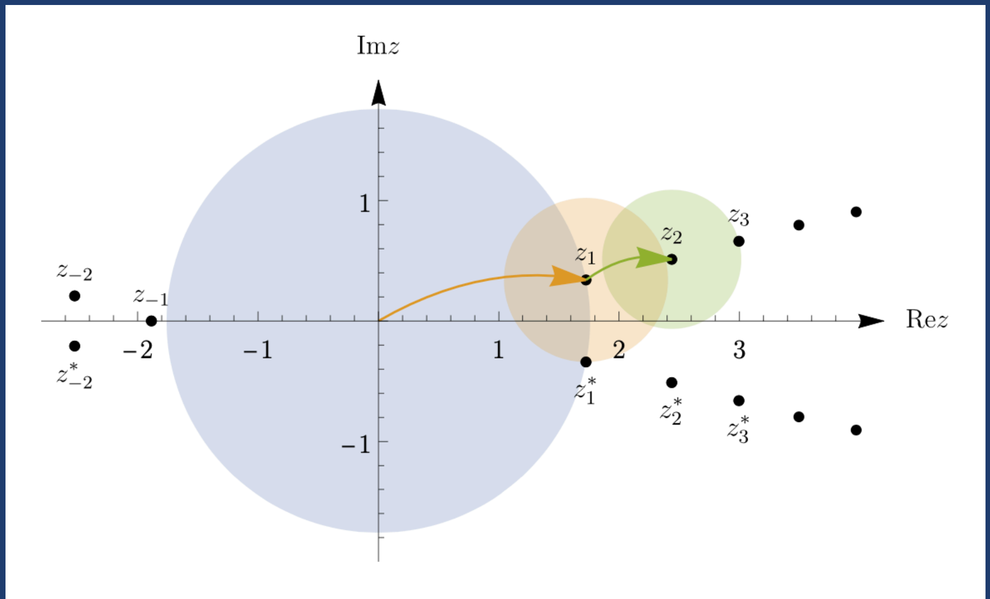
- start from 300 hydrodynamic coefficients $\omega_0(z) = \sum_{n=0}^{\infty} a_n z^n$
- use algorithm with 2 c.c. critical points, 'recover' 12 coefficients and compute the gap with analytic continuation on the same sheet (Padé approximant, …)

$$\mathfrak{w}_1(z) = \sum_{n=0}^{(N_1=12)-1} b_n (z-z_1)^{n/2} \quad \longrightarrow \quad \begin{aligned} \mathfrak{w}_1^{\text{calc}}(0) &= 1.23506 - 1.76338i \\ \mathfrak{w}(0) &= 1.23455 - 1.77586i \end{aligned}$$

- (re)compute the first 300 coefficients, use algorithm with 2 general critical points, 'recover' 12 coefficients and compute the gap

$$\mathfrak{w}_2(z) = \sum_{n=0}^{(N_2=12)-1} c_n (z-z_2)^{n/2} \quad \longrightarrow \quad \begin{aligned} \mathfrak{w}_2^{\text{calc}}(0) &= 2.16275 - 3.25341i \\ \mathfrak{w}_2(0) &= 2.12981 - 3.28100i \end{aligned}$$

- … exploration continues …
- conceptually useful and instructive, practically not (yet)…



II. SPECTRUM CAN BE RECONSTRUCTED FROM POLE SKIPPING

[SG, Lemut, Pedraza, PRD (2023)]

II. SPECTRUM FROM POLE SKIPPING

- pole skipping: ubiquitous feature of thermal correlators and black hole perturbations [SG, Schalm, Scopelliti, PRL (2017); Blake, Lee, Liu, JHEP (2018); Blake, Davison, SG, Liu, JHEP (2018); SG, JHEP (2019)]
- originally: all-order hydrodynamic sound mode $\omega(q) = \sum_{n=1}^{\infty} \alpha_n \left(T, \mu_i, \langle \mathcal{O}_j \rangle, \lambda\right) q^n$

  passes through a 'chaos point' at where the associated *2*-pt function is '0/0':

$$\boxed{\omega(q = i\lambda_L / v_B) = i\lambda_L = 2\pi T i}$$

$$\boxed{G_R = \frac{0}{0} = N(\delta\omega/\delta q)}$$

- relation to maximal quantum chaos as measured by the out-of-time-ordered correlation functions and entanglement wedge [...; Chua, Hartman, Weng (2025)]
- triviality of the Einstein equation at the horizon
- infinite number of such '0/0' points at negative Matsubara frequencies for $q \in \mathbb{C}$ [SG, Kovtun, Starinets, Tadić, JHEP (2019); Blake, Davison, Vegh, JHEP (2019)]

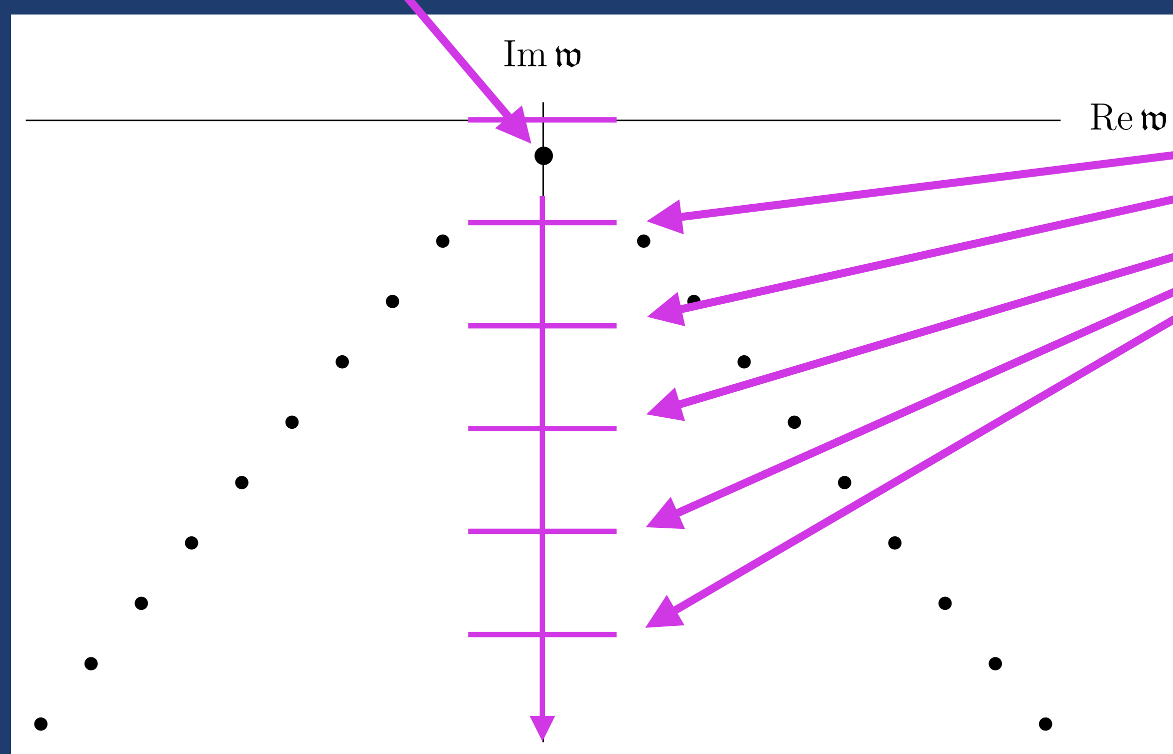
$$\boxed{\omega_n(q_n) = -2\pi T i n} \quad \boxed{n \geq 0}$$

II. SPECTRUM FROM POLE SKIPPING

- consider momentum diffusion in a neutral CFT dual to AdS-Schwarzschild black brane

$$\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \dots$$

for increasing real q



$$\omega_n(q_n) = -2\pi T i n$$

analytic result in AdS4/CFT3
[SG, PRL (2021)]

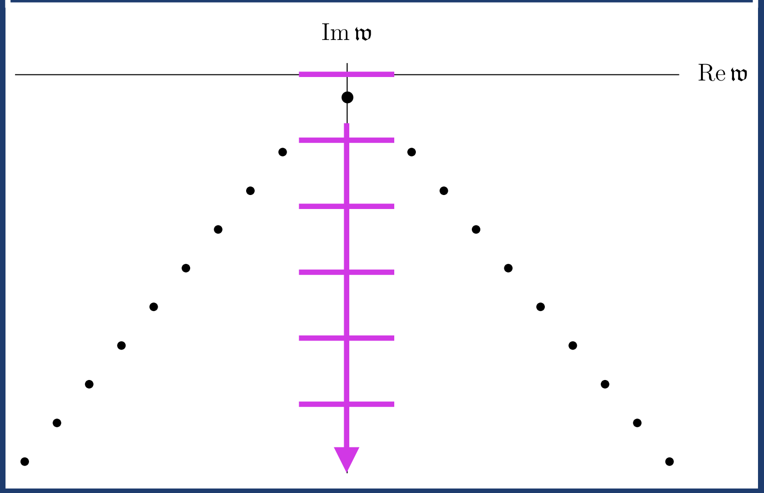
$$q_n = \frac{4\pi T}{\sqrt{3}} n^{1/4}, \;\; n = 0, 1, 2, \dots$$

II. SPECTRUM FROM POLE SKIPPING

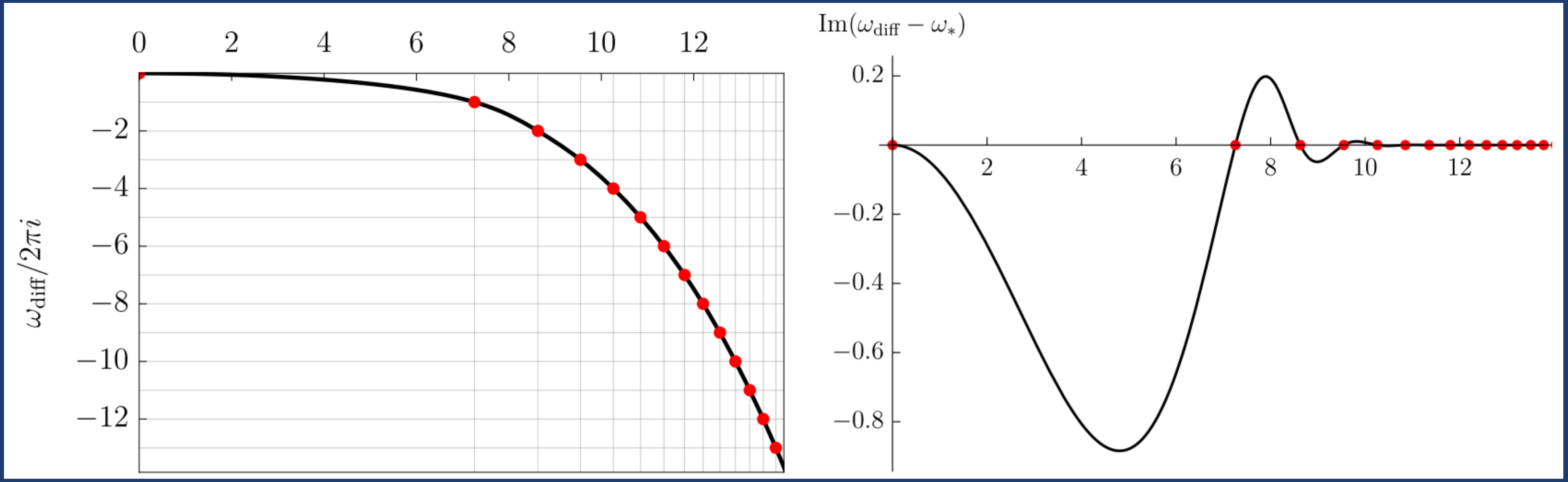
- consider momentum diffusion in a neutral CFT dual to AdS-Schwarzschild black brane

$$\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \ldots$$

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}$$



for increasing real q

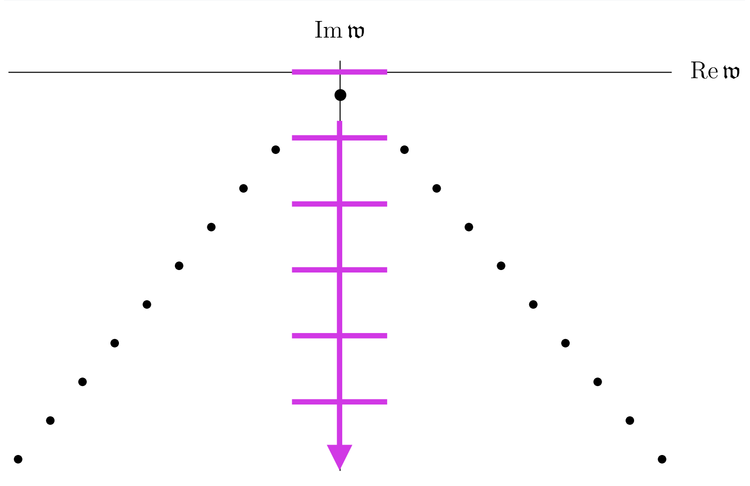


claim: in holographic theories of the type discussed here (N=4 SYM, M2, M5, …), the entire spectrum can be computed from only a discrete set of pole-skipping points

II. SPECTRUM FROM POLE SKIPPING

- interpolation problem: $\omega_n(q_n) = -2\pi T i n$

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}$$



for increasing real q

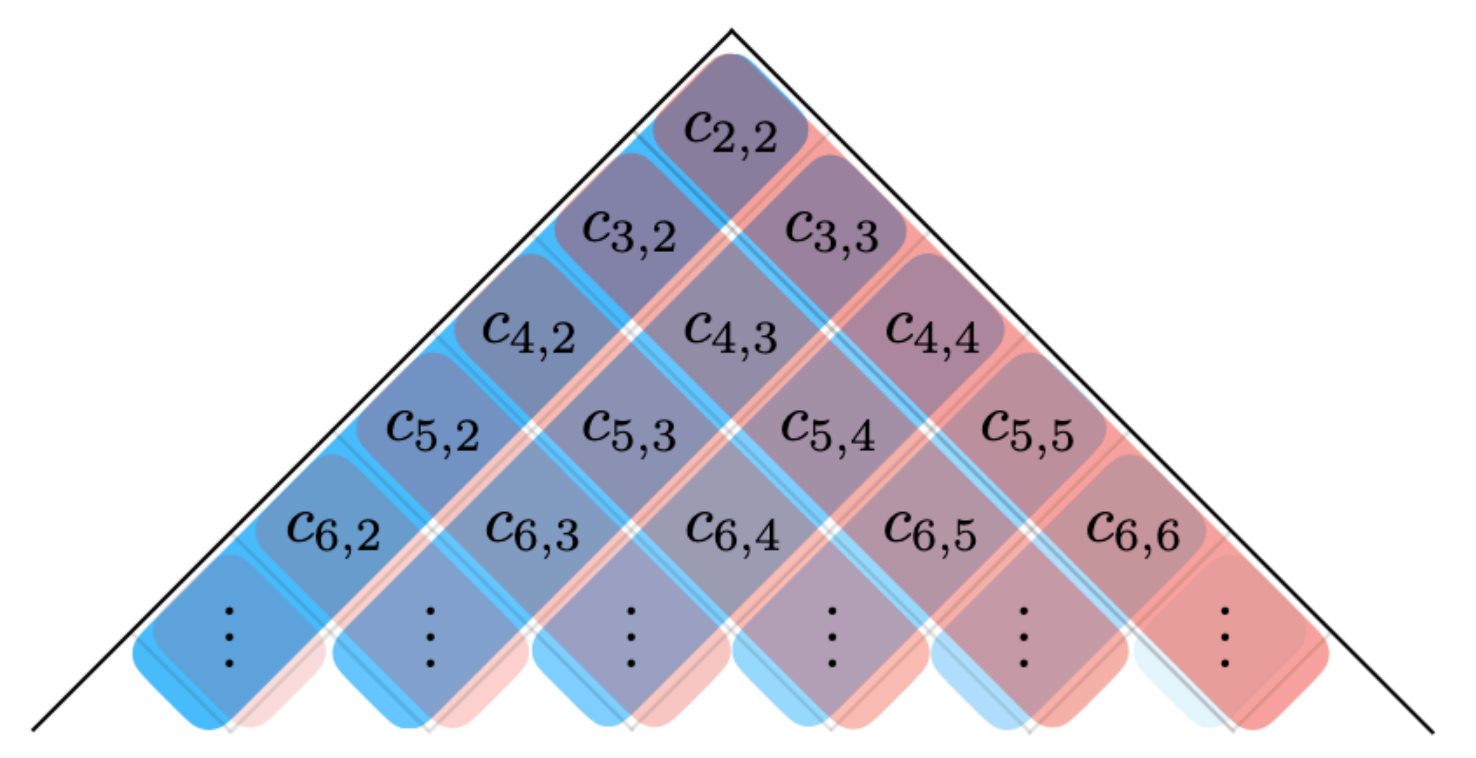
$$\mathfrak{w}_{\text{shear}} = -i\sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \dots$$

- unique solutions to interpolation problems are rare (Weierstrass-Hadamard, Nevanlinna-Pick)
- trick: 'analytic continuation' to *d* spacetime dimensions and expansion around infinite *d*
- general relativity in large *d* drastically simplifies: $V \sim 1/r^d$
- recall: large-*d* limit of quantum mechanics is useful in atomic physics (e.g., for Helium)
- convergence of such series depends on the details

II. SPECTRUM FROM POLE SKIPPING

- interpolation: $\omega_n(q_n) = -2\pi T i n$ → $\mathfrak{w}_{\text{shear}} = -i\sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \ldots$
- 'analytic continuation' to *d* spacetime dimensions and expansion around infinite *d*

$$\omega_0(q = Q\sqrt{d}) = -iQ^2 - \frac{i}{d^2}\left(c_{2,2}Q^4\right) - \frac{i}{d^3}\left(c_{3,2}Q^4 + c_{3,3}Q^6\right) - \frac{i}{d^4}\left(c_{4,2}Q^4 + c_{4,3}Q^6 + c_{4,4}Q^8\right) + \ldots$$



II. SPECTRUM FROM POLE SKIPPING

- interpolation: $\omega_n(q_n) = -2\pi T i n$ → $\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \dots$

- 'analytic continuation' to *d* spacetime dimensions and expansion around infinite *d*

$$\omega_0(q) = -i\left(\frac{q}{\sqrt{d}}\right)^2 - i\sum_{m=2}^{\infty}\frac{1}{d^m}\sum_{j=2}^{m} c_{m,j}\left(\frac{q}{\sqrt{d}}\right)^{2j}$$

$$\frac{q_n}{\sqrt{d}} = \sqrt{\frac{nd}{2}}\left(1 + \sum_{m=1}^{\infty}\frac{b_{n,m}}{d^m}\right)$$

→

$$b_{n,1} = -\sum_{m=2}^{\infty}\frac{n^{m-1}c_{m,m}}{2^m}$$

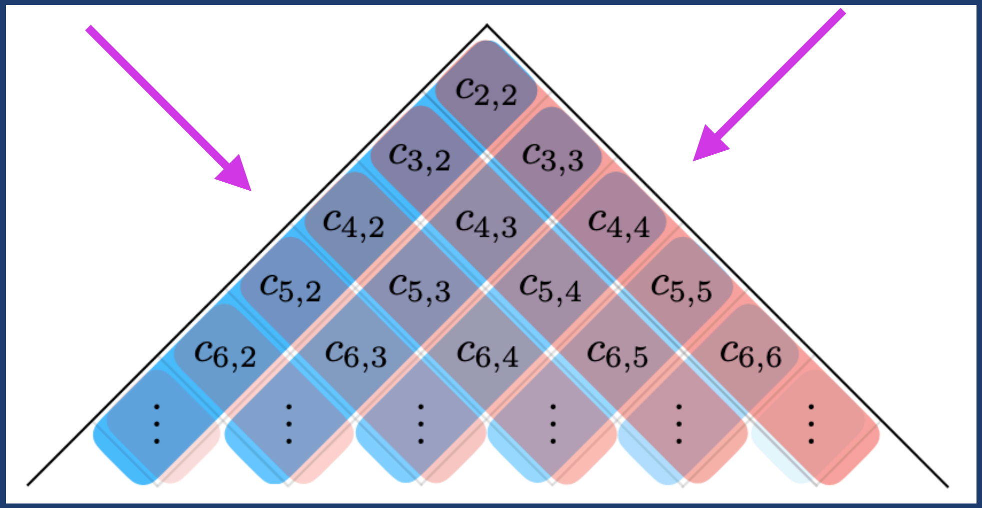
$$b_{n,2} = -\frac{b_{n,1}^2}{2} - \sum_{m=2}^{\infty}\frac{n^{m-1}\left(c_{m+1,m} + 2m b_{n,1} c_{m,m}\right)}{2^m}$$

second analytic continuation

$$n \in \mathbb{Z} \to x \in \mathbb{R}$$

hydrodynamics

pole skipping



symmetry?

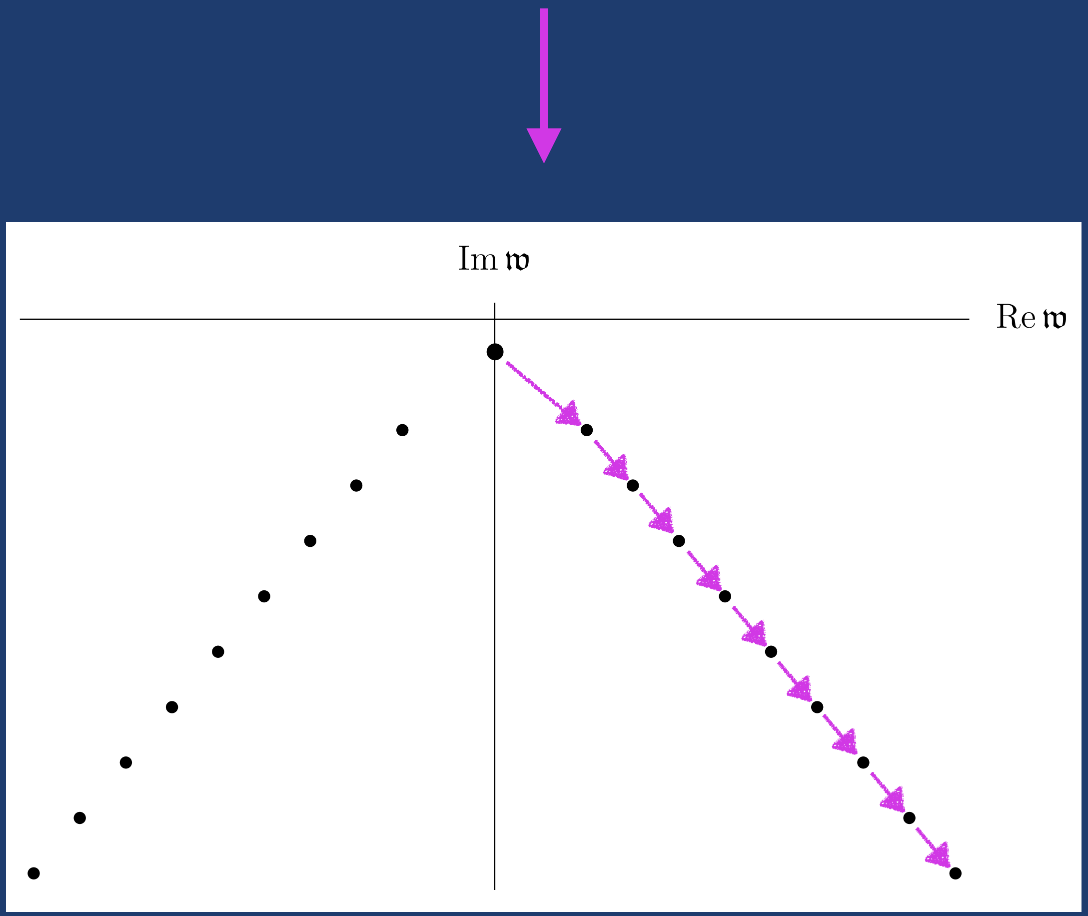
$$c_{m,m} = -\frac{2^m}{(m-1)!}\partial_x^{m-1} b_1(0)$$

$$c_{m+1,m} = -\frac{2^m \partial_x^{m-1} b_2(0)}{(m-1)!} + \sum_{j=2}^{m-1}\left(j - \frac{1}{4}\right) c_{j,j}\, c_{m-j+1,m-j+1}$$

generating functions

II. SPECTRUM FROM POLE SKIPPING

- the rest of the spectrum follows from a reconstruction in Fact I.



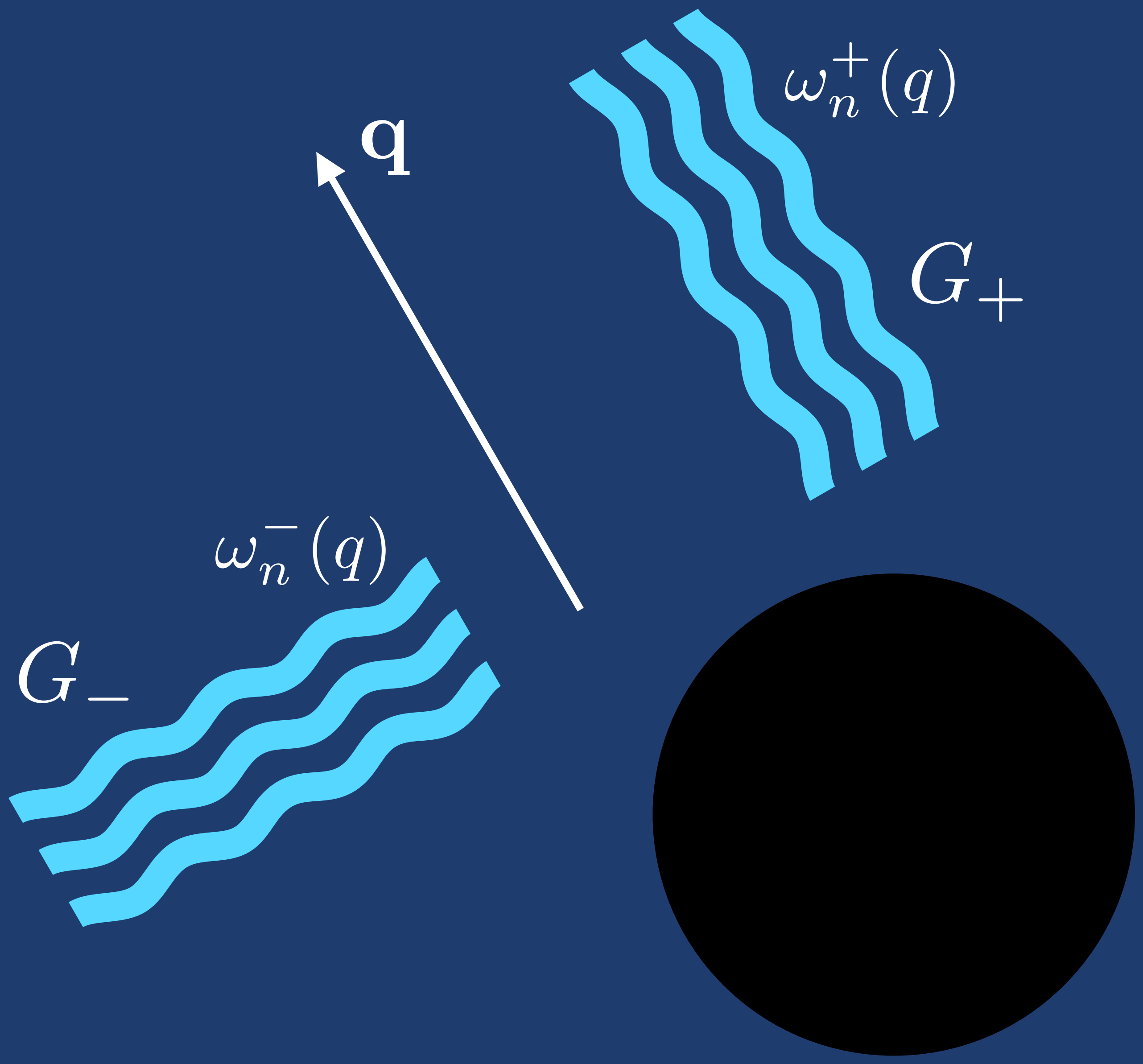
- complete reconstruction of the spectrum using only algebraic 'near-horizon' manipulations (local instead of global (ODE/PDE) analysis)
- thermal product formula: meromorphic correlators follow from poles (QNMs) and a function of q [Dodelson, Iosso, Karlsson, Zhiboedov (2023)]

III. SPECTRAL DUALITY RELATION

[SG, Vrbica, PRL (2024), 2505.14229
and another paper *very soon*]

III. SPECTRAL DUALITY RELATION

- two channels of black hole perturbations in AdS4/CFT3 cases: even (or sound) and odd (or shear)
- in each channel, we have QNMs $\omega_n^{\pm}(q)$ with associated dual correlators $G_{\pm}(\omega, q)$



isospectrality

in asymptotically flat black holes

[Chandrasekhar, Detweiler (1975)]

$$\omega_n^+(q) = \omega_n^-(q)$$

spectral duality relation

in asymptotically AdS black holes

$$\omega_n^+(q) \leftrightarrow \omega_m^-(q)$$

III. SPECTRAL DUALITY RELATION

- two channels of black hole perturbations in AdS4/CFT3 cases: even (or sound) and odd (or shear)
- two meromorphic CFT retarded correlators $G_\pm(\omega, q)$ with QNMs $\omega_n^\pm(q)$ (e.g. of $T^{\mu\nu}$ or J^μ)
- CFT3s have S-duality or particle-vortex duality
  gravity in 4*d* has Chandrasekhar duality, Darboux duality, EM duality

duality:

$$G_+(\omega, q)G_-(\omega, q) = \frac{\omega^2}{\omega_*^2(q)} - 1$$

self-duality: $\omega_*(q) \to \infty$

$$G_+(\omega, q)G_-(\omega, q) = -1$$

algebraically special frequencies
relation to pole skipping
[SG, Vrbica, EPJC (2023)]
easy to compute using the
Robinson-Trautman ansatz

III. SPECTRAL DUALITY RELATION

- define infinite convergent product

$$S(\omega, q) \equiv \left(1 + \frac{\omega}{\omega_*(q)}\right) \prod_n \left[1 - \frac{\omega}{\omega_n^+(q)}\right] \left[1 + \frac{\omega}{\omega_n^-(q)}\right]$$

- duality relations, the thermal product formula and details about QNM and Greens function asymptotics give a 'universal' relation:

$$S(\omega, k) - S(-\omega, k) = 2i\lambda(k) \sinh \frac{\beta\omega}{2}$$

$$\lambda(k) = \frac{2}{i\beta}\left[\frac{1}{\omega_*(k)} + \sum_n \left(\frac{1}{\omega_n^-(k)} - \frac{1}{\omega_n^+(k)}\right)\right]$$

- infinite towers of constraints

$$e_{2j+1}(\mathcal{W}) = \frac{i\lambda}{(2j+1)!}\left(\frac{\beta}{2}\right)^{2j+1}$$

elementary symmetric polynomials with $j \in \{0,1,\ldots,n\}$

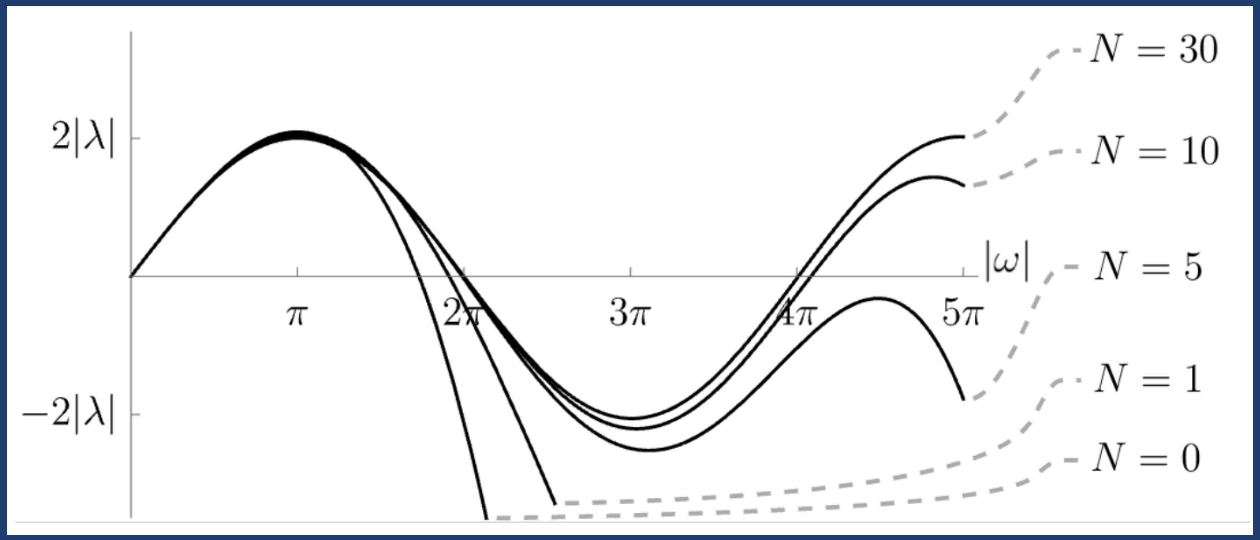
$$\mathcal{W} = \{1/\omega_*, 1/\omega_1^-, \ldots, 1/\omega_n^-, -1/\omega_1^+, \ldots, -1/\omega_n^+\}$$

III. SPECTRAL DUALITY RELATION

- AdS4-Schwarzschild black brane
  - $\langle J, J \rangle_R$ is self-dual

    poles must converge to Matsubara frequencies $\frac{iD_c\beta}{2} \lim_{q\to 0} k^2 S(\omega) = \sinh \frac{\beta\omega}{2}$
  - $\langle T, T \rangle_R$ has $\omega_* = i\frac{\gamma q^4}{6\bar{\epsilon}}$

    various hydro constraints follow, e.g.: $D/\Gamma = 2$



- AdS4-Reissner-Nordström black brane
  - channels are coupled with $\omega_* = i\frac{\gamma q^4}{6\bar{\epsilon}}\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + \left(\frac{2Q\gamma}{3\bar{\epsilon}}\right)^2 q^2}\right)^{-1}$

III. SPECTRAL DUALITY RELATION

- duality relation for any pair of meromorphic correlators:

$$G_+(\omega)G_-(\omega) = -1$$

$$S(\omega) = \prod_n \left(1 - \frac{\omega}{\omega_n^+}\right)\left(1 + \frac{\omega}{\omega_n^-}\right)$$

$$S(\omega) - S(-\omega) = 2i\lambda \sinh \frac{\beta\omega}{2}$$

- knowing one spectrum is sufficient for determining the other spectrum!

- spectral duality relation connects poles and zeros of a single correlator:

$$-G(\omega)G^{-1}(\omega) = -1$$

$$S(\omega) = \prod_n \left(1 - \frac{\omega}{\omega_n}\right)\left(1 + \frac{\omega}{z_n}\right)$$

thermal product formula →

$$r_n = \lambda G(0) \frac{\omega_n \sinh \frac{\beta\omega_n}{2}}{2 \prod_{\substack{m \\ m \neq n}} \left(1 - \frac{\omega_n^2}{\omega_m^2}\right)}$$

III. SPECTRAL DUALITY RELATION

- large-N field theory and double-trace deformed RG flow

$$Z_f[J] = e^{\Gamma[J,f]} = \left\langle e^{\int \mathcal{O} J - \frac{f}{2}\int \mathcal{O}^2} \right\rangle \qquad G(f) = \frac{G_0}{1 + f G_0}$$

$$\left[G(f_1)\left(f_2 - f_1\right) + 1\right]\left[G(f_2)\left(f_2 - f_1\right) - 1\right] = -1$$

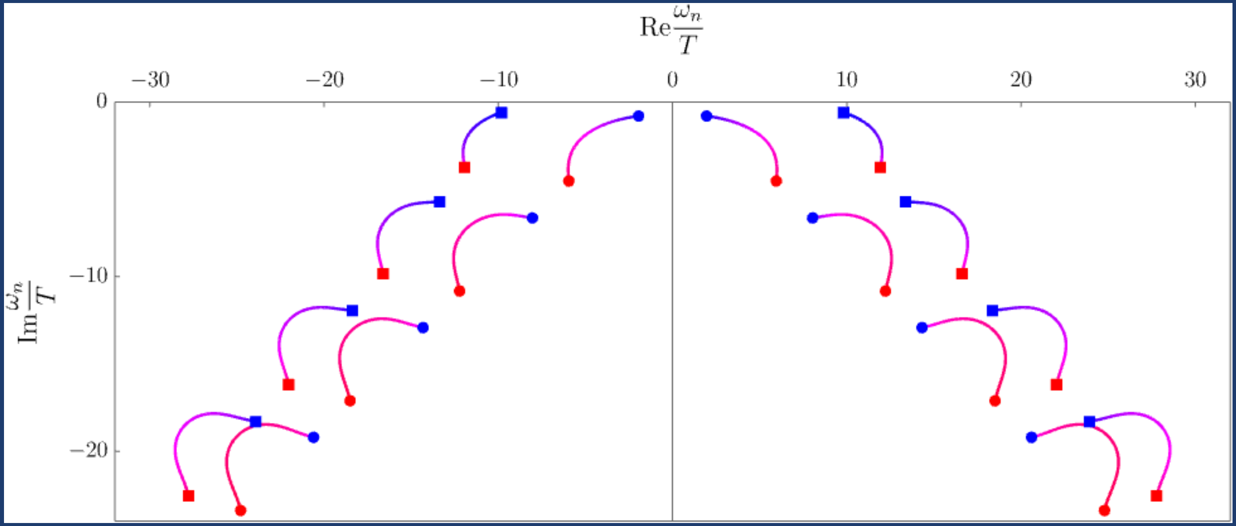
- using duality relations between UV and IR CFTs

$$Z_{f_-}[J_-] = e^{\Gamma_-[J_-, f_-]} = \left\langle e^{\int \mathcal{O}_- J_- - \frac{f_-}{2}\int \mathcal{O}_-^2} \right\rangle_-$$
$$Z_{f_+}[J_+] = e^{\Gamma_+[J_+, f_+]} = \left\langle e^{\int \mathcal{O}_+ J_+ - \frac{f_+}{2}\int \mathcal{O}_+^2} \right\rangle_+$$

$$\Delta_+ + \Delta_- = d$$
$$f_+ f_- = -1$$
$$G_-(f_-) = f_+\left[f_+ G_+(f_+) - 1\right]$$

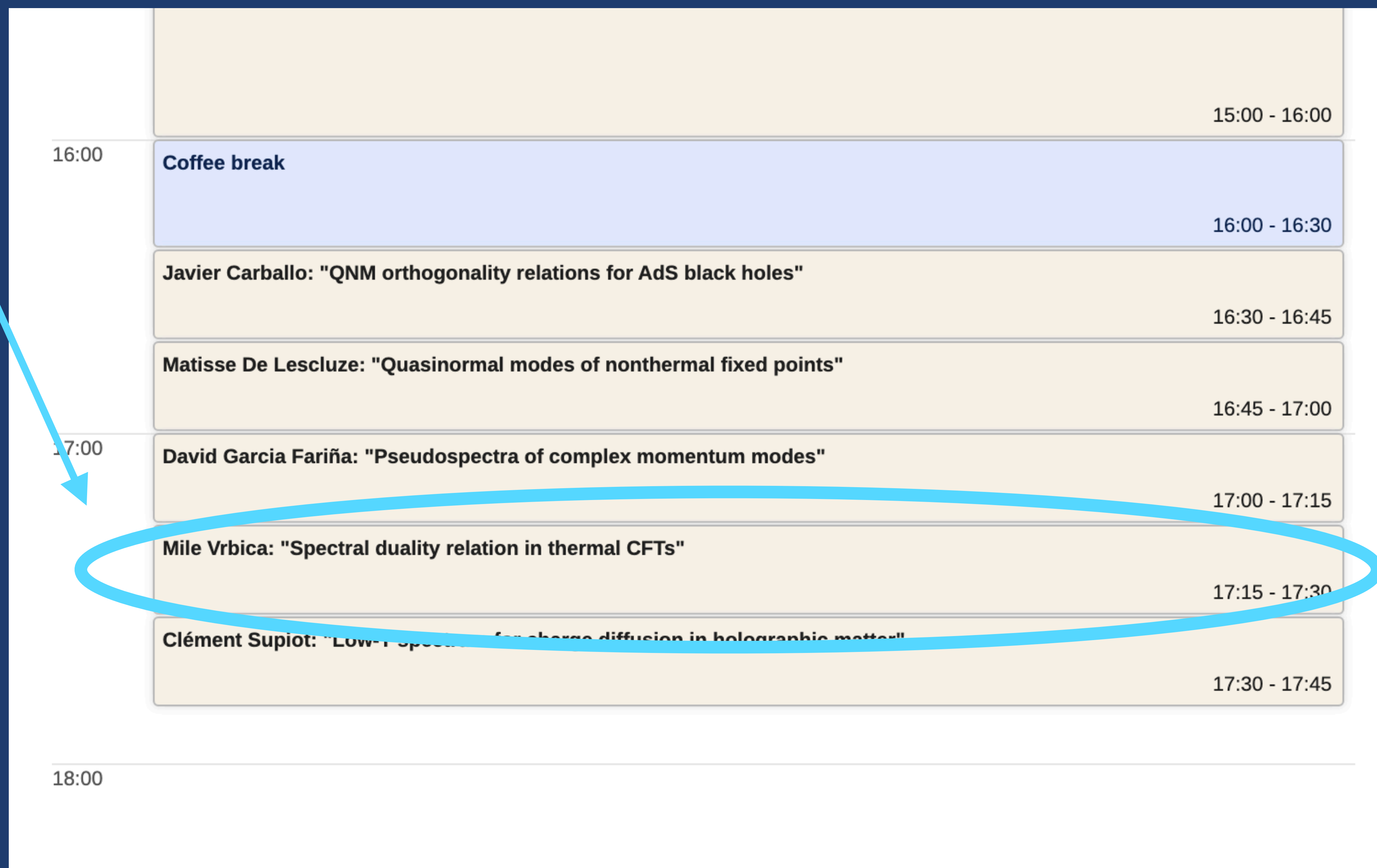
$$G_0^- G_0^+ = -1$$

- same statement applies to correlators in any pair of theories related by Legendre transform



III. SPECTRAL DUALITY RELATION

- for more details, stay around for Mile Vrbica's talk today



SUMMARY AND FUTURE DIRECTIONS

SUMMARY AND FUTURE DIRECTIONS

- QFTs and gravity exhibit extremely interesting and powerful (mathematical) structures
- large classes of large-N QFTs and black hole spectra exhibit stringent constraints
- results apply directly to gravity in AdS and the spectra of quasinormal modes
- keep exploring…
  … ideally in realistic QFTs

THANK YOU!

STUFF

MORE DETAILS ON II. SPECTRUM FROM POLE SKIPPING

$$\omega_0(q) = -i\left(\frac{q}{\sqrt{d}}\right)^2 - i\sum_{m=2}^{\infty}\frac{1}{d^m}\sum_{j=2}^{m} c_{m,n}\left(\frac{q}{\sqrt{d}}\right)^{2j}$$

$$\frac{q_n}{\sqrt{d}} = \sqrt{\frac{nd}{2}}\left(1+\sum_{m=1}^{\infty}\frac{b_{n,m}}{d^m}\right)$$

- first level: $b_{n,1} = -\frac{1}{2}H_n = -\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}$ ⟶ $H_n \to H(x) = \sum_{k=1}^{\infty}\frac{x}{k(x+k)}$

⟶ $c_{m,2} = c_{m,m} = (-1)^m 2^{m-1}\zeta(m)$ $\quad$ $\omega_0(q) = -i\bar{q}^2 - i\frac{\bar{q}^2}{d}H_{2\bar{q}^2/d} + \ldots$

- second level, …
- results: $b_{n,1}$ ⟶ $c_{m,2} = c_{m,m} = (-1)^m 2^{m-1}\zeta(m)$

  $b_{n,3}$ ⟶ $c_{4,2} \approx 1.000 \times 8\zeta(4),$ $c_{5,3} \approx -15.502 \times 16\zeta(5)$

  $b_{n,2}$ ⟶ $c_{3,2} \approx -1.000 \times 4\zeta(3),$ $c_{4,3} \approx 7.001 \times 8\zeta(4),$ $c_{5,4} \approx -15.548 \times 16\zeta(5),$ $c_{6,5} \approx 27.546 \times 32\zeta(6)$

- hidden symmetry?