Mixmaster dynamics in higher-dimensional AdS black holes

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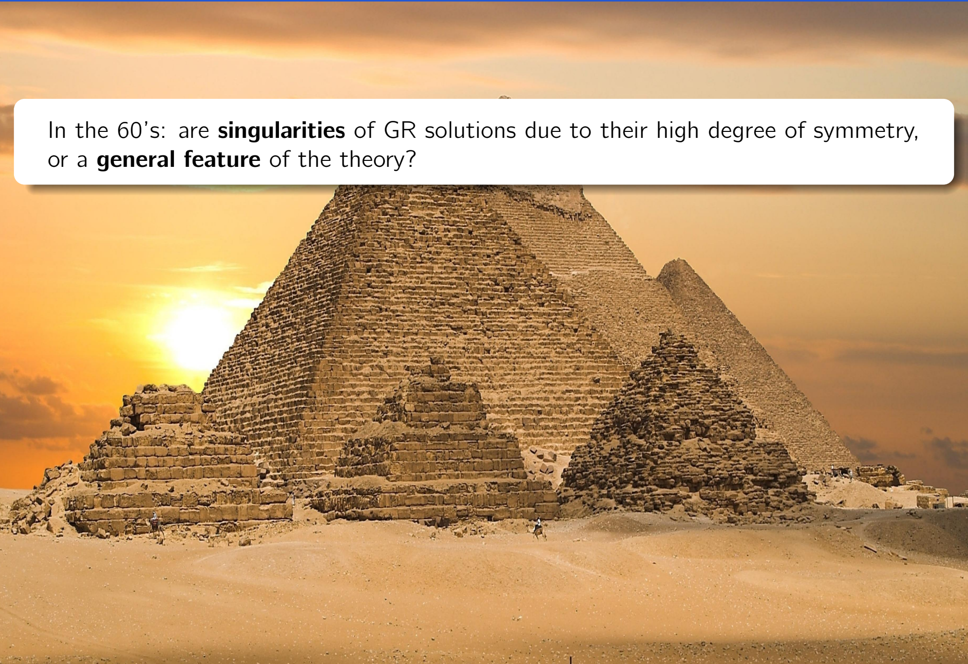
Work in progress w/ E. Cáceres, A. Patra & J. F. Pedraza

Oscillatory approach to singularity

In the 60's: are **singularities** of GR solutions due to their high degree of symmetry, or a **general feature** of the theory?



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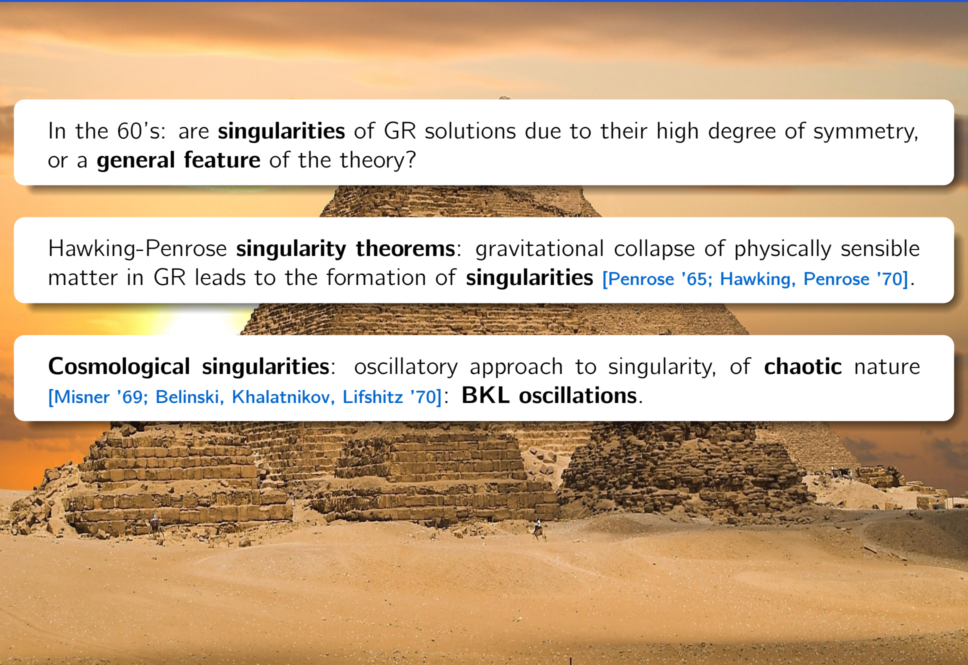

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Approaching the singularity: dynamics become **ultra-local** and spatial derivatives are negligible wrt temporal ones.

Kasner dynamics

Near singularity, space-time looks like **Kasner solution**:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{2p_1}\mathrm{d}x^2 + t^{2p_2}\mathrm{d}y^2 + t^{2p_3}\mathrm{d}z^2 \,, \quad p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \,.$$

The parameters p_i are **Kasner exponents**.

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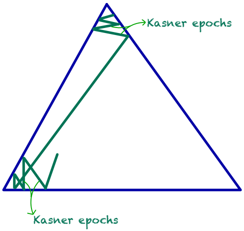
Motion restricted in parameter space to **equilateral triangle**:

1. Each straight line: **Kasner epoch** with fixed p_i.
2. Each bounce produces changes of exponents.



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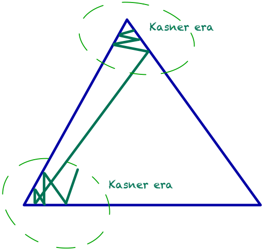

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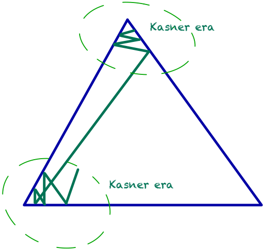

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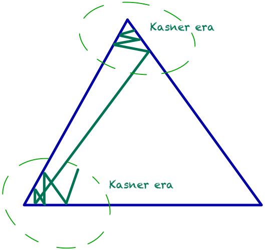

- Sequence of Kasner epochs: distances along two axes oscillate, decrease monotonically along the third.
- Change of Kasner era: axis along which distances decrease is exchanged.

BKL oscillations & billiards

BKL oscillations arose originally in pure gravitational context:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{i=1}^{3} t^{2p_i(x)} (\omega^i)^2 \,, \quad \omega^i = l^i_j \mathrm{d}x^j \,.$$

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Curvature walls are not required: billiard walls may be also due to **electric/magnetic walls** [Damour, Henneaux '00].

BKL oscillations in BHs

BKL dynamics have been thoroughly examined in cosmological backgrounds [*e.g.* **Erickson, Wesley, Steinhardt, Turok '03; Heinzle, Uggla, Rohr '07; Montani, Battisti, Benini, Imponente '07; Bakas, Bourliot, Lüst, Petropoulos '09...**]

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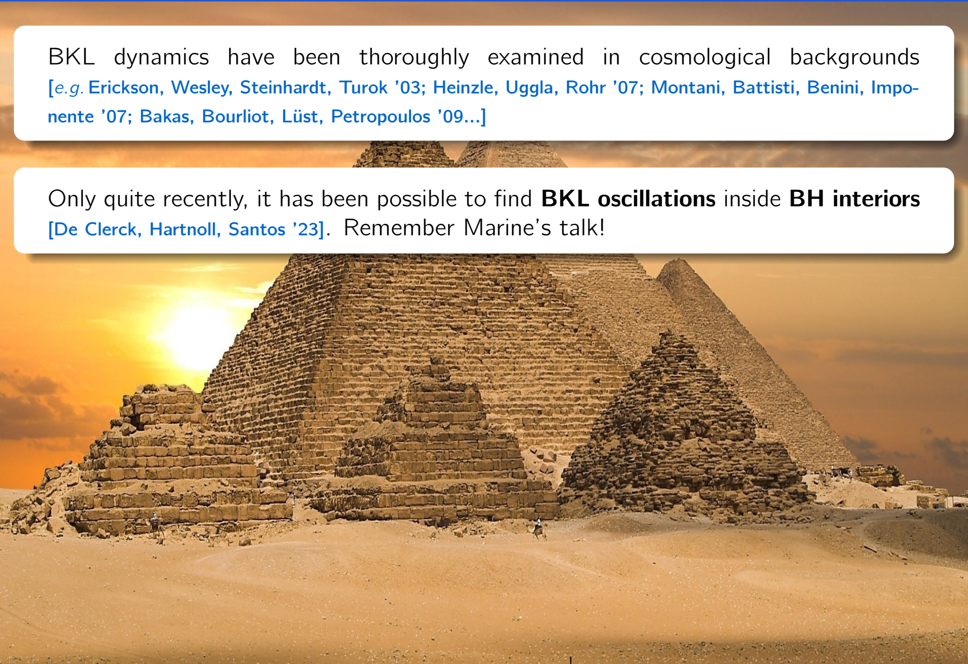

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$$\mathcal{L} = R - \frac{1}{4}\sum_{i=0}^{2}\left[F_i^2 + 2\mu_i^2 A_i^2\right] + \frac{6}{\ell^2}\,.$$

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and AdS BH ansatz:

$$\mathrm{d}s^2 = \frac{1}{z^2}\left(-F(z)e^{-2H(z)}\mathrm{d}t^2 + \frac{\mathrm{d}z^2}{F(z)} + e^{-2G(z)}\mathrm{d}x^2 + e^{2G(z)}\mathrm{d}y^2\right)\,,$$

$$A_0 = \phi_t(z)\mathrm{d}t\,,\quad A_1 = \phi_x(z)\mathrm{d}t\,,\quad A_2 = \phi_y(z)\mathrm{d}y\,.$$

Horizons: zeros of $F(z)$.

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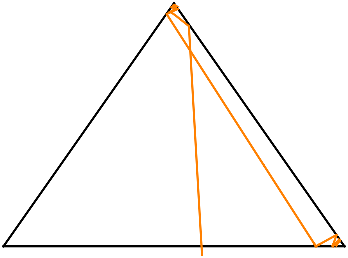

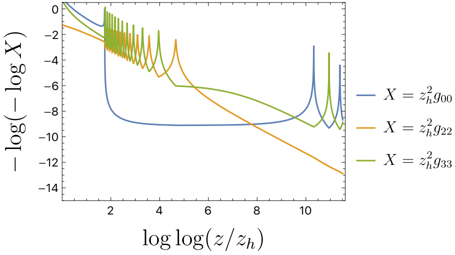

4-dimensional BKL oscillations in AdS black holes

Both in Mixmaster and BH models, **collision laws** for Kasner exponents before and after bounces are **exactly the same**.

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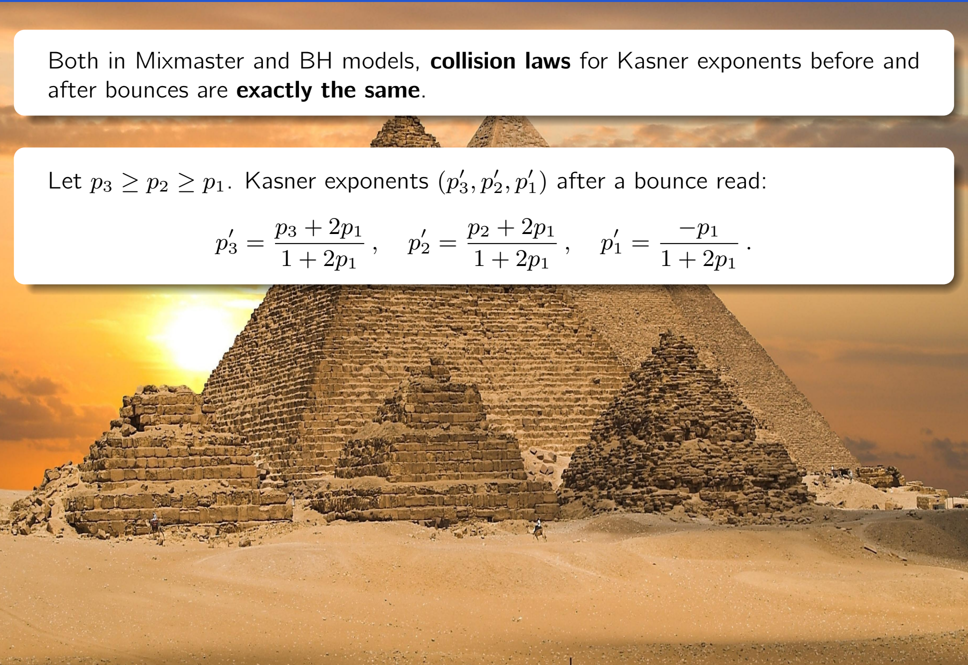

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Let $p_3 \geq p_2 \geq p_1$. Kasner exponents (p_3', p_2', p_1') after a bounce read:

$$p_3' = \frac{p_3 + 2p_1}{1 + 2p_1}\,, \quad p_2' = \frac{p_2 + 2p_1}{1 + 2p_1}\,, \quad p_1' = \frac{-p_1}{1 + 2p_1}\,.$$

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Using parameter $u \geq 1$ [Belinski, Khalatnikov, Lifshitz '70]:

$$\mathfrak{p}_3 = \frac{u(1+u)}{1+u+u^2} \,, \quad \mathfrak{p}_2 = \frac{1+u}{1+u+u^2} \,, \quad \mathfrak{p}_1 = \frac{-u}{1+u+u^2} \,, \quad \mathfrak{p}_3 \geq \mathfrak{p}_2 \geq \mathfrak{p}_1 \,.$$

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$(p_3, p_2, p_1) = (\mathfrak{p}_3(u), \mathfrak{p}_2(u), \mathfrak{p}_1(u))\,.$

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After bounce

- Within **same Kasner era** $(u \geq 2)$:

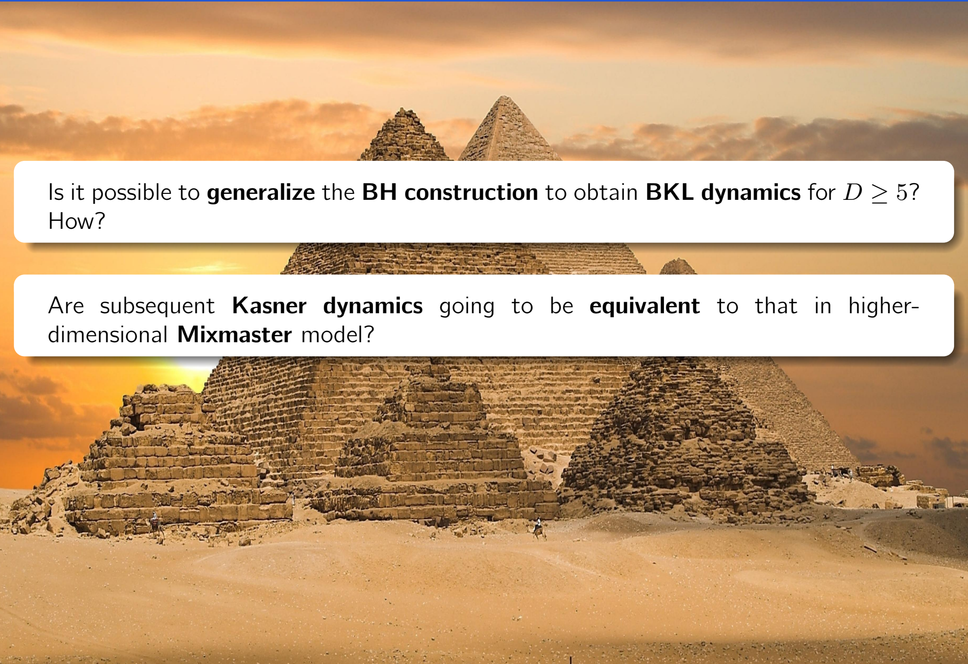
  $$(p_3, p_2, p_1) = (\mathfrak{p}_3(u-1), \mathfrak{p}_1(u-1), \mathfrak{p}_2(u-1)) \,.$$

- **Change** of **Kasner era** $(2 \geq u \geq 1)$:

  $$(p_3, p_2, p_1) = (\mathfrak{p}_2(u'), \mathfrak{p}_1(u'), \mathfrak{p}_3(u')) \,, \;\; u' = \frac{1}{u-1} \,.$$

Is it possible to **generalize** the **BH construction** to obtain **BKL dynamics** for $D \geq 5$? How?

Higher-dimensional BKL oscillations in AdS black holes



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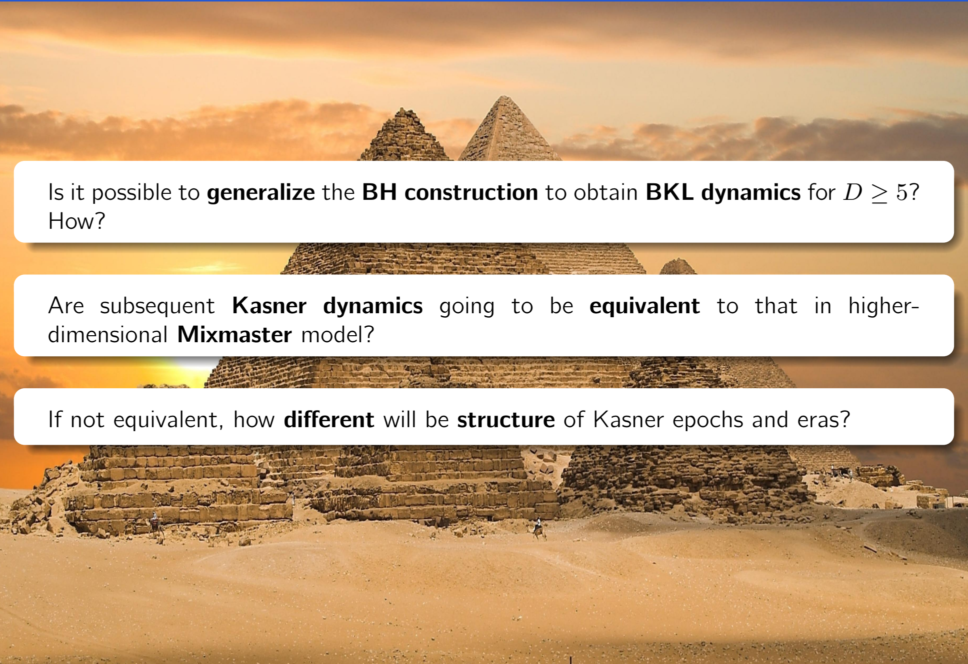

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If not equivalent, how **different** will be **structure** of Kasner epochs and eras?

Higher-dimensional Mixmaster model

For general space-time dimension $D \geq 5$:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{i=1}^{D-1} t^{2p_i(x)} (\omega^i)^2 \,, \quad \omega^i = l^i_j \mathrm{d}x^j \,.$$

Near the singularity: dynamics become **ultra-local** $\rightarrow$ **BKL dynamics** for $D \leq 10$.

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Kasner exponents are constant and satisfy **[Demaret, Henneaux, Spindel '85]**:

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Dynamics of Kasner exponents bounded by **curvature walls**. If $p_{D-1} \geq p_{D-1} \geq \ldots \geq p_1$, **collision law** for exponents before p_i and after p'_i [Demaret et al '85]:

$$p'_1 = \frac{-p_1 - P}{1 + 2p_1 + P} \,, \quad p'_{D-2} = \frac{p_{D-2} + P + 2p_1}{1 + 2p_1 + P} \,, \quad p'_{D-1} = \frac{p_{D-1} + P + 2p_1}{1 + 2p_1 + P}$$

$$p'_j = \frac{p_j}{1 + 2p_1 + P} \,, \quad j = 2, \ldots, D-3 \,, \quad P = \sum_{i=2}^{D-3} p_i \,.$$

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In appropriate parameter region, $\mathfrak{p}_{D-1} \geq \mathfrak{p}_{D-2} \geq \ldots \mathfrak{p}_1$.

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1. Transition of **Kasner epochs**. $p'_{D-1} = \mathfrak{p}_{D-1}(u_1-1, u_j)$ is largest Kasner exponent.

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1. Transition of **Kasner epochs**. $p'_{D-1} = \mathfrak{p}_{D-1}(u_1-1, u_j)$ is largest Kasner exponent.
2. Transition of **Kasner eras**, p'_{D-1} no longer largest Kasner exponent.

Let us find BKL dynamics in BH interiors for $D \geq 5$. Consider $(D-1)$-dimensional **massive vector fields**:

$$\mathcal{L} = R - \frac{1}{4}\sum_{j=0}^{D-2}\left[F_j^2 + 2\mu_j^2 A_j^2\right] + \frac{(D-1)(D-2)}{\ell^2}\,,$$

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and an **AdS BH ansatz**:

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- If at least one $\mu_i^2 < 0$ (and above BF bound) $\rightarrow$ **absence of inner horizon**.

Mixmaster dynamics inside higher-dimensional AdS BHs

Near BH singularity: **BKL oscillations arise**.
Dynamics governed by **Kasner regimes** alternated by jumps of Kasner exponents p_i:

$$\sum_{i=1}^{D-1} p_i = \sum_{i=1}^{D-1} p_i^2 = 1\,.$$

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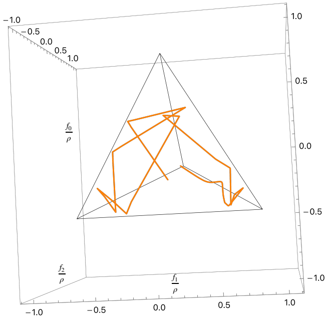
- In $D = 4$ an equilateral triangle, in $D = 5$ a tetrahedron...



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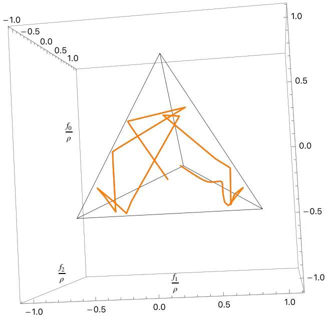

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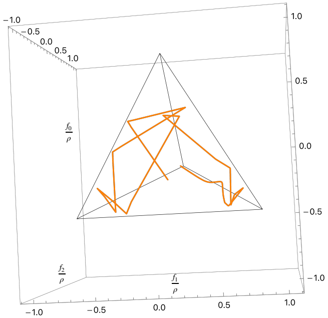
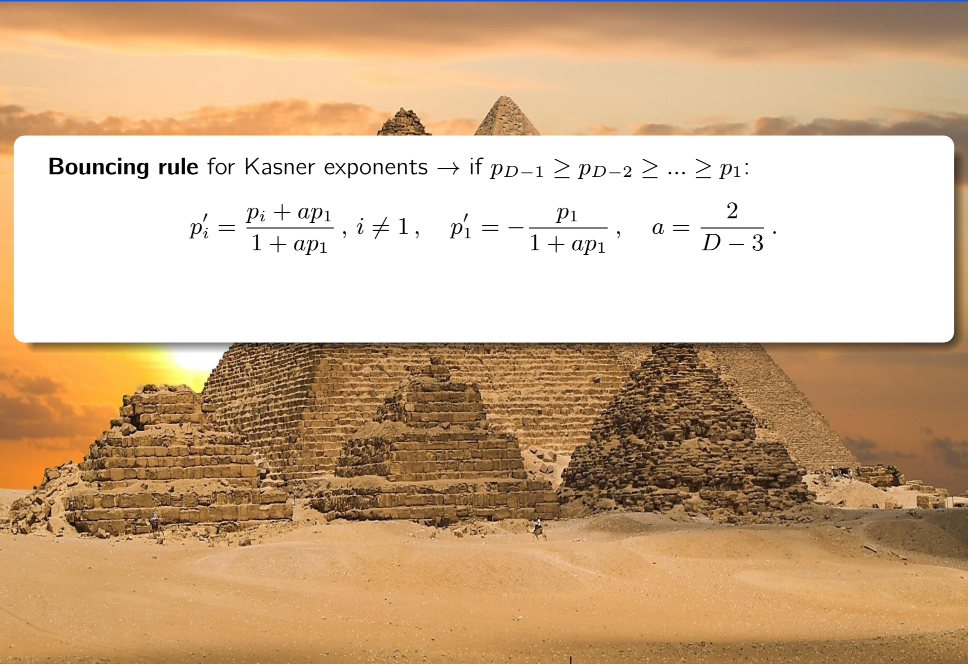


Bounces created by **electric walls**. No gravitational walls involved. Dynamics is **chaotic** for all D! [Damour, Henneaux '00].

Mixmaster dynamics inside higher-dimensional AdS BHs

Bouncing rule for Kasner exponents → if $p_{D-1} \geq p_{D-2} \geq \ldots \geq p_1$:

$$p_i' = \frac{p_i + a p_1}{1 + a p_1}\,,\ i \neq 1\,, \quad p_1' = -\frac{p_1}{1 + a p_1}\,, \quad a = \frac{2}{D-3}\,.$$



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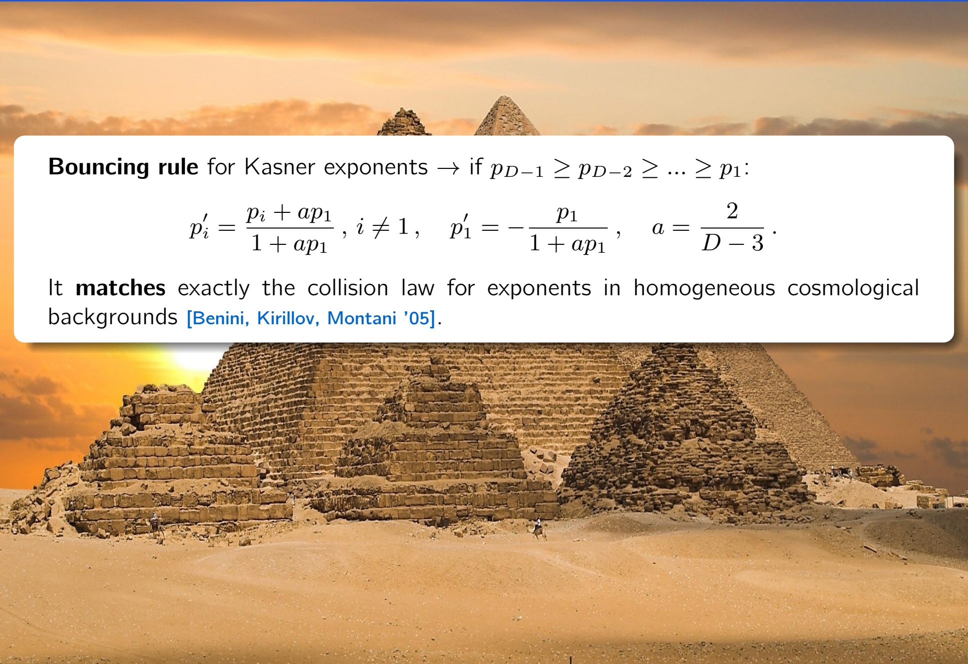

Bouncing rule for Kasner exponents $\rightarrow$ if $p_{D-1} \geq p_{D-2} \geq \ldots \geq p_1$:

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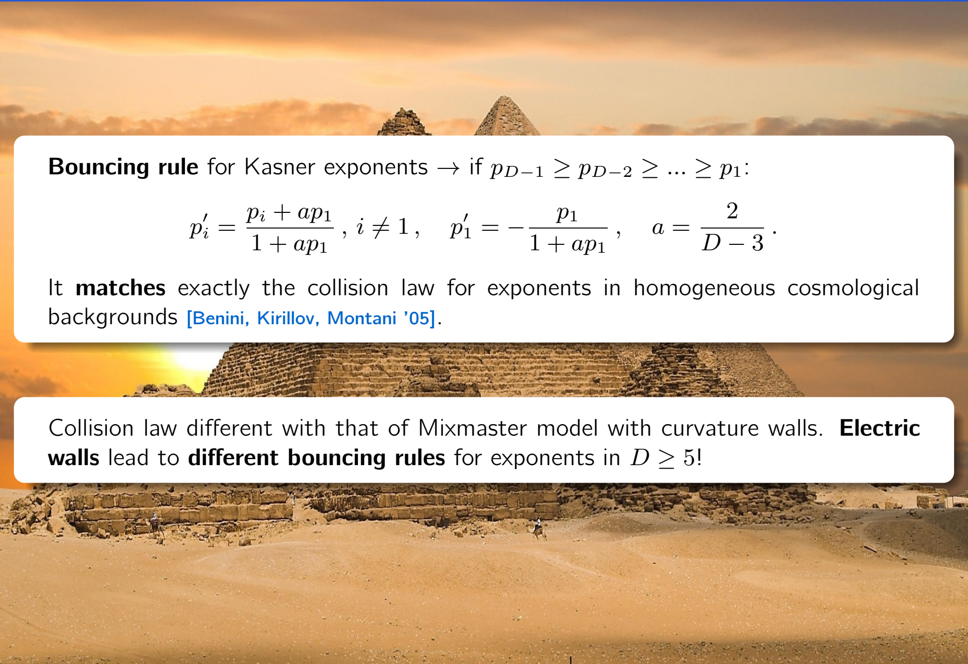

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Collision law different with that of Mixmaster model with curvature walls. **Electric walls** lead to **different bouncing rules** for exponents in $D \geq 5$!

Mixmaster dynamics in five-dimensional AdS BHs

Analyze Kasner dynamics in deep BH interior for $D = 5$. Assume Kasner epoch $p_4 \geq p_3 \geq p_2 \geq p_1$. Kasner exponents p_i' after bounce may satisfy:

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- Last ordering: direction of largest Kasner exponent is swapped $\rightarrow$ change of **Kasner era**.

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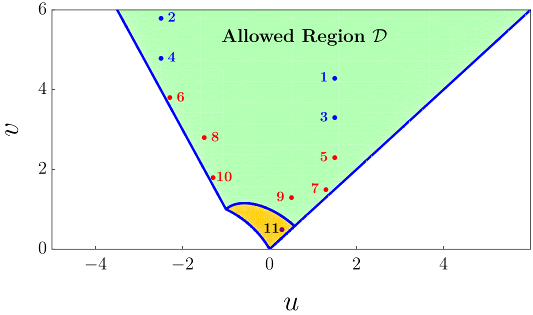
Using parameters (u, v) on appropriate domain $\mathcal{D}$ of definition:

$$p_4 = 1 - \frac{1}{s}\,, \quad p_3 = \frac{1+u+v}{s}\,, \quad p_2 = -\frac{u}{s}\,, \quad p_1 = -\frac{v}{s}\,,$$
$$s = 1 + u + v + u^2 + v^2 + uv\,.$$

Let us visualize **Kasner era** in $D = 5$ in terms of (u, v) parameters:



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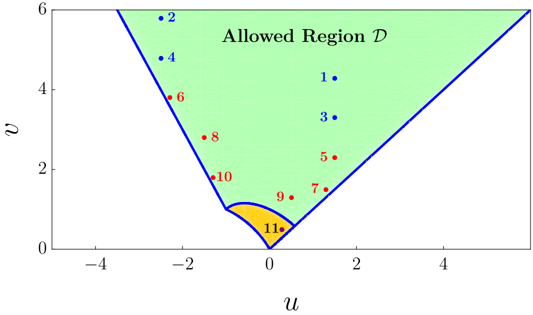

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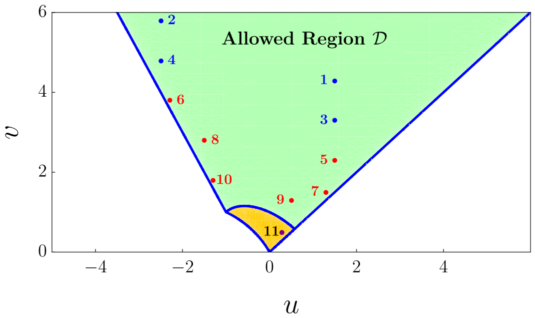

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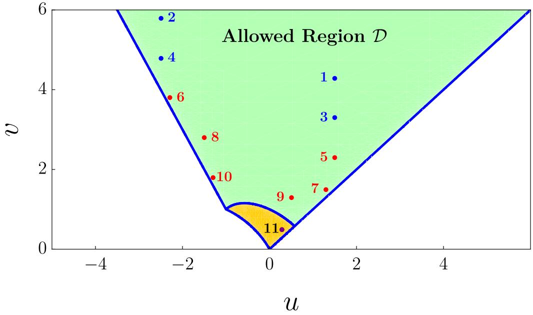

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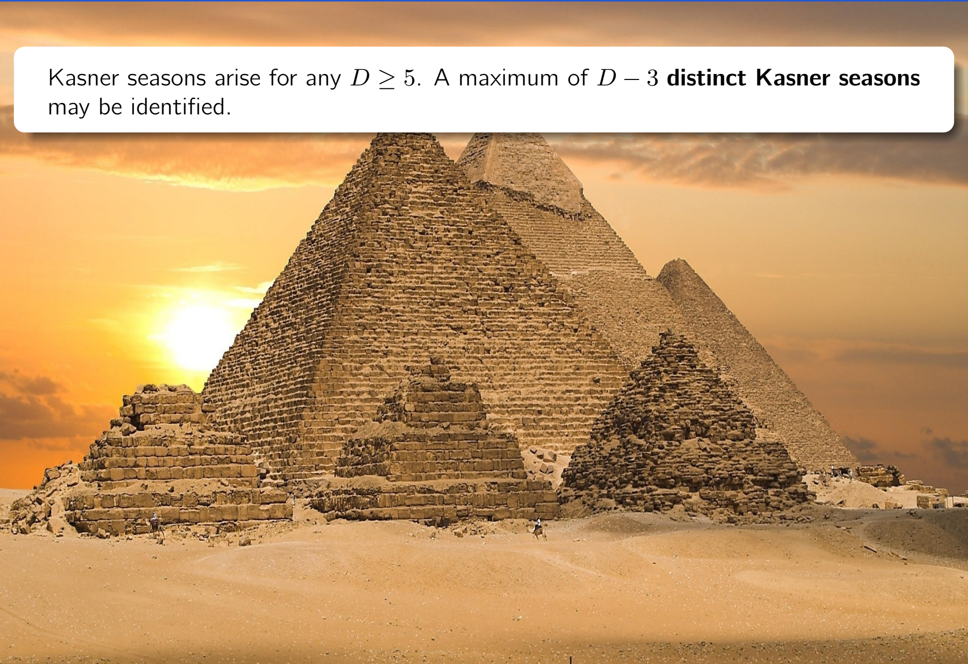
Mixmaster dynamics in higher-dimensional AdS BHs

Kasner seasons arise for any $D \geq 5$. A maximum of $D-3$ **distinct Kasner seasons** may be identified.



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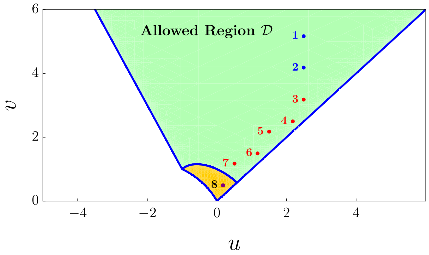

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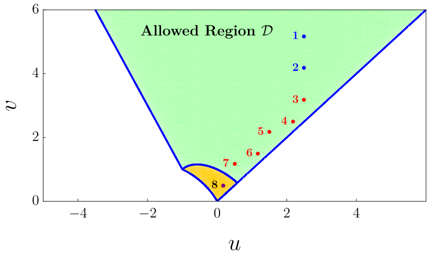

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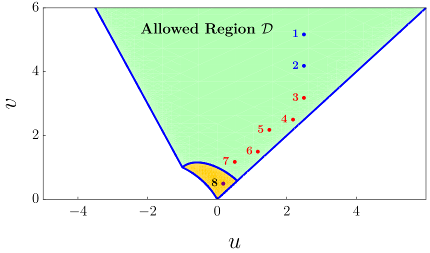

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Explicitly proved this is the case for bouncing rules of exponents in $D = 5$.

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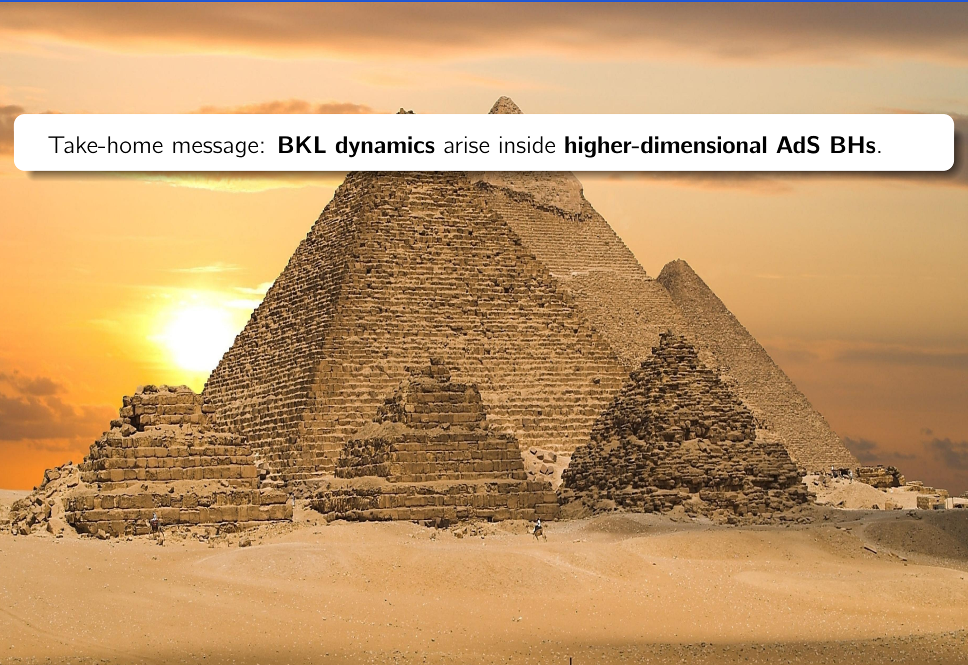

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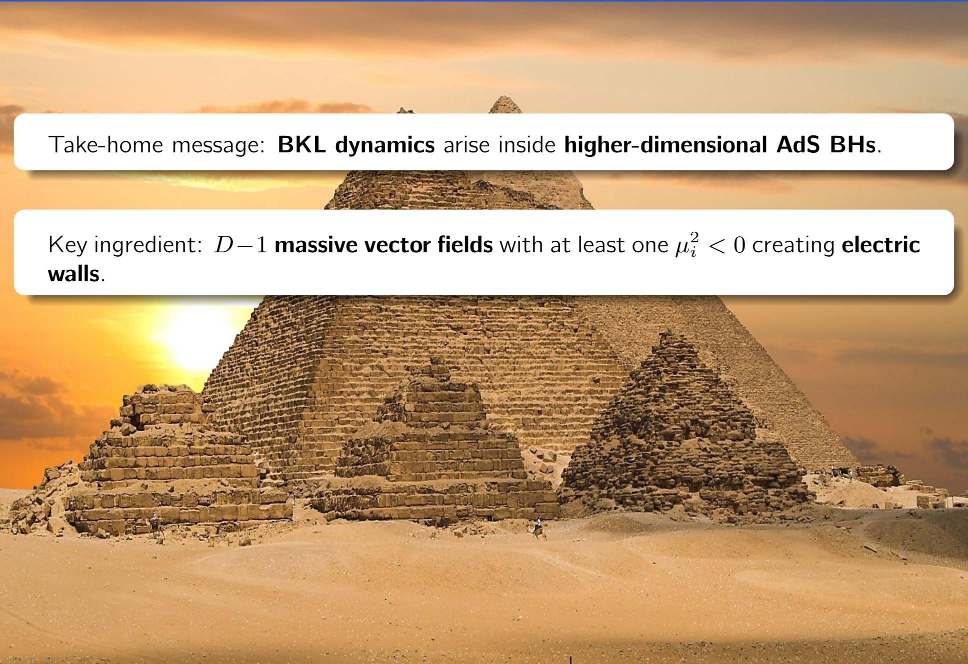

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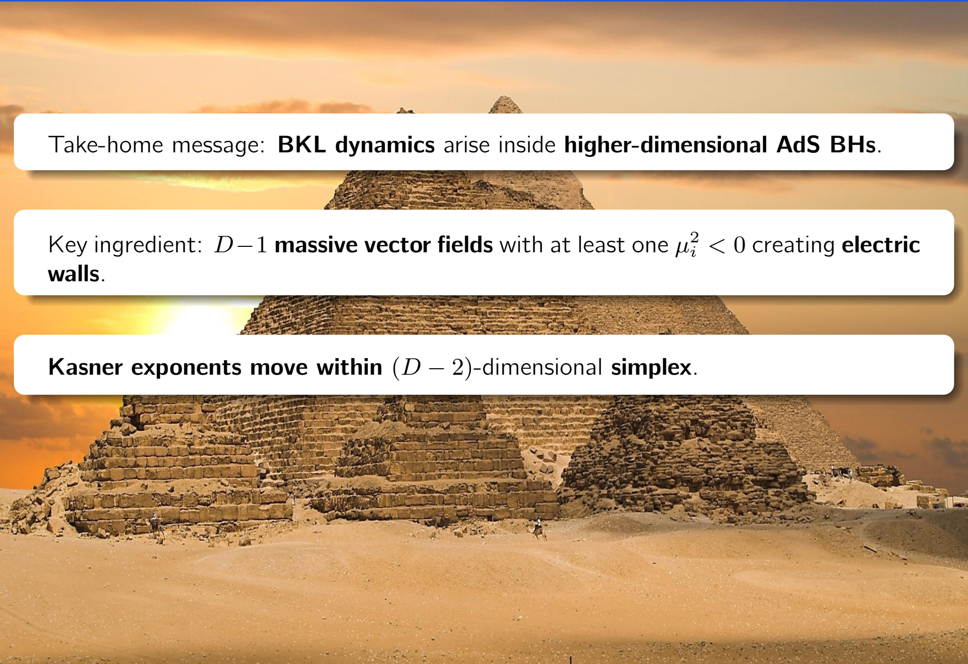

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Structure of Kasner eras reveals intriguing patterns that we have called **Kasner seasons**.

Outlook

Some future avenues:

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¡Muchas gracias!