Novel Nonperturbative Renormalization Scheme
(based on Gradient Flow)

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Lattice Gauge Theory Contributions to New Physics Searches

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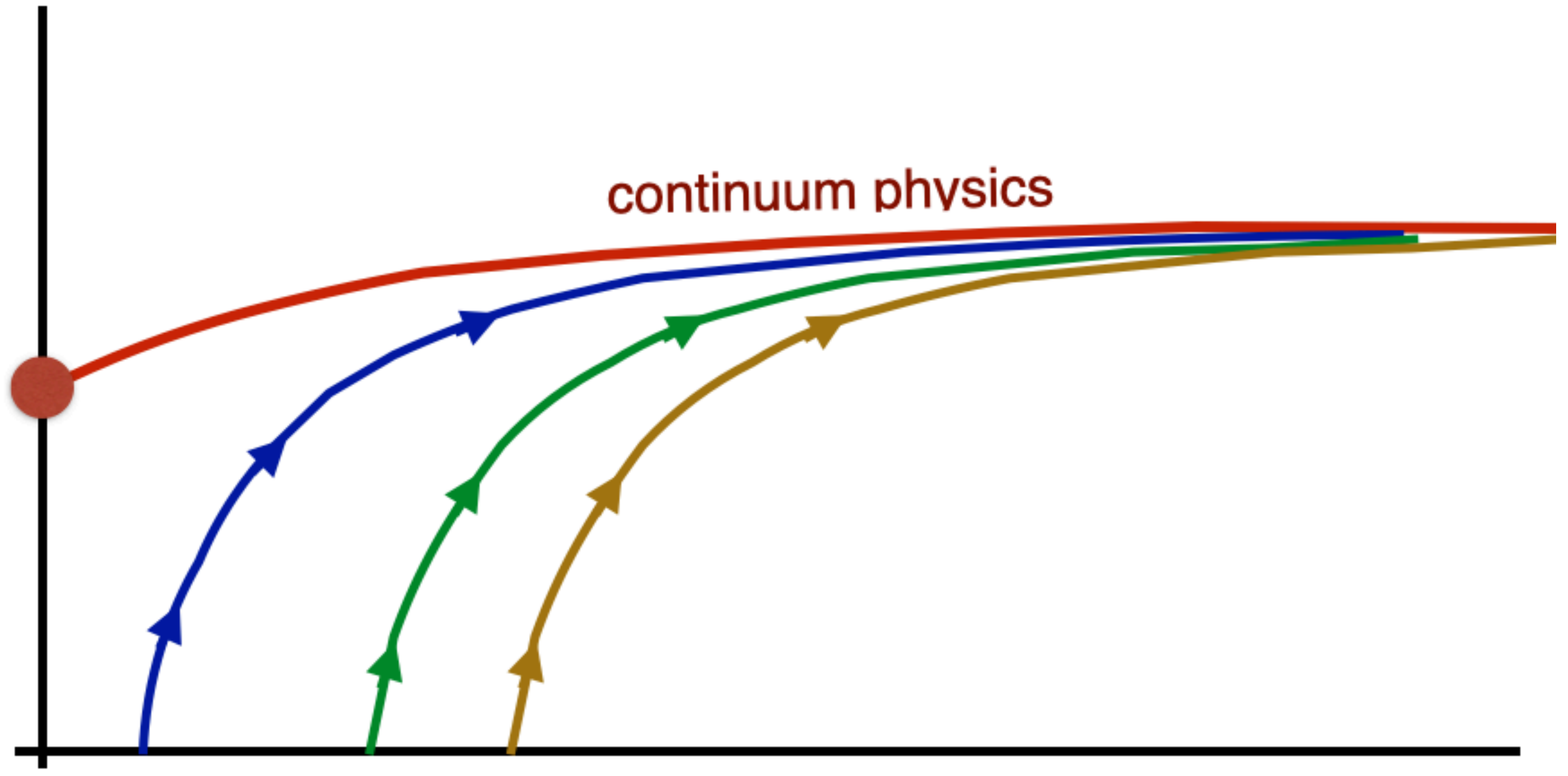

Gradient flow vs continuous RG transformations

A. Carosso, AH, E. Neil,
PRL 121,201601 (2018)

GF can be *interpreted* as continuous RG with $\mu \propto 1/\sqrt{8t}$
- in infinite volume
- for *local* operators

- $g_{GF}^2 = \mathcal{N} t^2 < E(t) > \quad \Longrightarrow \quad \beta_{GF}(a; g_{GF}^2) = -t \frac{d g_{GF}^2(a;t)}{dt}$

- $\mathcal{O} = \bar{\psi}(x)\Gamma\psi(x)$ or $G_{\mathcal{O}}(x_4, t) = \langle\, \mathcal{O}(\bar{p} = 0, x_4; \mathbf{t})\ \mathcal{O}(\bar{p} = 0, 0; \mathbf{t} = \mathbf{0})\,\rangle$,

$$\Longrightarrow \quad t \frac{d \log G_{\mathcal{O}}(t, x_4)}{d\, t} = \gamma_{\mathcal{O}}(t) + \eta_{\psi}(t)$$

New RG scheme for composite fermions

A.H.,C. Monahan, M. Rizik,
A. Shindler and O. Witzel
Lattice'21 (arXiv:2201:09740)

1. Simulations at bare coupling β_b - lattice spacing a
   define IR scale via GF : $\mu_{IR} = 1/\sqrt{8t_0}$, where $g^2_{GF}(t_0) = 0.3\mathcal{N} \approx 15.8...$

2. Define *matching factor* $Z^{GF}_{\mathcal{O}}(a;t)$ for operator *local bare operator* $\mathcal{O}(a)$
   traditionally $Z_{\mathcal{O}}(a;t_0)\mathcal{O}(a) = \mathcal{O}(a)^{\text{tree-level}}$; now

$$Z^{GF}_{\mathcal{O}}(a;t_0)\left(\frac{\mathcal{O}(a)}{\mathcal{O}(a;t_0)}\right) = \left(\frac{\mathcal{O}(a)}{\mathcal{O}(a;t_0)}\right)^{\text{tree-level}}$$

3. Connect IR to UV : $\lim_{a\to 0}\left[Z^{GF}_{\mathcal{O}}(a;t_0) \longrightarrow Z^{GF}_{\mathcal{O}}(a;\mu_{UV})\right]$

4. Match to $\overline{MS}$ in the UV : $c^{\overline{MS}\leftarrow GF}(\mu_{UV})$ (perturbative calculation)

Basic steps:

3) Run the energy scale (*fully nonperturbative*):

$$\frac{\bar{Z}_{\mathcal{O}}^{GF}(g_{UV}^2)}{\bar{Z}_{\mathcal{O}}^{GF}(g_{IR}^2)} = \exp\left\{ \int_{\mathrm{g_{IR}}}^{\mathrm{g_{UV}}} \mathrm{dg}' \frac{\gamma_{\mathcal{O}}(\mathrm{g}'^2)}{\beta(\mathrm{g}'^2)} \right\}$$

4) Connect to $\overline{MS}$: at tree level $c^{\overline{MS}\leftarrow GF}(\mu_{UV}) = \left(\frac{g_{GF}^2(\mu_{UV})}{g_{\overline{MS}}^2(\mu_{UV})} \right)^{-\gamma_{\mathcal{O}}^{(0)}/2b_0}$

At the end

$$\mathcal{O}_R^{\overline{MS}}(\mu_{UV}) = c^{\overline{MS}\leftarrow GF}(\mu_{UV}, \mu_{IR})\, Z_{\mathcal{O}}^{GF}(a; \mu_{IR})\mathcal{O}(a)$$

Basic steps:

3) Run the energy scale (*fully nonperturbative*):

$$\frac{\bar{Z}_{\mathcal{O}}^{GF}(g_{UV}^2)}{\bar{Z}_{\mathcal{O}}^{GF}(g_{IR}^2)} = \exp\left\{ \int_{g_{IR}}^{g_{UV}} dg' \frac{\gamma_{\mathcal{O}}(g'^2)}{\beta(g'^2)} \right\}$$

4) Connect to $\overline{MS}$: at tree level

$$c^{\overline{MS}\leftarrow GF}(\mu_{UV}) = \left(\frac{g_{GF}^2(\mu_{UV})}{g_{\overline{MS}}^2(\mu_{UV})}\right)^{-\gamma_{\mathcal{O}}^{(0)}/2b_0}$$

At the end

$$\mathcal{O}_R^{\overline{MS}}(\mu_{UV}) = \boxed{c^{\overline{MS}\leftarrow GF}(\mu_{UV}, \mu_{IR})} Z_{\mathcal{O}}^{GF}(a; \mu_{IR}) \mathcal{O}(a)$$

Basic steps:

3) Run the energy scale (*fully nonperturbative*):

$$\frac{\bar{Z}^{GF}_{\mathcal{O}}(g^2_{UV})}{\bar{Z}^{GF}_{\mathcal{O}}(g^2_{IR})} = \exp\left\{ \int_{g_{IR}}^{g_{UV}} dg' \frac{\gamma_{\mathcal{O}}(g'^2)}{\beta(g'^2)} \right\}$$

4) Connect to $\overline{MS}$: at tree level

$$c^{\overline{MS}\leftarrow GF}(\mu_{UV}) = \left(\frac{g^2_{GF}(\mu_{UV})}{g^2_{\overline{MS}}(\mu_{UV})} \right)^{-\gamma^{(0)}_{\mathcal{O}}/2b_0}$$

At the end

$$\mathcal{O}^{\overline{MS}}_R(\mu_{UV}) = \boxed{c^{\overline{MS}\leftarrow GF}(\mu_{UV}, \mu_{IR})}\boxed{Z^{GF}_{\mathcal{O}}(a; \mu_{IR})}\mathcal{O}(a)$$

$$Z^{GF}_{\mathcal{O}}(a; t_0) = \left(\frac{\mathcal{O}(a)}{\mathcal{O}(a; t_0)} \right)^{-1}$$

Basic steps:

3) Run the energy scale (*fully nonperturbative*):

$$\frac{\bar{Z}_{\mathcal{O}}^{GF}(g_{UV}^2)}{\bar{Z}_{\mathcal{O}}^{GF}(g_{IR}^2)} = \exp\left\{ \int_{g_{IR}}^{g_{UV}} dg' \frac{\gamma_{\mathcal{O}}(g'^2)}{\beta(g'^2)} \right\}$$

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At the end

$$\mathcal{O}_R^{\overline{MS}}(\mu_{UV}) = c^{\overline{MS}\leftarrow GF}(\mu_{UV}, \mu_{IR}) Z_{\mathcal{O}}^{GF}(a; \mu_{IR}) \mathcal{O}(a)$$

$$Z_{\mathcal{O}}^{GF}(a; t_0) = \left(\frac{\mathcal{O}(a)}{\mathcal{O}(a; t_0)}\right)^{-1}$$

The new scheme is
- nonperturbative
- gauge invariant
- requires only the calculation of flowed correlators

Numerical details

A.H.,C. Monahan, M. Rizik,
A. Shindler and O. Witzel
Lattice'21 (arXiv:2201:09740)

Pilot study:

- $N_f = 2$ Moebius domain wall fermions (stout smeared, Symanzik gauge)
- $24^3 \times 64$, $32^3 \times 64$ volumes
- $am_f = 0$ in the weak coupling, $am_f = 0.005,\ 0.010$ in confined regime
- Wilson fermion flow

- Determine $\beta(g^2)$ and $\gamma_{\mathcal{O}}(g^2)$ up to $g^2 \gtrsim 16$

RG β function

AH, O. Witzel, *Phys.Rev.D* 101 (2020) 3

Fodor et al,EPJWeb Conf. 175, 08027 (2018)

We use the continuous β function (CBF) method:

- GF renormalized coupling: $g^2_{GF}(t) = \mathcal{N} t^2 \langle E(t) \rangle$
- RG β function :

$$\beta(g_{GF}) = -t \frac{dg^2_{GF}}{dt}$$

- Infinite volume limit : $(a/L)^4 \to 0$ while $\sqrt{8t} \ll L$
- $am_f = 0$ chiral limit : $am_f \to 0$ (only in confining regime)
- Continuum limit : $t/a^2 \to \infty$ while keeping g^2_{GF} (or t) fixed

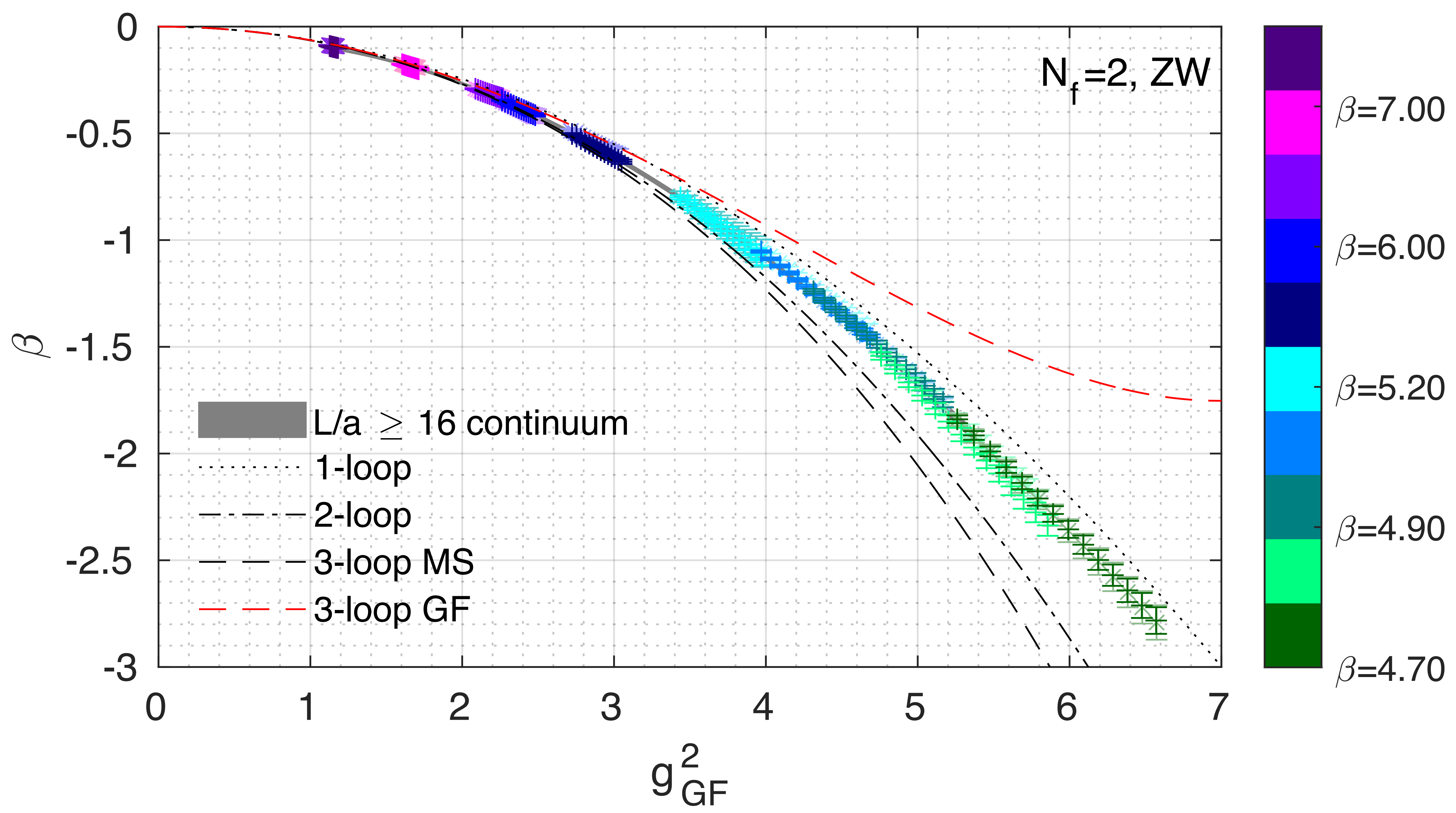
Details for $N_f = 0$ in

AH, C.Peterson, O.Witzel, J.VanSickle 2301.08274

The continuous β function (CBF) $N_f = 2$

Prior results : up to $g^2_{GF} \approx 6.0$ shows minimal cutoff effects

AH, O. Witzel, *Phys.Rev.D* 101 (2020) 3



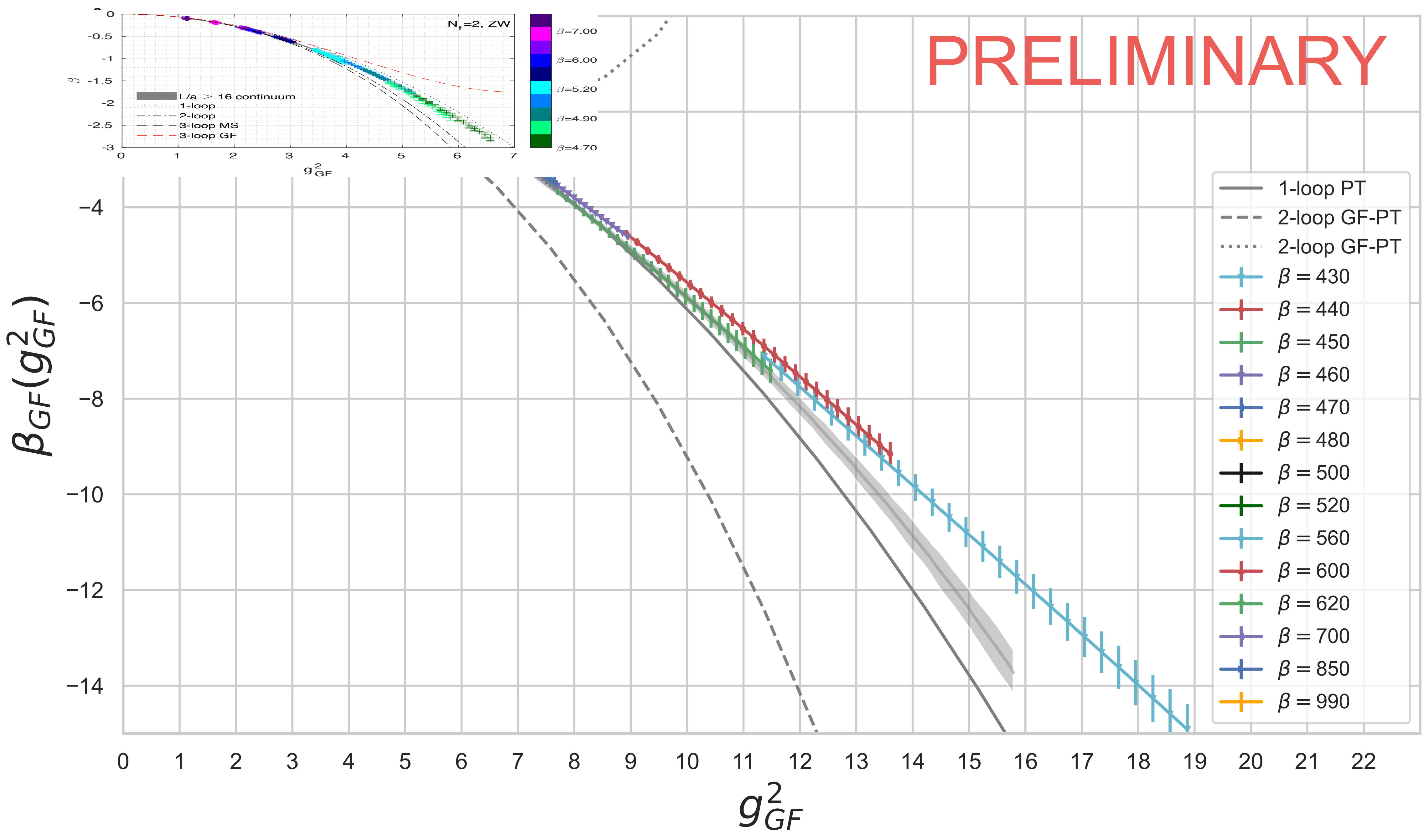
Colored points: raw data
Continuum limit: gray band

The continuous β function (CBF) $N_f = 2$

Prior results : up to $g^2_{GF} \approx 6.0$ shows minimal cutoff effects

AH, O. Witzel, *Phys.Rev.D* 101 (2020) 3

New results : extend existing data to the chirally broken regime, up to $g^2_{GF} \leq g^2_{GF}(t_0) \approx 15.9$

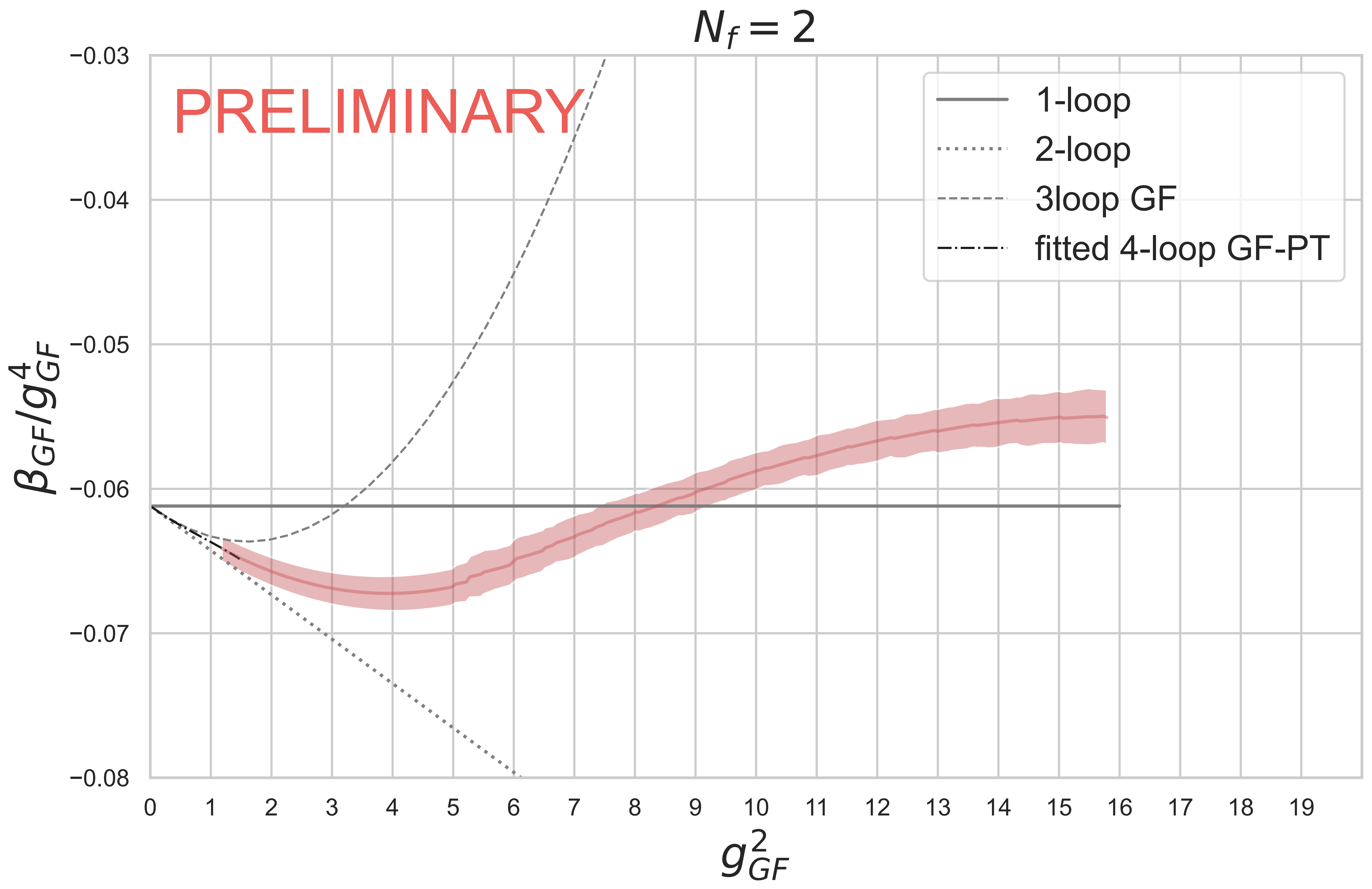


Colored points: raw data
Continuum limit: gray band

Λ_{GF} parameter, $N_f = 2$

Calculate the Λ parameter from the β function:

$$\Lambda_{GF}\sqrt{8t_0} = \left(b_0 g^2(t_0)\right)^{-\frac{b_1}{b_0^2}} \exp\left(-\frac{1}{b_0 g(t_0)^2}\right) \times \exp\left[-\int_0^{g^2(t_0)} \mathrm{d}x \left(\frac{1}{\beta_{GF}(x)} + \frac{1}{b_0 x^2} - \frac{b_1}{b_0^2 x}\right)\right], \quad g^2(t_0) \approx 15.9$$



Need β_{GF} precisely at small g^2_{GF}

smallest $g^2 \approx 1$

— find an effective "4-loop" β function

(a single fit coefficient)

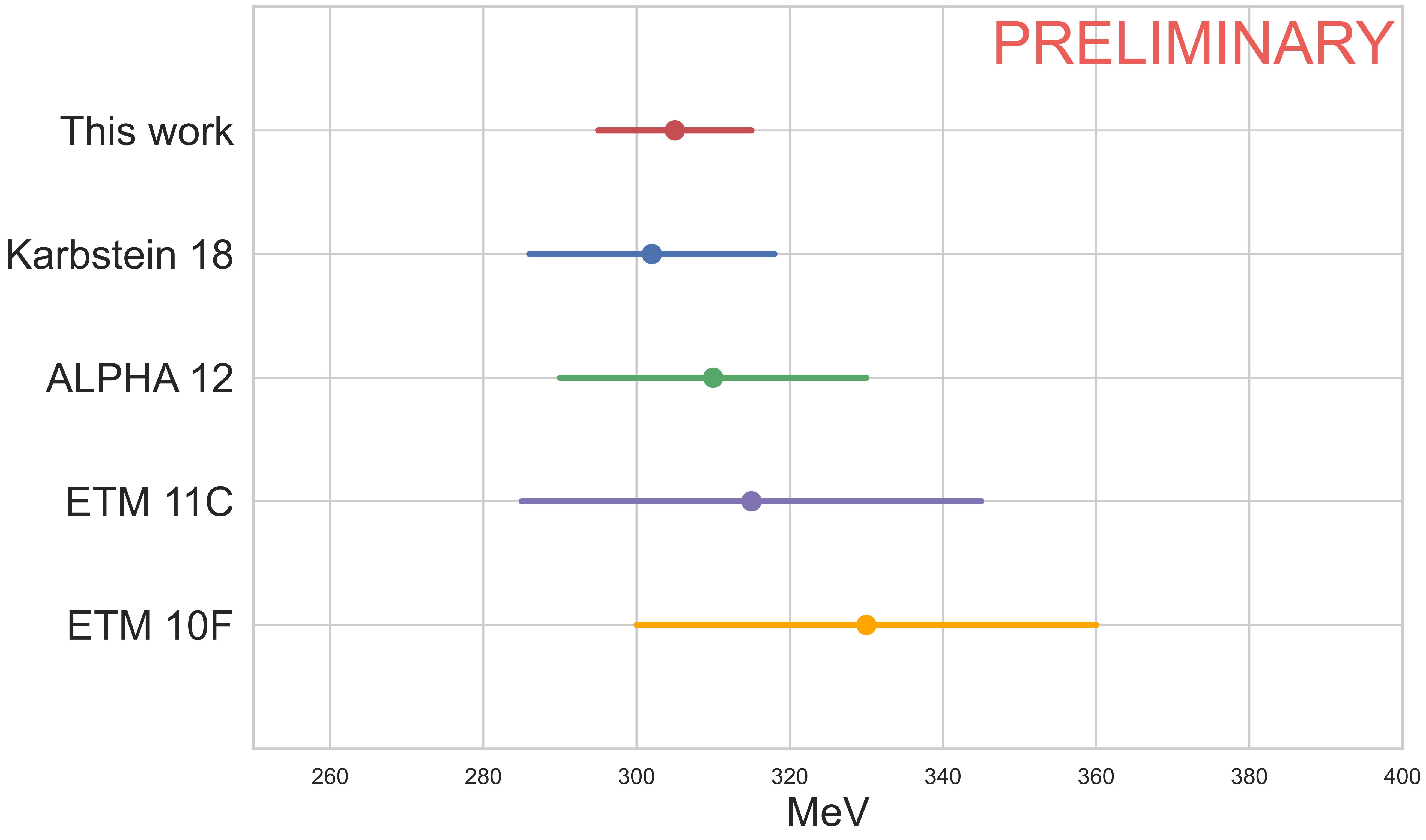
$\Lambda_{\overline{MS}}\sqrt{8t_0} = 0.675(20)$

$\Lambda_{\overline{MS}} = 305(10)$ MeV (statistical errors only)

Compare to FLAG 2021: $(N_f = 2)$

$\Lambda_{\overline{MS}} = 305(10)$ MeV (statistical errors only)

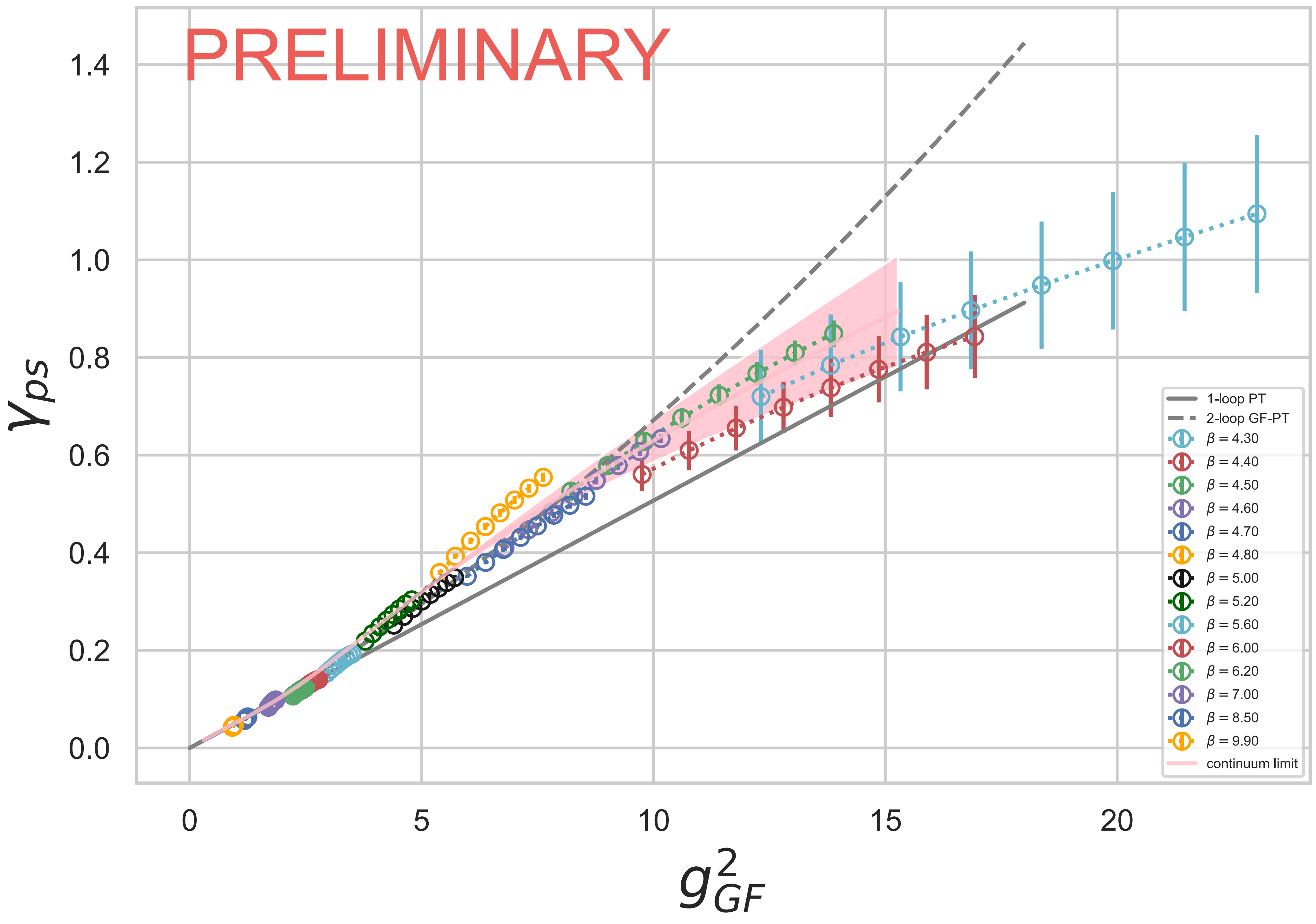
Consistent with FLAG 2021 values



Anomalous dimension and $Z_{\mathcal{O}}$

Correlator of local operator $\mathcal{O}$: $t\dfrac{d\log G_{\mathcal{O}}(t, x_4)}{dt} = d_{\mathcal{O}} + \gamma_{\mathcal{O}}(t) + \eta_{\psi}(t)$;

Vector correlator : $t\dfrac{d\log G_{\mathcal{V}}(t, x_4)}{dt} = d_{\mathcal{O}} + \eta_{\psi}(t)$



colored points show raw data:
at fixed bare coupling, changing flow time

Combine $\gamma_{\mathcal{O}}(a;t)$ and $g^2_{GF}(a;t)$ to predict the running anomalous dimension $\gamma_{\mathcal{O}}(g^2_{GF})$

Continuum limit:

- $a^2/t \to 0$ at fixed g^2_{GF} (pink band)

Anomalous dimension closely tracks the 1-loop line

- need more data at strong coupling to cover $g^2_{GF} \leq g^2_{GF}(t_0)$

So far:

- CBF method allows calculation of RG β function even in the confining regime
- Preliminary prediction for $N_f = 2$

  $\Lambda_{\overline{MS}} = 305(10)$ (statistical error only)
- Calculation of meson $\gamma_{\mathcal{O}}(g^2)$ is similar

Put it together:

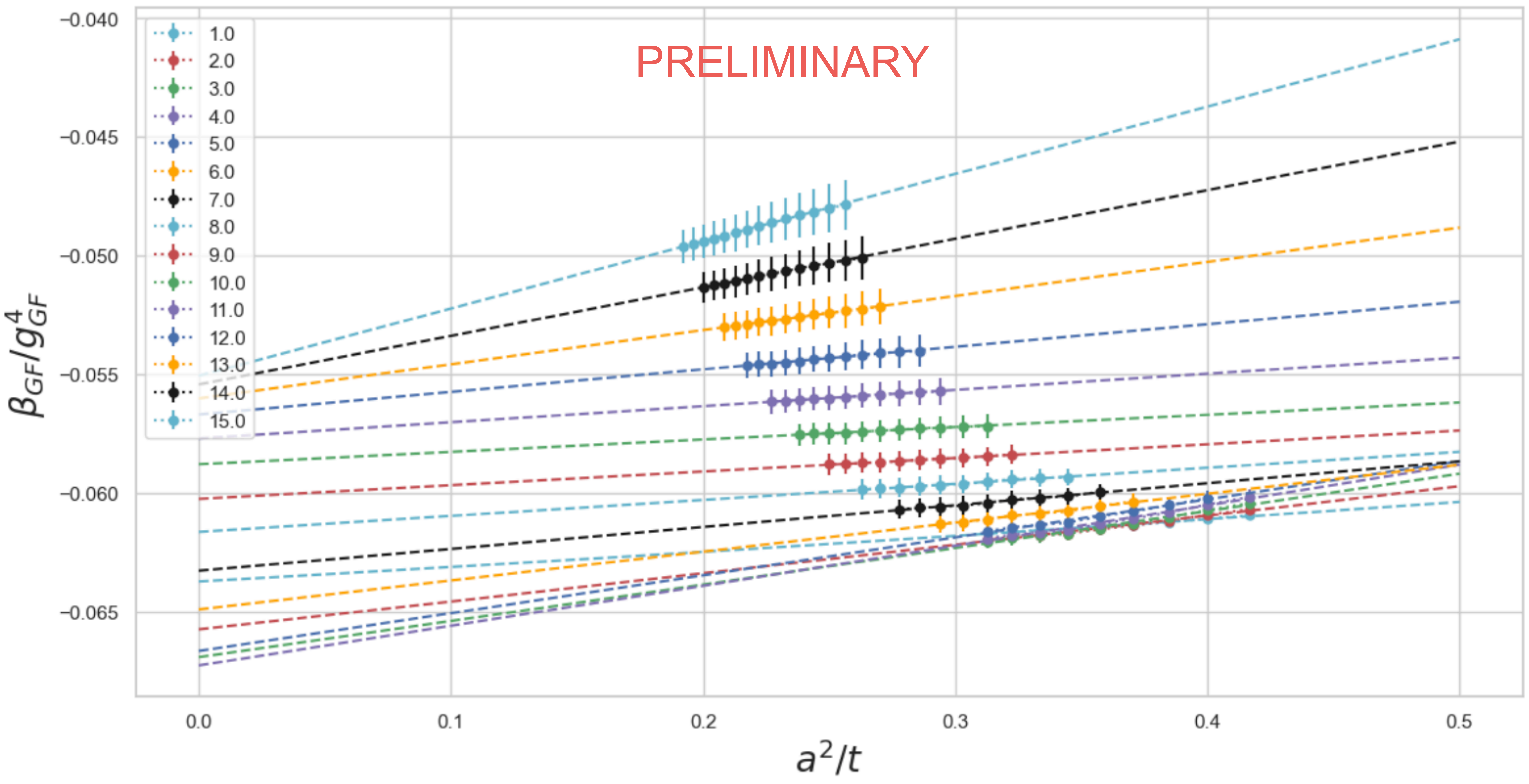
- Integrate $\int dg^2 \gamma(g^2)/\beta(g^2)$
- Connect to $\overline{MS}$: 1-loop is straightforward; 2-loop needs PT
- Calculate the IR matching factor (look up)

  - $N_f = 3$ and/or 4 is straightforward
  - calculation with *Wilson fermions* could be easier; would provide check and could even be combined with the DWF result

EXTRA SLIDES

β function , $N_f = 2$

Continuum extrapolation



Anomalous dimension and $Z_{\mathcal{O}}$

Often $\langle \mathcal{O}_\Gamma(t) \rangle = 0$; consider a GF two-point function

$$G_{\mathcal{O}}(x_4, t) = \int d^3x d^3x' \langle \mathcal{O}(\mathbf{x}, x_4; t) \mathcal{O}(\mathbf{x}', 0; t=0) \rangle,$$

Only one operator is flowed, so coarse graining is OK

If $\mathcal{O}$ is a scaling operator, an RG transformation with scale change $b \propto \sqrt{8t/a^2}$ predicts

$$G_{\mathcal{O}}(g_i, x_4) = b^{-\Delta_{\mathcal{O}}} G_{\mathcal{O}}(g_i^{(b)}, x_4/b), \qquad x_4 \gg b, \qquad (g_i^{(b)} \text{ are RG flowed couplings})$$

The scaling dimension is

- $\Delta_{\mathcal{O}} = d_{\mathcal{O}} + \gamma_{\mathcal{O}} + \eta$ for a linear RG (fermions)
  ($\eta/2$ is the anomalous dimension of the fermion)

A. Carosso, AH, E. Neil, PRL 121,201601 (2018)

Anomalous dimension and $Z_{\mathcal{O}}$

A. Carosso, AH, E. Neil,
PRL 121,201601 (2018)

The scaling dimension of $G_{\mathcal{O}}(x_4, t) = \int d^3x d^3x' \langle \mathcal{O}(\mathbf{x}, x_4; t) \mathcal{O}(\mathbf{x}', 0; t=0) \rangle$ is $\Delta_{\mathcal{O}} = d_{\mathcal{O}} + \gamma_{\mathcal{O}} + \eta$

In the ratio $\mathcal{R}_{\mathcal{O}}(x_4; t) = \dfrac{G_{\mathcal{O}}(x_4; t)}{G_V(x_4; t)}$ both Z_χ and $d_{\mathcal{O}}$ cancel

Use the double ratio $\overline{\mathcal{R}}_{\mathcal{O}}(a; x_4, t) = \dfrac{\mathcal{R}_{\mathcal{O}}(a; x_4, t=0)}{\mathcal{R}_{\mathcal{O}}(a; x_4, t)}$ to

define the GF scheme

$$\bar{Z}_{\mathcal{O}}^{GF}(a; t_0) \overline{\mathcal{R}}_{\mathcal{O}}(a; x_4, t_0) = \overline{\mathcal{R}}_{\mathcal{O}}^{\text{tree-level}}(a; x_4, t_0) \longrightarrow 1 \text{ as } x_4/\sqrt{8t_0} \to \infty$$

and

$$\gamma_{\mathcal{O}}(a; t) = \mu \frac{d \log \bar{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mathrm{a}; \mu)}{d\mu} = 2t \frac{d \log \mathcal{R}_{\mathcal{O}}(\mathrm{a}; \mathrm{x}_4, \mathrm{t})}{dt}, \qquad \mu = 1/\sqrt{8t}$$

All we need to calculate $\overline{\mathcal{R}}_{\mathcal{O}}(a; x_4, t)$ are the flowed correlators

Anomalous dimension and $Z_{\mathcal{O}}$

$\gamma_{\mathcal{O}}$ is the logarithmic derivative

$$\gamma_{\mathcal{O}}(a;t) = \mu \frac{d \log \bar{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mathrm{a};\mu)}{d\mu} = 2t \frac{d \log \mathscr{R}_{\mathcal{O}}(\mathrm{a};\mathrm{x}_4,\mathrm{t})}{dt} \quad ; \quad \mathscr{R}_{\mathcal{O}}(x_4;t) = \frac{G_{\mathcal{O}}(x_4;t)}{G_V(x_4;t)}$$

Typical correlator

$$G_{\mathcal{O}}(t) = A_1(t)e^{-m_1 x_4} + A_2(t)e^{-m_2 x_4} + \ldots$$

$$2t\frac{\mathrm{dlogG}_{\mathcal{O}}(\mathrm{t})}{dt} = \frac{\mathrm{dlogA}_1(\mathrm{t})}{dt} + \mathcal{O}\left(e^{-(m_2-m_1)x_4}\right)$$

- $\gamma_{\mathcal{O}}(t)$ is independent of x_4 if $x_4 \ll \sqrt{8t}$
- $\gamma_{\mathcal{O}}(t)$ corresponds to the lightest state; all others die out