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Black hole interiors and arithmetic chaos

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June 17, 2025

Based on arXiv:2312.11622 with S. A. Hartnoll and J. E. Santos, and WIP with S. A. Hartnoll and M. Yang.

Introduction: BKL chaos near a spacelike singularity

Belinski, Khalatnikov, Lifshitz (and others) taught us that the generic dynamics of the metric components near a spacelike singularity is chaotic and displays an infinite sequence of oscillations as the overall spacelike volume shrinks to zero.

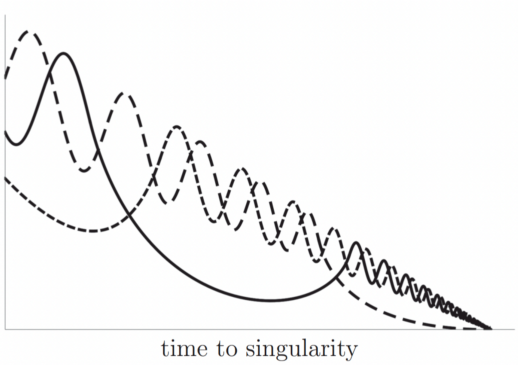

Figure: Belinski, Henneaux (2018)

Plan of the talk

- Review of BKL theory & cosmological billiards
- Mixmaster dynamics in the interior of an AdS black hole
- BKL dynamics and arithmetic chaos
- Number theory near a singularity

Brief review of BKL theory in vacuum

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1. **Ultra-local dynamics**: most of the time, the spatial Ricci tensor, and hence all spatial derivatives are subdominant compared to time derivatives
   - the evolution of nearby spatial points decouple $\Rightarrow$ PDEs become ODEs
   - the dynamics is simple and described by an anisotropic Kasner-like spacetime

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \eta_{\bar{\alpha}\bar{\beta}} l^{\bar{\alpha}}_{\alpha} l^{\bar{\beta}}_{\beta} \mathrm{d}x^{\alpha} \mathrm{d}x^{\beta}$$

   with $\eta_{\bar{\alpha}\bar{\beta}} = \mathrm{diag}(t^{2p_1(x)}, t^{2p_2(x)}, t^{2p_3(x)})$ such that $\sum_i p_i = 1$ and $\sum_i p_i^2 = 1$.

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2. **Oscillatory, chaotic dynamics** of metric components:
   - Kasner regime breaks down when some spatial derivative terms become important

Toy model for oscillatory solution: Bianchi IX or the "mixmaster" universe

The mixmaster universe is a homogeneous but anisotropic solution of the vacuum EE. The metric takes the form

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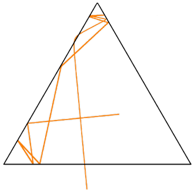
A Kasner epoch terminates whenever ${}^{(3)}R \approx 0$ is no longer a good approximation. This happens as a result of the growth of one of the metric components. The effect on the evolution is the break-down of the current Kasner epoch, and the start of a new and different Kasner evolution:

$$\{p_i^{(n)}\} \to \{p_i^{(n+1)}\}.$$

Reformulation of the chaotic dynamics as billiard motion

There is an equivalent formulation of this chaotic dynamics in terms of so-called *cosmological billiards*.

[Damour, Henneaux, Nicolai (2002)]

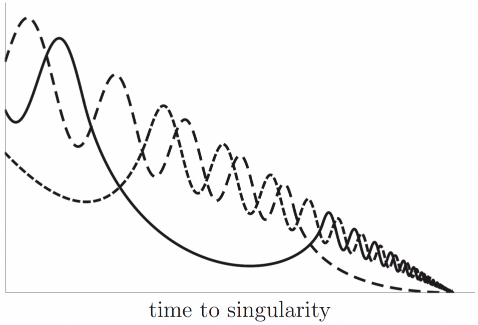


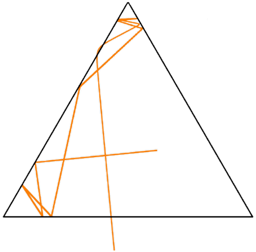
This picture is obtained in the Hamiltonian formulation of GR. In the case of the mixmaster model, the Hamiltonian splits into:

- a **kinetic** part, describing Kasner evolution, and
- **potential** terms (originating from ${}^{(3)}R$) that define exponential walls that eventually forces a break-down of the free Kasner motion, and leads to a different Kasner universe

Take-away message: the mixmaster chaotic dynamics is compactly formulated as motion on a 2d *hyperbolic equilateral triangle*.

BKL chaos near a spacelike singularity





Can one find realizations of this chaos inside black holes?

The interior of analytically known black hole geometries, such as the Schwarzschild solution, are fine-tuned. Indeed, one can work out that the near-singularity scaling of Schwarzschild metric components $\vec{p} = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

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Goal of our work

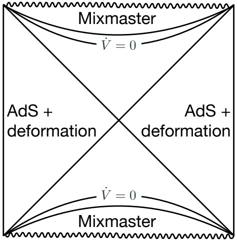
We provide a holographic realisation of the BKL chaos by constructing an AdS black hole with BKL-like near-singularity chaotic dynamics.

Holographic setup: mixmaster chaos in an AdS BH interior

Can the fine-tuned near-singularity behavior of the Schwarzschild black hole turn into a more generic case by **deforming the CFT** state dual to an AdS black hole by relevant operators?

$$S = S_{CFT} + \int \mathrm{d}^4 x \phi_0 \mathcal{O}$$

Within the framework of AdS/CFT, this sources the bulk geometry at asymptotic infinity and results in a different (exterior and interior) bulk geometry.



Previous work: scalar field deformation

[Frenkel, Hartnoll, Kruthoff, Shi (2020)]

This fact was illustrated in 2020 by sourcing a scalar field at the boundary

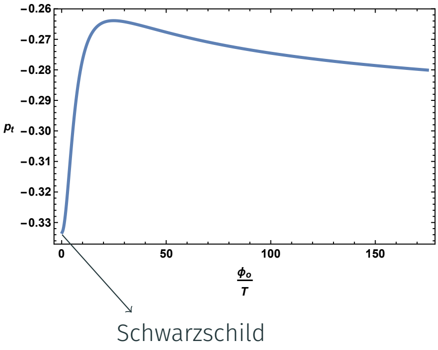
$$\mathcal{L} = R + 6 - \frac{1}{2}\left(g^{ab}\partial_a\phi\partial_b\phi - m^2\phi^2\right)$$

with the following ansatz for the metric

$$\mathrm{d}s^2 = \frac{1}{z^2}\left(-F(z)e^{-2H(z)}\mathrm{d}t^2 + \frac{\mathrm{d}z^2}{F(z)} + \mathrm{d}x^2 + \mathrm{d}y^2\right).$$

Near the singularity, they found a modification of the Schwarzschild Kasner scaling:

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \tau^{2p_t}\mathrm{d}t^2 + \tau^{2p_x}(\mathrm{d}x^2 + \mathrm{d}y^2)$$



Using vector fields to obtain mixmaster chaotic interior dynamics

Consider now the addition of 3 massive vector fields to the bulk action [MDC, Hartnoll, Santos (2023)]

$$S = \int \mathrm{d}x^4 \sqrt{-g}\left[R + 6 - \sum_{i=1}^{3}\left(\frac{1}{4}F_i^2 + \frac{\mu_i^2}{2}A_i^2\right)\right].$$

We use the following consistent ansatz for the fields

$$\mathrm{d}s^2 = \frac{1}{z^2}\left(-F(z)e^{-2H(z)}\mathrm{d}t^2 + \frac{\mathrm{d}z^2}{F(z)} + e^{-2G(z)}\mathrm{d}x^2 + e^{2G(z)}\mathrm{d}y^2\right),$$

$$A_1 = \phi_t(z)dt, \quad A_2 = \phi_x(z)dx, \quad A_3 = \phi_y(z)dy.$$

See also e.g. [Cai,Duan,Yang (2024)] for a different setup

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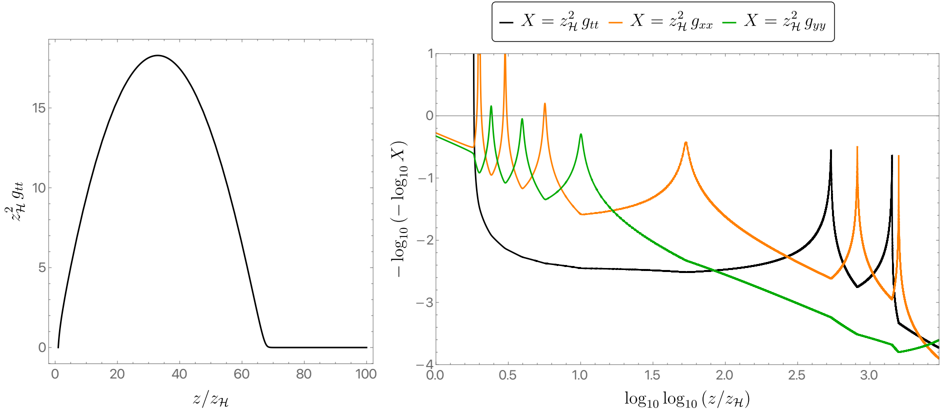
The masses are chosen so as to preserve the AdS asympotics, here $\mu_i^2 = -3/16$, leading to the boundary conditions

$$F \to 1, \quad G, H \to 0, \quad \phi_i \to \phi_i^{(0)} z^{1/4} + \phi_i^{(1)} z^{3/4}.$$

This *homogeneous* ansatz leads to ODEs that can be solved numerically.

Evolution of metric components across and past the horizon



Main result

This type of setup realizes the near-singularity oscillating dynamics **embedded** in a black hole with a regular horizon and exterior geometry.

Hamiltonian formalism & billiard picture

Billiard picture and the equilateral triangle

To verify that this interior reproduces the mixmaster geometry it is useful to analyze the *Hamiltonian constraint* parameterized in terms of scale factors

$$\mathrm{d}s^2 = e^{-\Omega} e^{-2h} \mathrm{d}t^2 - n^2 \mathrm{d}\rho^2 + e^{-\Omega} e^{h} \left(e^{-\sqrt{3}g} \mathrm{d}x^2 + e^{\sqrt{3}g} \mathrm{d}y^2 \right).$$

At late times, the masses and cosmological constant are irrelevant and the Hamiltonian constraint is given by

$$\mathcal{H} \equiv -\pi_\Omega^2 + \pi_g^2 + \pi_h^2 + 3e^{-\Omega} \left(e^{-2h} \pi_t^2 + e^{h-\sqrt{3}g} \pi_x^2 + e^{h+\sqrt{3}g} \pi_y^2 \right) = 0$$

with $\pi_A = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_A}$. Upon interpreting this as an operator equation on wavefunctions of the form $\Psi = \Psi(g, h, \Omega) e^{\frac{i}{\sqrt{3}} \sum_i f_i \phi_i}$, one finds the Wheeler-DeWitt equation:

$$\left[\frac{\partial^2}{\partial \Omega^2} - \frac{\partial^2}{\partial g^2} - \frac{\partial^2}{\partial h^2} + e^{-\Omega} \left(e^{-2h} f_t^2 + e^{h-\sqrt{3}g} f_x^2 + e^{h+\sqrt{3}g} f_y^2 \right) \right] \Psi(g, h, \Omega) = 0.$$

Billiard picture and the equilateral triangle

In order to interpret the dynamics in terms of a billiard motion, one must take the "BKL limit". For this, let us change variables and consider the Chitre-Misner coordinates:

$$\Omega = e^{\tau}\cosh R\,, \qquad g = e^{\tau}\sinh R\cos\phi\,, \qquad h = e^{\tau}\sinh R\sin\phi,$$

so that the WDW equation becomes

$$\left[e^{-2\tau}\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial}{\partial\tau}-\nabla^2_{H^2}\right)+f_t^2 e^{-e^{\tau}W_t(R,\phi)}+f_x^2 e^{-e^{\tau}W_x(R,\phi)}+f_y^2 e^{-e^{\tau}W_y(R,\phi)}\right]\Psi(\tau,R,\phi)=0\,,$$

with $\nabla^2_{H^2}$ the hyperbolic 2d laplacian. At late times, i.e. small volume $e^{-3/2\Omega}\to 0$ or $\tau\to\infty$, the exponential terms are well-approximated by **infinite walls** that restrict the null motion to the domain $W_i(R,\phi)>0$ on hyperbolic space.

Note: in the BKL limit $\tau\to\infty$, π_τ is an emergent conserved Hamiltonian!

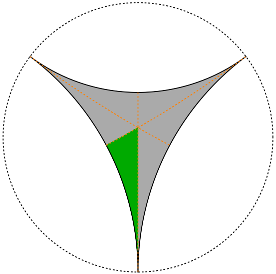
Billiard domain: the equilateral triangle in hyperbolic space

One finds that the above walls precisely coincide with the mixmaster equilateral triangle on hyperbolic space.

The WDW equation in the BKL limit is solved by separation of variables $\Psi(\tau, R, \phi) = \Psi_n(R, \phi)e^{[-\frac{1}{2}\pm i\varepsilon_n]\tau}$ where

$$-\nabla^2_{H^2}\Psi_n(R, \phi) = \left(\frac{1}{4} + \varepsilon_n^2\right)\Psi_n(R, \phi)\,,$$

with boundary condition $\Psi_n = 0$ outside of the triangle.

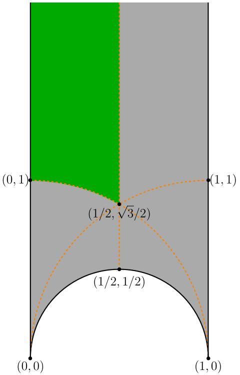


Poincaré disk

Arithmetic quantum chaos of the late-time dynamics

The equilateral billiard: an arithmetic triangle

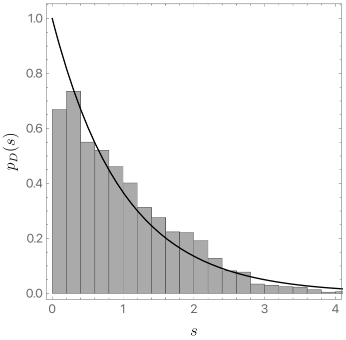
The equilateral triangle on hyperbolic space has an enhanced symmetry structure. When mapped to the upper half plane, it is half of the fundamental domain of a subgroup of the modular group $SL(2,\mathbb{Z})$, called $\Gamma(2)$. This leads to unusual characteristics (see [Bogomolny, Georgeot, Giannoni, Schmit (1997)] for a review).



Upper half plane

Anomalous level spacing statistics for arithmetic chaos

- Nearest-neighbor correlations in the spectrum have a universal character and reveal the integrable features of the model
- Level spacing statistics for integrable models is Poissonian while chaotic models follow the Wigner-surmise (cf. RMT)
- However, **arithmetic chaos is characterized by a Poisson statistics** due to the existence of an infinite tower of (Hecke) operators that commute with the Hamiltonian



Chaos in the wavefunctions and the Sato-Tate conjecture

On the upper half plane (x, y) with $y > 0$, the wavefunctions Ψ_k can be expanded in a Fourier basis

$$\Psi_k(x, y) = \sum_{m=1}^{\infty} c_m^k \sqrt{y} K_{i\varepsilon_k}(2m\pi y) \sin(2m\pi x)$$

The Hecke symmetries constrain the coefficients to satisfy the Hecke relations

$$c_{vp}^k = c_v^k c_p^k - c_{v/p}^k$$

with $v \in \mathbb{N}$ and p prime. Therefore, only the primed Fourier coefficients are independent.

The **Sato-Tate conjecture** asserts that the primed Fourier coefficients follow a Wigner semicircle distribution (can be verified numerically).

Arithmetic chaos: a signature of gravity?

- **Until now** we focused on an idealized, homogeneous set-up, leading to an equilateral triangular billiard.
- It turns out that **pure gravity in 4d** also gives rise to an arithmetic cosmological triangle: half of the fundamental domain of $SL(2,\mathbb{Z})$.
- Remarkably, when applying the BKL analysis to **11 dimensional supergravity** [Damour, Henneaux,Kleinschmidt,Koehn,Nicolai], the wavefunctions of the associated nine dimensional cosmological billiard display modular invariance involving **integers in the octonions**
- In fact, each modular group over some set of integers of a normed division algebra $(\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O})$ can be realized in supergravity theories

Number theory near the singularity

WDW wavefunctions and their L-function

Consider a WDW wavefunction at energy level ε_k

[Hartnoll, Yang (2025)]

$$\Psi_k(x,y) = \sum_{m=1}^{\infty} c_m^k \sqrt{y} K_{i\varepsilon_k}(2m\pi y) \sin(2m\pi x)$$

with coefficients satisfying the Hecke relations

$$\begin{aligned} c_{nm}^k &= c_m^k c_n^k \quad \text{for} \quad (n,m) = 1 \\ c_{p^{n+1}}^k &= c_p^k c_{p^n}^k - c_{p^{n-1}}^k \quad \forall p \in \mathbb{P}. \end{aligned}$$

From the Mellin transform of Ψ_k, one can extract an L-function constructed from these Fourier coefficients

$$L(s) = \sum_{n=1}^{\infty} \frac{c_n^k}{n^s}. \tag{1}$$

Intermezzo: Chaos in the Riemann ζ function

The most famous example of an L-function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Via the Euler product, one can interpret the Riemann ζ function as a thermal partition function at inverse temperature s [Julia (1990)]:

$$\begin{aligned} \zeta(s) &= \prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} \frac{1}{p^{ks}} = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}} \\ &= \prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} e^{-sn \log p} \equiv \operatorname{tr} e^{-sH} \qquad \text{with} \qquad H = \sum_{p} \log p \, a_p^{\dagger} a_p \end{aligned}$$

[Hartnoll, Yang (2025)]

In a similar way, the L-function associated to a WDW wavefunction can be written as

$$L(s) = \sum_{n=1}^{\infty} \frac{c_n^k}{n^s} = \prod_{p\in\mathbb{P}} \sum_{m=0}^{\infty} \frac{c_{p^m}^k}{p^{ms}} = \prod_{p\in\mathbb{P}} \frac{1}{1 - c_p^k p^{-s} + p^{-2s}} \tag{2}$$

Then, using the parameterization $c_p^k = 2\cos\theta_p^k$, one writes

$$L(s) = \prod_{p\in\mathbb{P}} \frac{1}{1 - e^{i\theta_p^k} p^{-s}} \frac{1}{1 - e^{-i\theta_p^k} p^{-s}} \equiv \mathrm{tr} e^{-sH + i\theta\cdot Q} \tag{3}$$

with

$$H = \sum_{p\in\mathbb{P}} \log p \left(a_p^\dagger a_p + b_p^\dagger b_p \right) , \qquad \theta\cdot Q = \sum_{p\in\mathbb{P}} \theta_p^k \left(a_p^\dagger a_p - b_p^\dagger b_p \right)$$

[MDC, Hartnoll, Yang (2506.xxxxx)]

Is there a similar dual perspective for higher dimensional theories?

We can work our way up to eleven dimensional supergravities (related to $\mathbb{O}$) and start with theories where billiards domains (in $\mathbb{H}_3$) are part of the fundamental domain of $SL(2, \mathcal{O})$ with $\mathcal{O}$ a set of integers in $\mathbb{C}$.

- Hecke operators still exist and lead to Hecke relations for the Fourier coefficients of the wavefunctions
- One obtains primon gases of oscillators labeled now by Gaussian or Eisenstein primes (depending on the theory)

Summary

- BKL have described the generic evolution of metric components near a space-like singularity in gravity (Schwarzschild metric is fine-tuned)
- The oscillatory mixmaster universe can be obtained in a BH interior by adding vector-like boundary sources, providing a holographic setting for BKL chaos
- Arithmetic chaos emerges in the billiard formulation of the mixmaster universe and pure gravity in 4d and leads to a dual perspective of the WDW states in terms of primon partition functions
- Arithmetic chaos also crucially appears in 11 dimensional supergravity and is related to octonions

Open questions

- Do inhomogeneities allow for microstates counting?
- What does the primon partition function look like in 11 dimensional supergravity?
- What can BKL dynamics and arithmetic chaos teach us about quantum and stringy effects near the singularity?
- Do higher curvature corrections modify the arithmetic nature of the near-singularity chaos? WIP with Hartnoll, Santos

Thank you!
Questions?