Flowing to the interior

by

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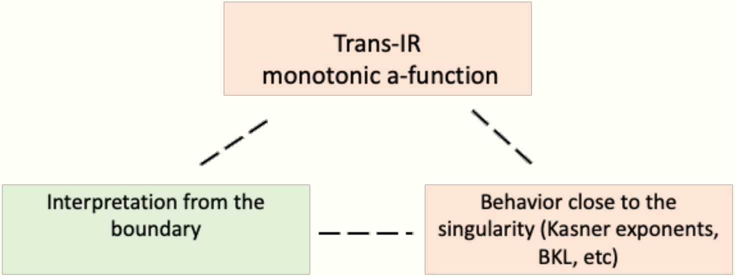
- Black hole interiors
    - conceptual aspects of classical and quantum gravity
    - physics at the singularity (classical)
    - instabilities
    - Holographic implications?

Trans-IR
monotonic a-function

Interpretation from the boundary

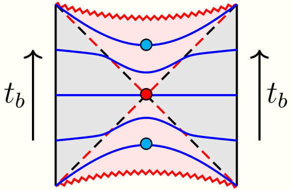
Behavior close to the singularity (Kasner exponents, BKL, etc)


Trans-IR
monotonic a-function
Interpretation from the boundary
Behavior close to the singularity (Kasner exponents, BKL, etc)

Holographic probes of the black hole interior

- 2-point functions encoded by geodesics [Balasubramanian, Ross]
- RT/HRT proposal: encoded by minimal codimension-2 bulk surfaces [Ryu, Takayanagi 2006] [Hubeny, Rangamani, Takayanagi 2007]
- Complexity = Volume (CV): encoded by maximal codimension-1 bulk slice [Susskind 2014] [Susskind, Stanford 2014]
- Complexity = Action (CA): encoded by the action evaluated on Wheeler-deWitt patch [Brown, Roberts, Susskind, Swingle, Zhao 2016]
- Complexity =Anything [Belin, Myers, Ruan, Sárosi, Speranza 2022]

- Not all of these quantities "see" the full interior –up to the singularity
- In $d \geq 2$ for CV and $d \geq 3$ for EE : **extremal surface barriers** block surfaces from reaching singularity [Wall 2014] [Engelhardt, Wall 2014]



Today:

- Holographic thermal a-function as a probe of the interior
- Near singularity behavior
- Early stage idea: Wheeler de Witt states and RG flows in the interior

Holographic RG flows into a black hole

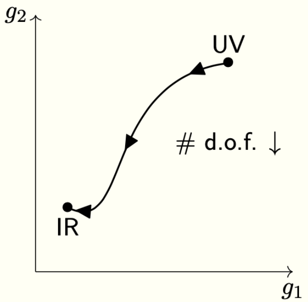
Holography:

- Radial direction $\leftrightarrow$ energy
- Holographic RG flow UV-IR / boundary -horizon [de Boer, Verlinde, Verlinde 1999]
- RG flows at finite temperature

  [Gursoy, Kiritsis, Nitti, Silva Pimenta 2018][Bea, Mateos 2018]
- "RG" flow into the black hole? Trans-IR
  - Here: NEC $\rightarrow$ monotonic function in the interior
  - QFT interpretation? Open question.

Counting Degrees of Freedom

- Count degrees of freedom along flow with a monotonically decreasing function of energy
- Zamolodchikov c-theorem $(d = 2)$
  - Evaluates to central charge at fixed points. $c_{UV} > c_{IR}$
- Cardy a-theorem $(d = 2n)$ Proven for $d = 4$ [Komardoski,Schwimmer 2011]
  - Evaluates to trace anomaly coefficient at fixed points



Holographic History RG Flows

- In AdS/CFT, adding matter sector to bulk is dual to deforming boundary CFT
  - Example: Scalar field ϕ is dual to operator $\mathcal{O}$

$$\int d^{d+1}X\sqrt{|g|}\,[\nabla_\mu\phi\nabla^\mu\phi + V(\phi)] \longleftrightarrow \int d^dx\,\phi_0\mathcal{O}$$

- Relevant deformations trigger RG flows
  - Flow is encoded by classical bulk dynamics

[Balasubramanian, Kraus 1999] [de Boer, Verlinde, Verlinde 2000]

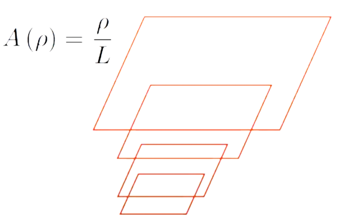
- a-function for **vacuum** states

  [Freedman, Gubser, Pilch, Warner 1999][Henningson,Skenderis 1998]

  - Consider $d+1$ metric Poincare invariant in d dim.

$$ds^2 = e^{2A(\rho)}\left(-dt^2 + d\vec{x}^2\right) + d\rho^2, \;\; ((t,\vec{x}) \in \mathbb{R}^d, \;\; \rho \geq 0)$$

AdS$_{d+1}$ with curvature ℓ when $A(\rho) = \rho/\ell$



$$a_{\mathsf{UV}} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{\ell}{\ell_P}\right)^{d-1} = \boxed{\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right) \ell_P^{d-1}} \left[\frac{1}{A'(\rho)}\right]^{d-1} \equiv a(\rho)}$$

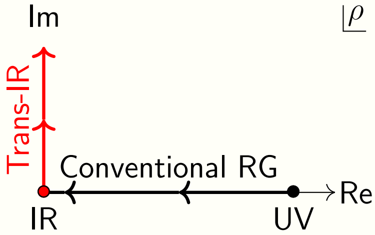
- For general $e^{A(\rho)}$ and Einstein gravity, Null Energy Condition (NEC) [Myers, Sinha 2010–11]

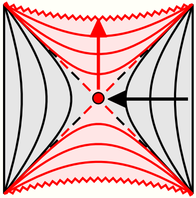
$$T_{\mu\nu}\, k^\mu\, k^\nu \geq 0$$

implies monotonicity of $a(\rho)$

RG flows into the black hole

- AdS/CFT: energy scale is the bulk radial extra dimension
- black holes
  - RG flow of some UV thermal state (bdry.) to IR (horizon)
  - In the interior the radial coordinate becomes timelike $\implies$ trans-IR [Frenkel, Hartnoll, Kruthoff, Shi 2020]





[Caceres, Kundu, Patra, Shashi]

- To probe trans-IR, need to use a black hole geometry
- Consider

$$ds^2 = e^{2A(\rho)}\left(-f(\rho)^2 dt^2 + d\vec{x}^2\right) + d\rho^2,$$

  - Exterior is $(t, \vec{x}) \in \mathbb{R}^d$, $\rho \geq 0$, with horizon at $\rho = 0$

- The interior is accessed via analytic continuation [Fidkowski, Hubeny, Kleban, Shenker 2003] [Grinberg, Maldacena 2021]

$$\rho = i\kappa$$

- Using NEC we can prove that

$$\boxed{a_T(\rho) \equiv \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\ell_P^{d-1}}\left[\frac{f(\rho)}{A'(\rho)}\right]^{d-1}}$$

is monotonic along all the flow

Monotonicity

Use Schawrzchild-like coordinates

$$ds^2 = \frac{1}{r^2}\left[-F(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{F(r)} + d\vec{x}^2\right]$$

$$e^{2A(\rho)} = \frac{1}{r^2}, \quad f(\rho)^2 = F(r)e^{-\chi(r)}, \quad \frac{dr}{d\rho} = -r\sqrt{F(r)}$$

$\Rightarrow$

$$\boxed{a_T(r) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\ell_P^{d-1}} e^{-(d-1)\chi(r)/2}}$$

- Bdy. at $r = 0$
- Horizon at r_h and $F(r)|_{r\leq r_h} \geq 0, \quad F(r)|_{r\geq r_h} \leq 0$

Monotonicity

- For Einstein gravity $G_{\mu\nu} - \frac{d(d-1)}{2} g_{\mu\nu} = \ell_P^{d-1} T_{\mu\nu}$, NEC implies

$$\frac{da_T}{dr} \leq 0 \text{ for all r}$$

Example: Einstein + Free Scalar Theory

$$I_S = -\frac{1}{4\ell_P^{d-1}} \int d^{d+1}x\sqrt{-g}\left(\nabla^\alpha \phi \nabla_\alpha \phi + m^2\phi^2\right)$$

- $\phi = \phi(r)$ dual to relevant operator $\mathcal{O}$ with dimension Δ,

$$I_{\mathcal{O}} = \int d^d x \, \phi_0 \, \mathcal{O}$$

- Schwarzschild-like metric ansatz with horizon $r = r_h$

$$ds^2 = \frac{1}{r^2}\left[-F(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{F(r)} + d\vec{x}^2\right], \quad (r \in \mathbb{R})$$

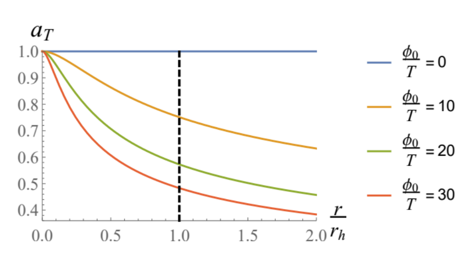
Exterior: $r \leq r_h$ $(F \geq 0)$; Interior: $r \geq r_h$ $(F \leq 0)$.
Bdry at $r = 0$

- Solve numerically for F, χ, ϕ
- Solutions labeled by "strength" of deformation measured by the dimensionless parameter $\phi_0/T^{d-\Delta}$

Plotting a_T

- Monotonicity of a_T

$$a_T(r) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\ell_P^{d-1}} e^{-(d-1)\chi(r)/2}$$



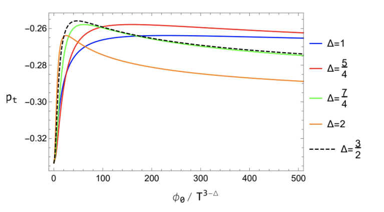
$d = 3$, $\Delta = 2$, a_T, for various deformations

- As we approach singularity ($r \to \infty$) we have $a_T \to 0$

$$a_T(r) \sim r^{-(d-1)^2 q^2/2}$$

where q is a function of p_t and d

- Near-singularity geometry, Kasner exponents
$ds^2 \sim -d\tau^2 + \tau^{2p_t} dt^2 + \tau^{2p_x} d\bar{x}$ –more later



Is the monotonic trans IR a-function an artifact of having too much symmetry? Study Non-isotropic RG flows [D.Giataganas, U. Gürsoy, J.F. Pedraza 2018][C.S. Chu, D. Giataganas 2020]

Non-isotropic Trans-IR Consider a background that breaks the rotational symmetry of the constant-ρ slices,

$$ds^2 = e^{2A(\rho)} \left[-f(\rho)^2 dt^2 + e^{2\mathcal{X}(\rho)} d\vec{x}_1^2 + d\vec{x}_2^2 \right] + d\rho^2$$

Asymptotically AdS

$$A(\rho) \sim \frac{\rho}{\ell} \;\; (\rho \to \infty), \quad \lim_{\rho\to\infty} \mathcal{X}(\rho) = 0, \quad \lim_{\rho\to\infty} f(\rho) = 1.$$

Key observation: if NEC along $k^\mu = \dfrac{e^{-A(\rho)}}{f(\rho)}\partial_t^\mu + \partial_\rho^\mu$ can be written as

$$\boxed{\mathcal{C}(\rho)\frac{d}{d\rho}\left[\tilde{a}(\rho)\right] - \mathcal{K}(\rho)^2 \geq 0,}$$

where $\mathcal{C}(\rho)$ is manifestly positive outside the horizon.

$\Longrightarrow$

$$\frac{d}{d\rho}\left[\tilde{a}(\rho)\right] \geq \frac{\mathcal{K}(\rho)^2}{\mathcal{C}(\rho)} \geq 0.$$

$\Longrightarrow \tilde{a}(\rho)$ is a candidate a-function.

Consider $d+1$ dimensional space, $d_1 + d_2 = d - 1$

$$ds^2 = e^{2A(\rho)}\left[-f(\rho)^2 dt^2 + e^{2\mathcal{X}(\rho)} d\vec{x}_1^2 + d\vec{x}_2^2\right] + d\rho^2$$

we have,

$$\mathcal{C}(\rho) = \frac{1}{(d-1)f(\rho)}\left[d_1\big(A'(\rho) + \mathcal{X}'(\rho)\big) + d_2 A'(\rho)\right]^2 e^{d_1\mathcal{X}(\rho)/(d-1)},$$

$$\mathcal{K}(\rho) = \sqrt{\frac{d_1 d_2}{d-1}}\mathcal{X}'(\rho).$$

$\rightarrow$ the monotonic a-function is,

$$\boxed{a(\rho) \sim e^{-d_1\mathcal{X}(\rho)}\left[\frac{(d-1)f(\rho)}{d_1\big(A'(\rho)+\mathcal{X}'(\rho)\big)+d_2 A'(\rho)}\right]^{d-1}}$$

Example: p-wave superfluid

[M. Ammon, J. Erdmenger, V. Grass, P. Kerner and A. O'Bannon 2010]

$(4+1)$-dimensional Einstein-Yang-Mills theory with $SU(2)$ gauge symmetry whose bulk action is (setting $\ell = 1$)

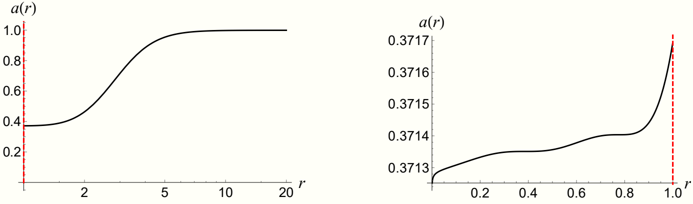
$$I_{\mathsf{EYM}} = \int d^5x\sqrt{-g}\left[\frac{1}{2\ell_{\mathsf{P}}^3}\left(R+12\right) - \frac{1}{4\hat{g}^2}F^a_{\mu\nu}F^{a\mu\nu}\right].$$

$a = 1, 2, 3$ are $SU(2)$ indices

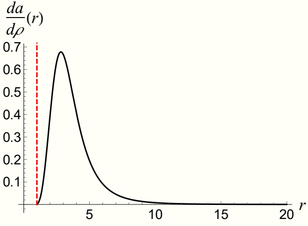
$$ds^2 = e^{2A(\rho)} \left[-f(\rho)^2 dt^2 + e^{2\mathcal{X}(\rho)} dx^2 + dy^2 + dz^2 \right] + d\rho^2$$

The a-function is

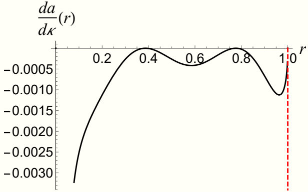
$$a_T^{(1)}(\rho) \sim e^{-\mathcal{X}(\rho)} \left[\frac{f(\rho)}{A'(\rho) + \frac{1}{3}\mathcal{X}'(\rho)} \right]^3$$



(a) Exterior a-function **(b)** Interior a-function



(a) Derivative, exterior . $\dfrac{da_T}{d\rho}$



(b) Derivative, interior, $\dfrac{da_T}{d\kappa}$

- Approach to the singularity [Cai, Ge, Li, Yang 2022][Sword, Vegh 2022]

Near singularity

In the simple case of Einstein gravity + free scalar

$$I = \int d^4x\sqrt{-g}\left(R + 6 - \partial^\alpha\phi\partial_\alpha\phi - m^2\phi^2\right)$$

with solution, $ds^2 = \frac{1}{r^2}\left[-F(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{F(r)} + d\vec{x}^2\right]$

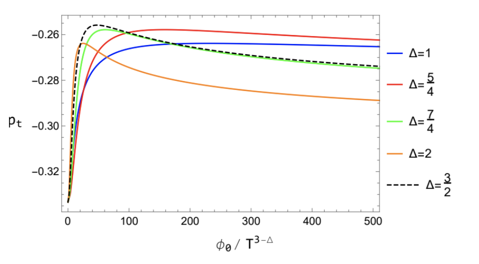
- Near-singularity ($r \to \infty$) geometry is Kasner-like

$$ds^2 \sim -d\tau^2 + \tau^{2p_t}dt^2 + \tau^{2p_x}(dx^2 + dy^2)$$

$$p_t + 2p_x = 1, \quad p_\phi^2 + p_t^2 + 2p_x^2 = 1$$

$$\phi \sim -\sqrt{2}p_\phi \log\tau$$

- (free) Kasner flows [Frenkel, Hartnoll, Kruthoff, Shi] [Caceres, Kundu, Patra, Shashi]



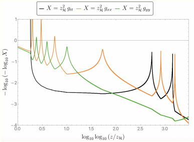
- AdS-Schwarzchild corresponds to $p_t = -\frac{1}{3}$, $p_x = p_y = \frac{2}{3}$ and $p_\phi = 0$

Toward Generic Singularities: BKL Picture

- **Belinski, Khalatnikov, Lifshitz (BKL)**: structure of generic spacelike singularities.
- Kasner universe with changing parameters
- Evolution is **oscillatory and chaotic** ("Mixmaster behavior").

Many developments:

- Bouncing universes; super-exponential potential [Hartnoll,Neogi 2022]. Thermal a-function in bouncing interiors [Caceres, Patra, Pedraza 2023]
- Mixmaster in $D = 4$, 3 gauge fields –Marine's talk



[De Clerck, Hartnoll, Santos 2023]

- Lovelock gravity, no matter [Bueno, Cano, Hennigar, Li 2024]
- Mixmaster behavior with higher derivatives? Open question
- First: Mixmaster chaos in higher dimensions –Angel's talk [Caceres, Murcia, Patra, Pedraza]

Kasner exponents in Lovelock theories

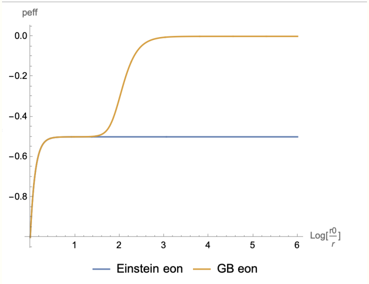
- In vacuum higher derivative theories:
  - Periods where a specific Higher derivative term dominates : **eons** [Bueno,Cano, Hennigar 2024]
  - For example, for Gauss-Bonnet gravity,

$$\mathcal{L} = \frac{\Lambda_0}{\ell^2} + R + \lambda_2 \ell^2 (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho})$$

  - close to the singularity

$$ds = -d\tau^2 + \tau^{2p_t} + \sum_{i=1}^{d-1} \tau^{2p_i}\, dx_i^2$$

Example:
in Gauss Bonnet gravity in 5 dimensions, $p_{eff} \equiv \frac{rf'(r)}{2f(r)-rf'(r)}$



- If we are interested in the a-function → add matter

Thermal RG flows in higher derivative theories

In higher derivative theories we have two different possible conditions to impose,
NEC

$$k^{\mu}k^{\nu}T_{\mu\nu} \geq 0$$

NCC

$$k^{\mu}k^{\nu}R_{\mu\nu} \geq 0$$

$\Longrightarrow$ two different a-functions

Let's look at a concrete case

- In quasi-topological gravities, $d \geq 4$ [Oliva,Ray 2010][Myers,Robinson 2010]

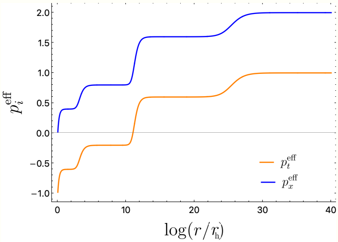
$$I = \int d^{d+1}x\sqrt{-g}(R + d(d-1) + \sum_{n=2}^{\infty} \lambda_n \, \mathcal{Z}_{(n)})$$

HD terms

the coefficients of HD terms are chosen such that for one-function, $f(r)$, static bh the eom are first-order in derivatives [Bueno, Cano, Moreno, Murcia 2019][Moreno, Murcia 2023]

$$-\frac{\lambda_0}{\ell^2}+\sum_{n=1}^{\infty}\lambda_n(-1)^n(d+1-2n)(d\ell^2 f(r)(d+1))^{n-1}\left(nrf'(r)-d\,f(r)\right)=0$$

Quasi-topological gravity of order 5, without matter, $n=5,\ d=5$



Goal:

- Add matter $\rightarrow$ we expect the behavior close to the singularity will change
- study NEC and NCC a-function, do both see the change close to the singularity

We want to add matter $\rightarrow$ look for a subset of QT gravities such that the solutions involve $f(r)$ and $\chi(r)$ and eom are still first order (solve numerically)

$$I = \int d^{d+1}x\sqrt{-g}[R + d(d-1) + \sum_{n=2}^{\infty} \lambda_n \mathcal{Z}_{(n)} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2]$$

$$\mathrm{d}s^2 = \frac{1}{r^2}\left(-f(r)e^{-\chi(r)}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + \mathrm{d}\vec{x}_{d-1}^2\right), \qquad \phi = \phi(r)$$

The coefficients of $\mathcal{Z}_{(n)}$ can be chosen so that the EOMs are first and second order. Solve numerically Close to the singularity,

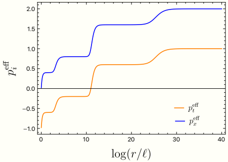
$$
\begin{aligned}
r^2 f\phi'' &= \frac{1}{2} r\,\phi' \left(f\left(r\chi' + 2(d-1)\right) - 2rf'\right) + V'(\phi(r))\,. \\
\chi' &= \frac{r(\phi')^2}{\sum_{n=1}^{\infty} \tilde{\lambda}_n \ell^{2(n-1)} f^{n-1}}\,, \\
f' &= \frac{-2\lambda_0 + 2\ell^2 V(\phi(r)) + 2d\sum_{n=1}^{\infty} \frac{\ell^{2n}\tilde{\lambda}_n}{n} f^n + \ell^2 f r^2 (\phi')^2}{2r\sum_{n=1}^{\infty} \ell^{2n}\tilde{\lambda}_n f^{n-1}}\,,
\end{aligned}
$$

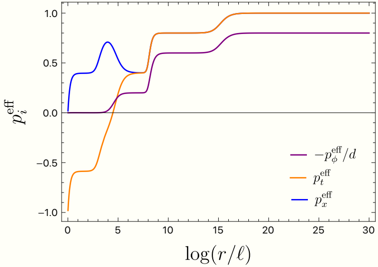
Close to the singularity,

$$
\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \tau^{2p_t}\mathrm{d}t^2 + \tau^{2p_x}\mathrm{d}x_{d-1}^2
$$

Matter significantly changes the behavior close to the singularity

[Caceres, Murcia, Patra, Pedraza 2024]





Thermal RG-flows [Caceres, Murcia, Patra, Pedraza 2024]

- Imposing NEC,

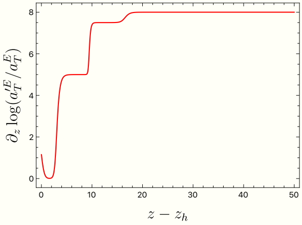
$$a_T^{\mathsf{E}}(r) \propto \exp\left(-\frac{(d-1)}{2}\chi - \frac{1}{2}\sum_{n=2}^{\infty}\tilde{\lambda}_n \int f^{(n-1)}\chi' dr \right)$$

  where $\tilde{\lambda}_n$ are rescaled couplings

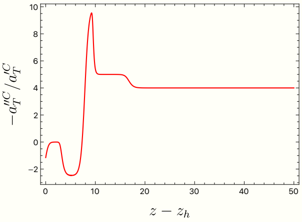
- Imposing NCC,

$$a_T^{\mathsf{C}}(r) \propto \exp\left(-\frac{(d-1)}{2}\chi \right).$$

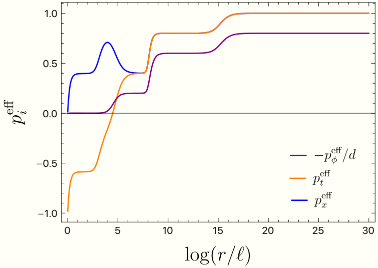
- Can the thermal a-function "see" the behavior of Kasner exponents? Yes!



$\partial_z \log(a_T'^E / a_T^E)$
$z - z_h$



$-a_T''^C / a_T'^C$
$z - z_h$



p_i^{eff}
$-p_\phi^{\mathrm{eff}}/d$
p_t^{eff}
p_x^{eff}
$\log(r/\ell)$

- the a-function is a diagnostic tool of HD effects close to the singularity

The interior

- Wheeler-DeWitt states of the AdS-Schwarzschild interior
  [Hartnoll 2022]
  - Solved the Wheeler-DeWitt equation for the planar AdS-Schwarzschild interior
  - Constructed a Gaussian wavepacket that peaks on the classical interior
  - .....

- To make contact with RG flows, add a scalar
  [Early stage WIP with Hare Krishna 2025]

Brief review of the setup in [Hartnoll 2022]

$$S = \int d^4x\sqrt{-g}(R+6) + 2\int d^3x\,\sqrt{\gamma}K$$

Ansatz,

$$ds^2 = -N dz^2 + v^{2/3}(e^{4k/3}dt^2 + e^{-2k/3}(dx^2+dy^2))$$

Lagrangian density,

$$\mathcal{L} = 6Nv + \frac{2}{3}\frac{v^2(\partial_r k)^2 - (\partial_r v)^2}{Nv}\,.$$

- N imposes the a constraint, in terms of momenta

$$\boxed{-\ \pi_k^2 + v^2\pi_v^2 + 16v^2 = 0}$$

- Hamilton-Jacobi, let $\pi_k = \frac{\partial \mathcal{S}}{\partial k}, \pi_v = \frac{\partial \mathcal{S}}{\partial v} \Rightarrow$ Hamiltonian constraint is

$$-(\partial_k S)^2 + v^2(\partial_v S)^2 + 16v^2 = 0$$

Solve,

$$S(v, k; k_o) = 4v \sinh[k + k_o]$$

Aside:As usual in Hamilton-Jacobi theory, the general solution to eoms is obtained by solving $\partial_{k_o} S = \varepsilon_o$
$\rightarrow v = \frac{\varepsilon_o}{4} \operatorname{sech}[k + k_o] \Leftarrow \text{this is Schwz-AdS}$

Wheeler-de Witt equation: canonical quantization of Hamiltonian constraint

- Promote $\pi_k \to i\frac{\partial}{\partial k}, \quad i\pi_v \to \frac{\partial}{\partial v}$

$$\partial_k^2 \Psi - v\partial_v \left(v\partial_v \Psi \right) + 16v^2\Psi = 0$$

$$\Psi = e^{iS(v,k;k_0)},$$

Build wave packet, etc.

To make contact with RG flows, add matter

$$S = \int d^4x\sqrt{-g}R + 2\int d^3x\,\sqrt{\gamma}K + S_m$$

$$V(\phi) \sim -\frac{6}{\ell^2} + \frac{1}{2}m^2\phi^2 + \cdots$$

Ansatz,

$$ds^2 = -e^{\alpha(z)}dz^2 + \gamma_{\mu\nu}dx^\mu dx^\nu$$

$$\gamma_{\mu\nu}dx^\mu dx^\nu = e^{2\beta_x(z)}dx^2 + e^{2\beta_y(z)}dy^2 + e^{2\beta_t(z)}dt^2$$

$$S = -2 \int d^4x \, \sqrt{\gamma} e^{2\alpha_z(z)} \left(\beta_t'(z) \left(\beta_x'(z) + \beta_y'(z)\right) + \beta_x'(z)\beta_y'(z)\right) + S_{matter}$$

Redefine linear combinations, $\beta_i \to \bar{\beta}_i$.

$$S = -2 \int d^4x \, \sqrt{\gamma} e^{2\alpha_z(z)} (\bar{\beta}_t'(z)^2 + \bar{\beta}_x'(z)^2 + \bar{\beta}_y'(z)^2) + S_{matter}$$

Conjugate momenta,

$$\pi_{\bar{\beta}_t} = \frac{\partial L}{\partial \, \bar{\beta}_t'} = -2\bar{\beta}_t', \quad \pi_{\bar{\beta}_x} = \frac{\partial L}{\partial \, \bar{\beta}_x'} = 2\bar{\beta}_x', \quad \pi_{\bar{\beta}_y} = \frac{\partial L}{\partial \, \bar{\beta}_y'} = 2\bar{\beta}_y'$$

The Hamiltonian constraint is

$$\boxed{-\frac{1}{4}\pi_{\bar{\beta}_t}^2 + \frac{1}{4}\pi_{\bar{\beta}_x}^2 + \frac{1}{4}\pi_{\bar{\beta}_y}^2 + \frac{1}{2}\pi_{\phi}^2 + V(\phi) = 0}$$

Use Hamilton-Jacobi to write conjugate momenta as the variation of the on-shell action with respect to dynamical fields,

$$\pi_{\bar{\beta}_t} = \frac{\delta S}{\delta\,\bar{\beta}_t}, \quad \pi_{\bar{\beta}_x} = \frac{\delta S}{\delta\,\bar{\beta}_x}, \quad \pi_{\bar{\beta}_y} = \frac{\delta S}{\delta\,\bar{\beta}_y}, \quad \pi_{\phi} = \frac{\delta S}{\delta \phi}$$

$$\frac{1}{4}\Big(\frac{\delta S}{\delta\,\bar{\beta}_t}\Big)^2+\frac{1}{4}\Big(\frac{\delta S}{\delta\,\bar{\beta}_x}\Big)^2+\frac{1}{4}\Big(\frac{\delta S}{\delta\,\bar{\beta}_y}\Big)^2+\frac{1}{2}\Big(\frac{\delta S}{\delta\phi}\Big)^2+V(\phi)=0$$

Note that,

- Close to the singularity $\beta_i \sim \mathrm{Kasner\ exponents} p_i$
- Quantum cosmology with a scalar (1980's) re-interpreted in holographic context.

  .....WIP

Conclusions and takeaways

- Trans-IR flows seems rather abstract in typical QFT.
    - Naturally emerge in holographic RG framework as black hole interiors
- We can define an a-function that is monotonic along the flows, even in anisotropic backgrounds and higher derivative theories
- Trans-IR flow and singularity structure

- We need a better holographic understanding of black hole interiors/near-singularity geometries with more complicated matter profiles (BKL analysis)
  - Scalars with non-minimal couplings
  - Mixmaster in higher dimensions
  - Mixmaster and higher derivatives
- What does is the trans-IR flow from the boundary perspective? Connection to WDW states?

Many more questions...

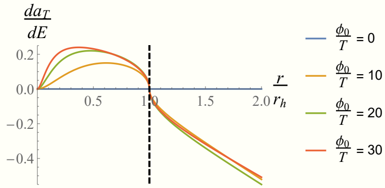
a_T satisfies

- We can prove the following
  - Stationary at horizon:

$$\left.\frac{da_T}{d\rho}\right|_{\text{hor}} = 0$$

  - Monotonicity condition:

$$\text{UV} \to \text{IR} : \frac{da_T}{d\rho} \geq 0,$$
$$\text{Trans-IR} : \frac{da_T}{d\kappa} \leq 0$$


$\frac{da_T}{dE}$
0.2
0.0
−0.2
−0.4
0.5
1.0
1.5
2.0
$\frac{r}{r_h}$
$\frac{\phi_0}{T}$ = 0
$\frac{\phi_0}{T}$ = 10
$\frac{\phi_0}{T}$ = 20
$\frac{\phi_0}{T}$ = 30

- Equations of motion with this ansatz:

$$\phi'' + \left(\frac{F'}{F} - \frac{d-1}{r} - \frac{\chi'}{2}\right)\phi' + \frac{\Delta(d-\Delta)}{r^2 F}\phi = 0 \quad (4.1)$$

$$\chi' - \frac{2F'}{F} - \frac{\Delta(d-\Delta)\phi^2}{(d-1)rF} - \frac{2d}{rF} + \frac{2d}{r} = 0 \quad (4.2)$$

$$\chi' - \frac{r}{d-1}(\phi')^2 = 0, \quad (4.3)$$

  - Solve numerically for F, χ, ϕ

$$\beta_t = \frac{1}{\sqrt{6}}(\bar{\beta}_t - \bar{\beta}_x - \sqrt{3}\bar{\beta}_y) \tag{4.4}$$

$$\beta_x = \frac{1}{\sqrt{6}}(\bar{\beta}_t - \bar{\beta}_x + \sqrt{3}\bar{\beta}_y) \tag{4.5}$$

$$\beta_y = \frac{1}{\sqrt{6}}(\bar{\beta}_t + 2\bar{\beta}_x). \tag{4.6}$$

A couple of remarks

- Both a_T^C and a_T^E are monotonic but only a_T^E approaches the zero temperature a-function proposed by [Myers, Paulos, Sinha] for QTG

$$a_T^{\mathsf{E}}(\rho) = \left(\frac{h(\rho)}{A'(\rho)}\right)^{d-1} e^{-\mathcal{F}(h(\rho),h'(\rho),A'(\rho),A''(\rho))}$$

zero temperature case $\rightarrow h = 1$

$$a_T^{\mathsf{E}} \approx \left(\frac{1}{A'(\rho)}\right)^{d-1}\left(1 - \sum_{n=2}^{\infty} \frac{\tilde{\lambda}_n}{2n-2} A'^{2(n-1)}\right)$$