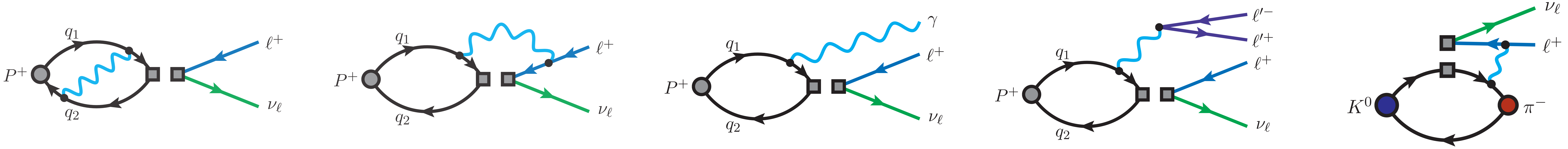


Radiative corrections to weak decays on the lattice

Matteo Di Carlo

14th June 2023

Centre for Theoretical Physics Higgs
THE UNIVERSITY of EDINBURGH

Lattice Gauge Theory Contributions to New Physics Searches

ift Instituto de Física Teórica UAM-CSIC

Outline of the talk

1. **Why** are radiative corrections **relevant for new physics searches?**

2. **How** are radiative corrections **included in lattice calculations?**

3. **What observables** have been computed?

1. Why

Indirect searches of new physics using CKM matrix unitarity constraints

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

in the Standard Model:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

1. Why

Indirect searches of new physics using CKM matrix unitarity constraints

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

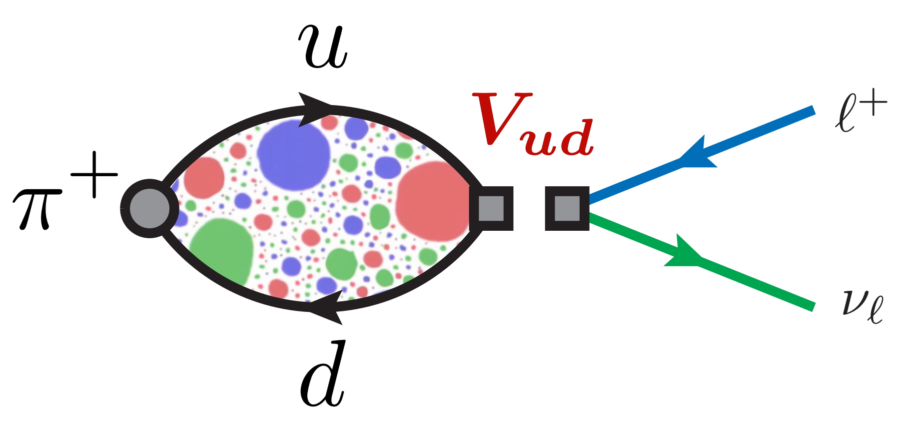
in the Standard Model:

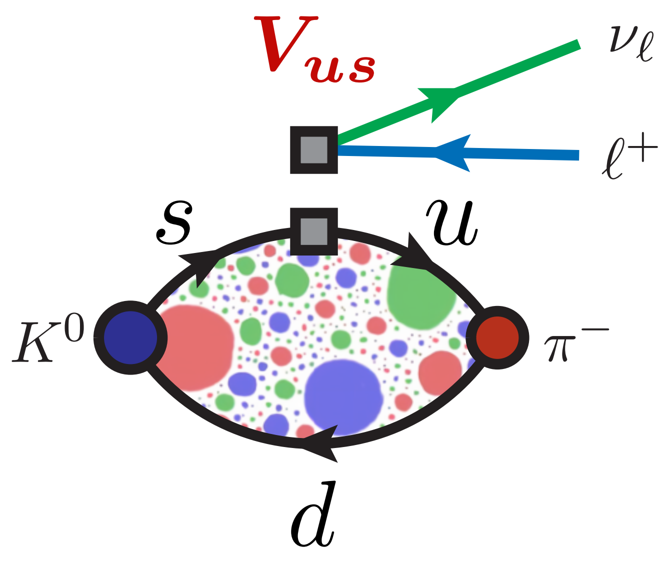
$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

Matrix elements can be extracted e.g. from leptonic and semileptonic decays of mesons

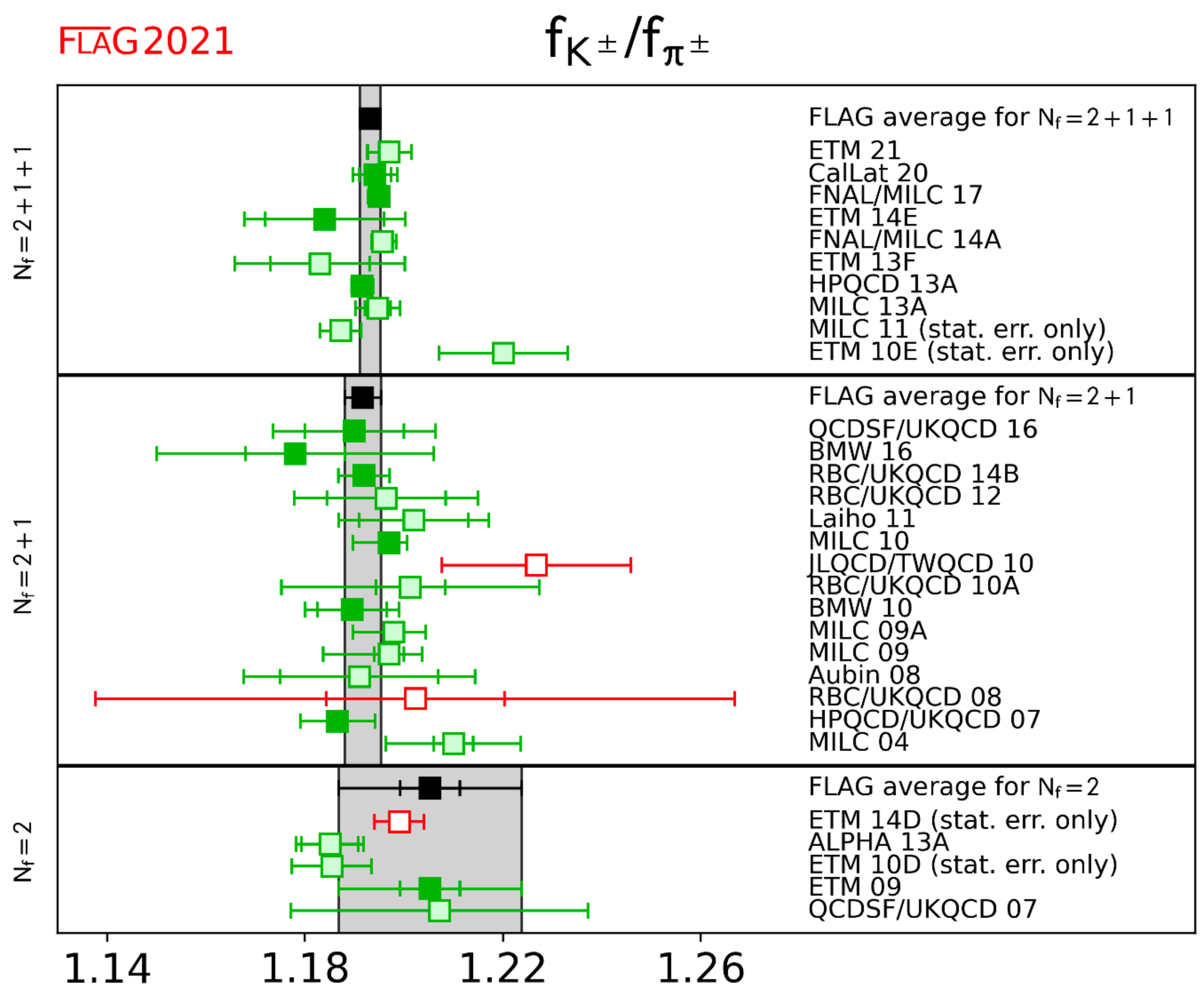
$$\underbrace{\frac{\Gamma[K \to \ell\nu_\ell(\gamma)]}{\Gamma[\pi \to \ell\nu_\ell(\gamma)]}}_{\text{experiments}} \propto \left|\frac{V_{us}}{V_{ud}}\right|^2 \underbrace{\left(\frac{f_K}{f_\pi}\right)^2}_{\text{QCD}}$$

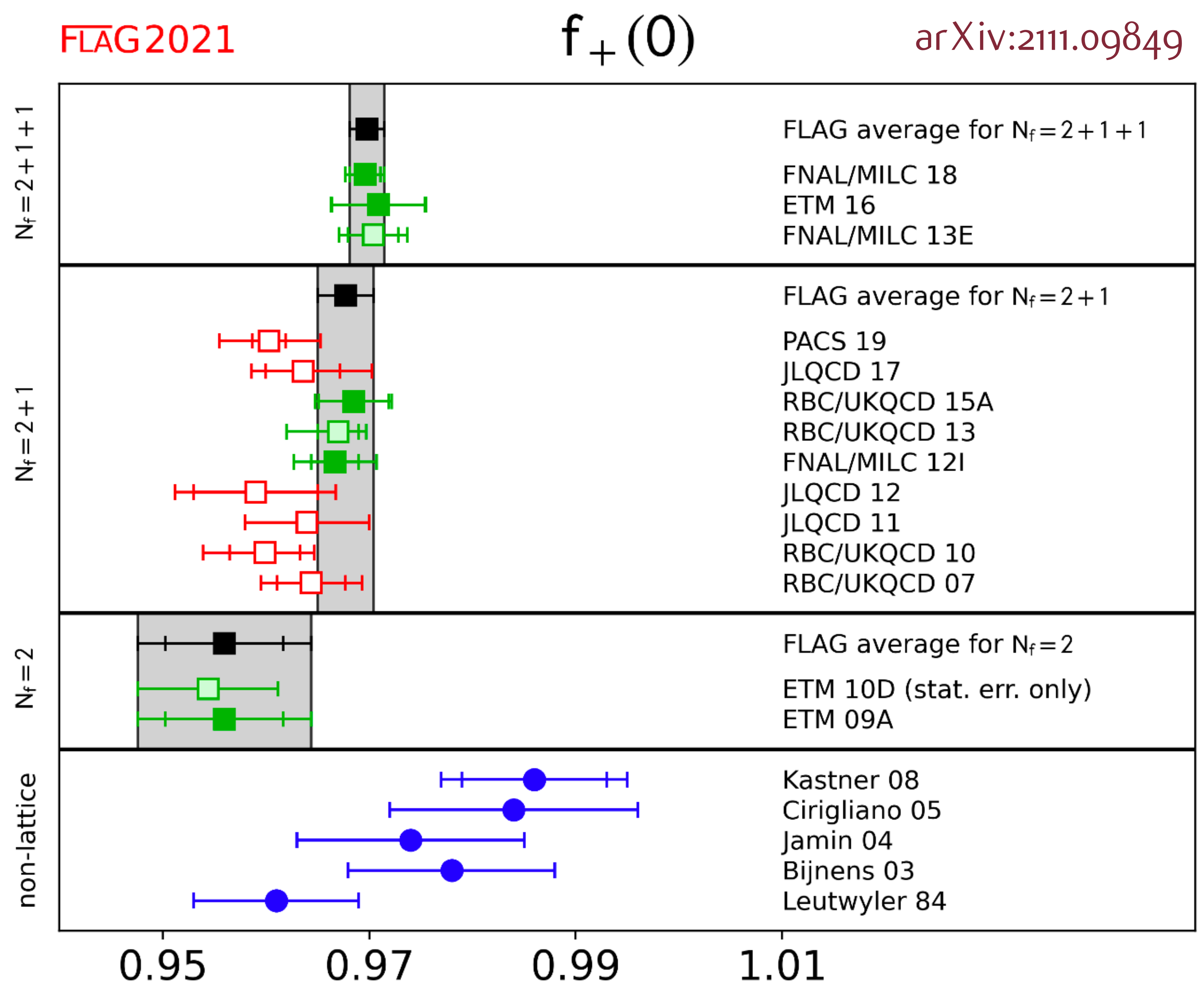
$$\underbrace{\Gamma[K \to \pi\ell\nu_\ell(\gamma)]}_{\text{experiments}} \propto |V_{us}|^2 \underbrace{|f_+^{K\pi}(0)|^2}_{\text{QCD}}$$





Leptonic and semi-leptonic decays from lattice QCD



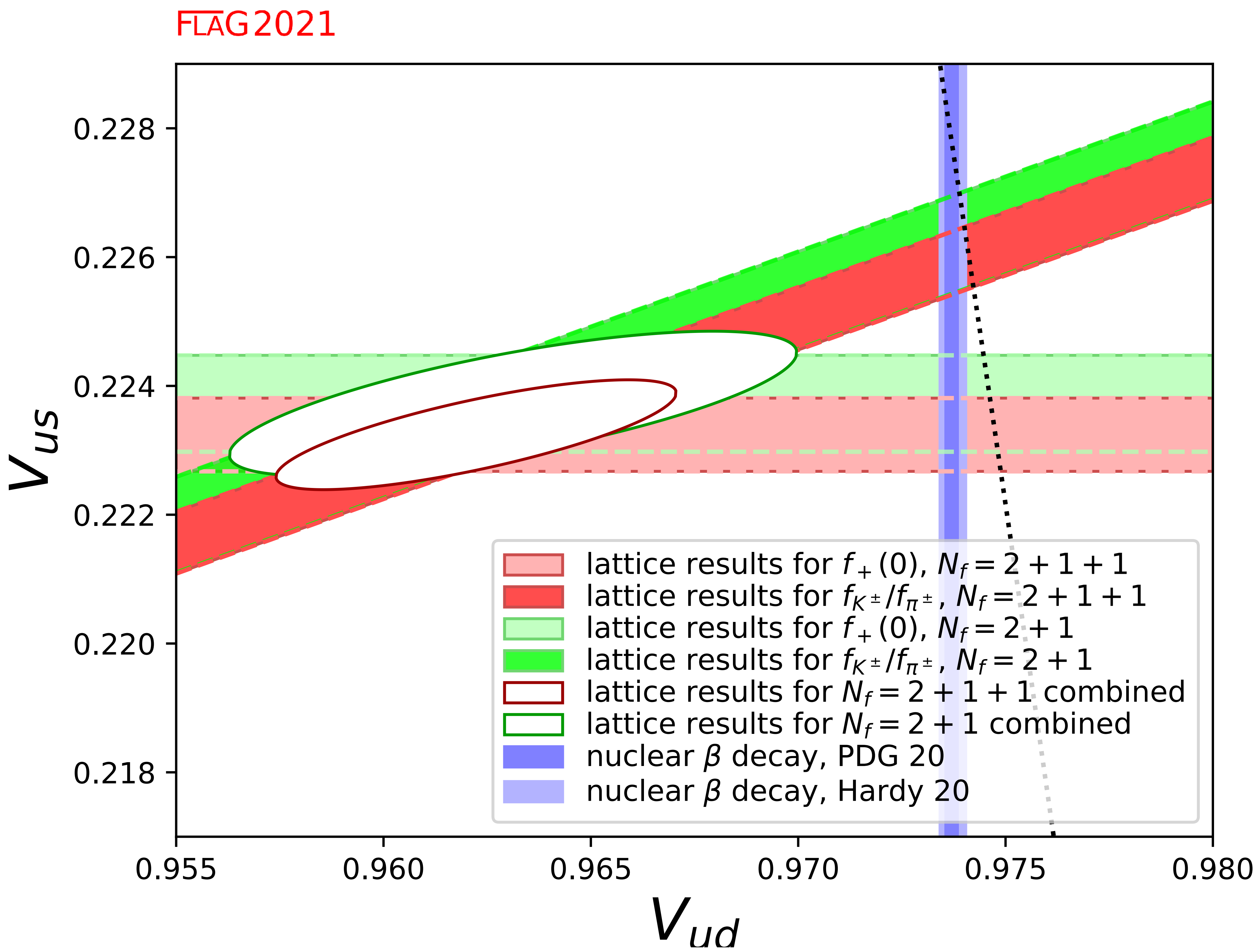


$$f_{K^\pm}/f_{\pi^\pm} = 1.1934\,(19) \qquad f_+^{K\pi}(0) = 0.9698\,(17)$$

FLAG 2021 Flavour Lattice Averaging Group

f_K/f_π and $f_+^{K\pi}(0)$ determined from lattice QCD with **sub percent precision!**

Tests of the Standard Model



Different **tensions** in the V_{us}-V_{ud} plane:

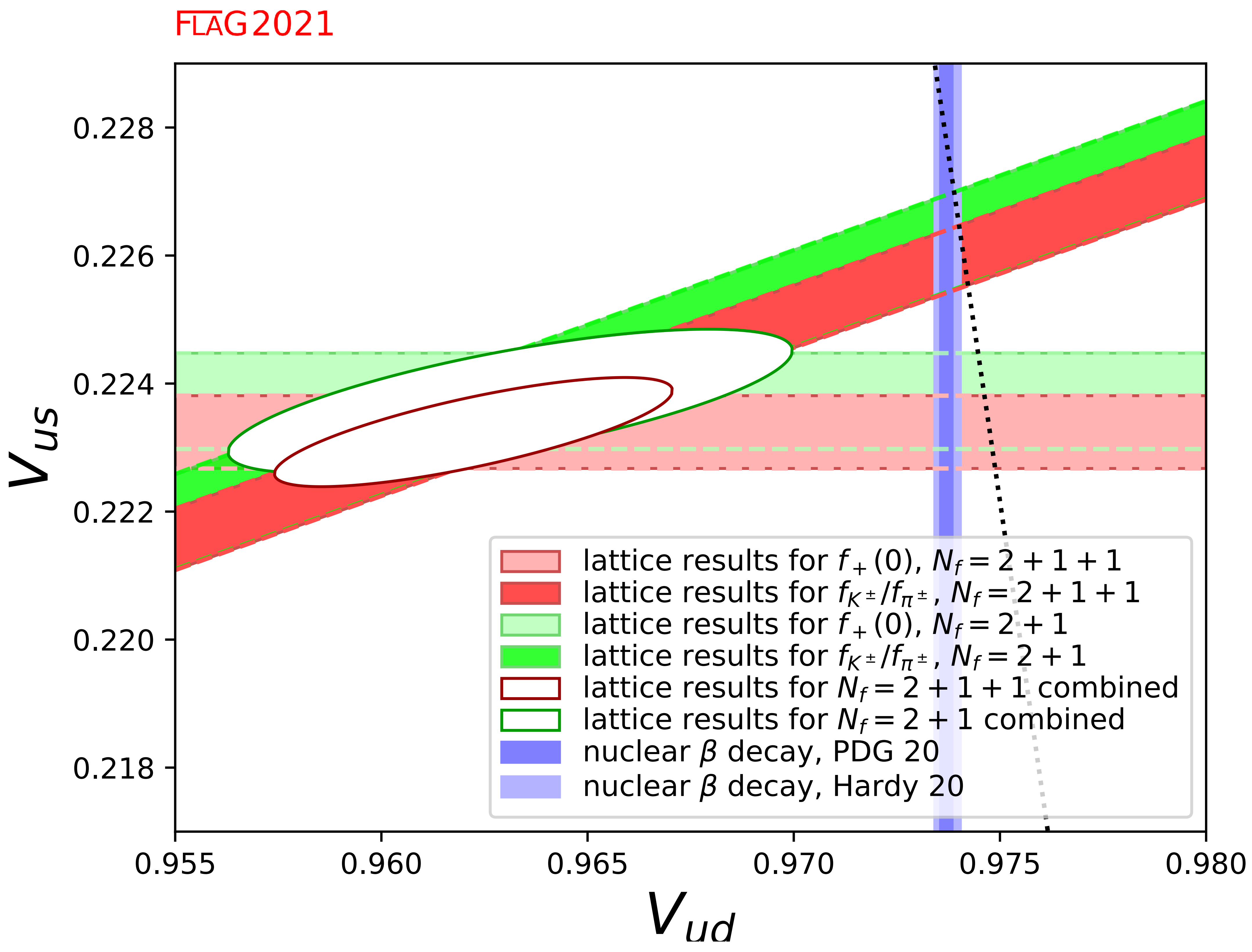
$$|V_u|^2 - 1 = 2.8\sigma$$

$$|V_u|^2 - 1 = 5.6\sigma \qquad |V_u|^2 - 1 = 3.3\sigma$$

$$|V_u|^2 - 1 = 3.1\sigma \qquad |V_u|^2 - 1 = 1.7\sigma$$

Experimental and theoretical control of these quantities is of crucial importance to solve the issue

Tests of the Standard Model



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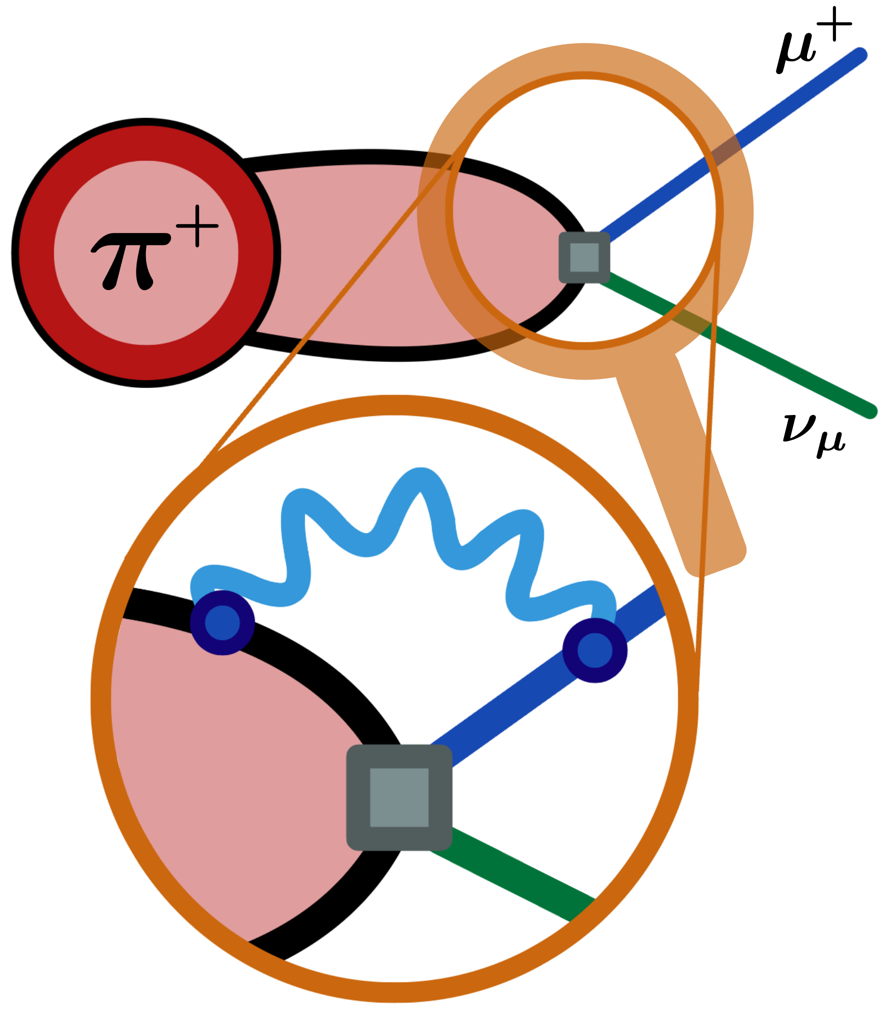
$$|V_u|^2 - 1 = 3.1\sigma \qquad |V_u|^2 - 1 = 1.7\sigma$$

Experimental and theoretical control of these quantities is of crucial importance to solve the issue

- **new measurements** (e.g. at NA62)
  (recent proposal in [V.Cirigliano et al., 2208.11707]: $K_{\mu 3}/K_{\mu 2}$)
- improve predictions of **radiative corrections and isospin-breaking effects**

QED and isospin-breaking effects

Current level of precision requires the inclusion of isospin breaking (IB) corrections



- strong effects $[m_u - m_d]_{\mathrm{QCD}} \neq 0$
- electromagnetic effects $\alpha \neq 0$

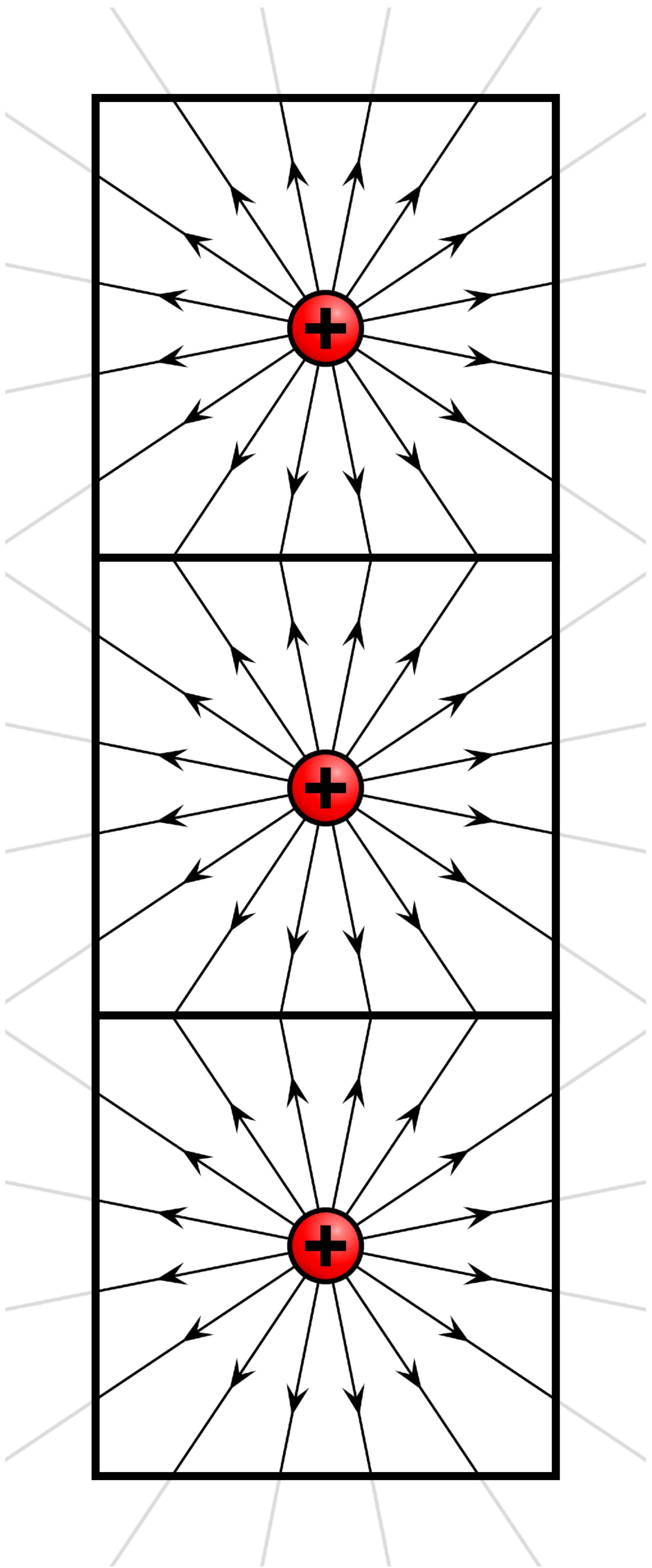
$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|\boldsymbol{V_{us}}|^2}{|\boldsymbol{V_{ud}}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi}) \qquad \Gamma(K \to \pi\ell\nu_\ell) \propto |\boldsymbol{V_{us}}|^2 \, |f_+^{K\pi}(0)|^2 \, (1 + \delta R^\ell_{K\pi})$$

- currently quoted corrections in the PDG come from χPT — V.Cirigliano & H.Neufeld, PLB 700 (2011)
- these are fully non-perturbative (structure dependent)
- first-principle lattice calculations are possible!

2. How

Computing **QED corrections on the lattice** is challenging:

- **long-range interactions** don't like **finite volumes** with **boundary conditions**
- **finite-volume effects** can be sizeable and power-like
- logarithmic **infrared divergences** arise in virtual/real decay rates
- QCD and QCD+QED are **different theories** which require separate renormalisation and scale-setting A.Portelli @Monday



Charged states in a finite box

Gauss law: only zero net charge is allowed in a finite volume with periodic boundary conditions

$$Q = \int \mathrm{d}^3\mathbf{x}\; j_0(t,\mathbf{x}) = \int\; \mathrm{d}^3\mathbf{x}\; \boldsymbol{\nabla}\cdot\boldsymbol{E}(t,\mathbf{x}) \stackrel{!}{=} 0$$

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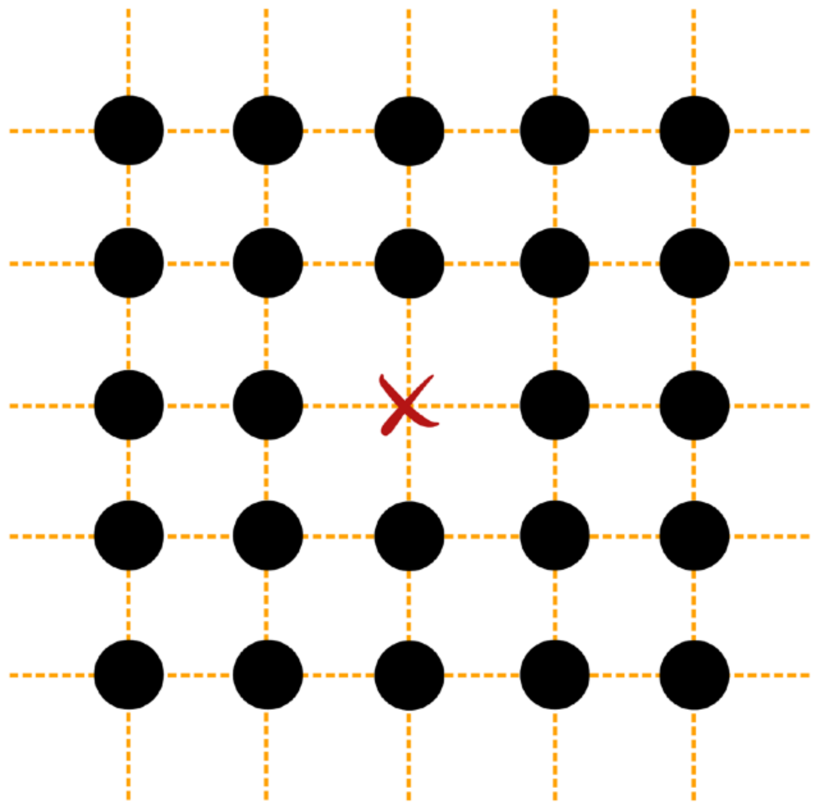
Possible solutions:

$\mathrm{QED_L}$

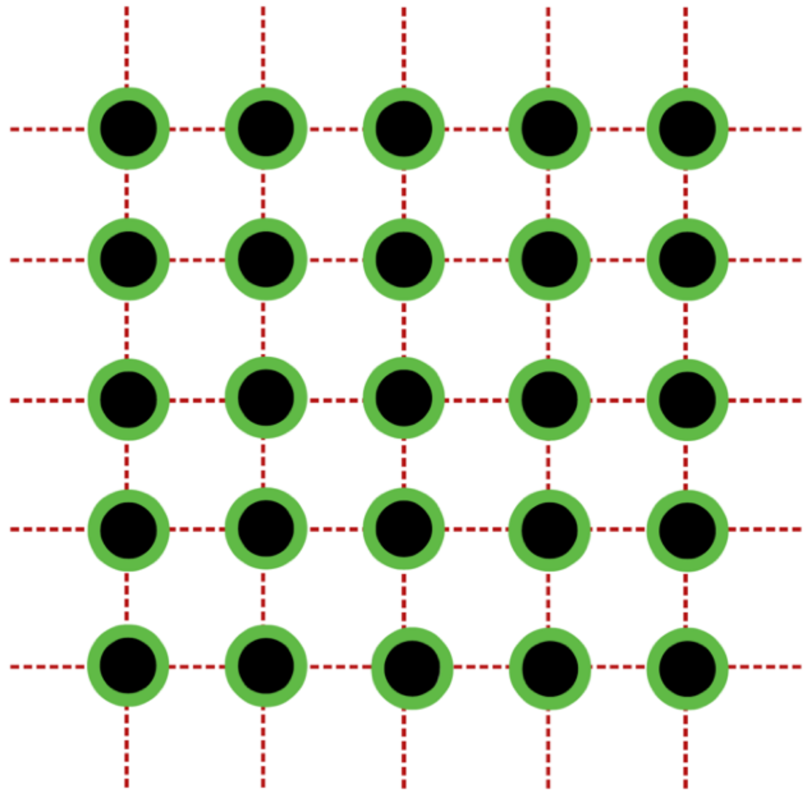


$\Omega_3 = 2\pi\mathbb{Z}^3/L$

remove spatial zero-mode of the photon field

M.Hayakawa & S.Uno, PTP 120 (2008)

$\mathrm{QED_m}$

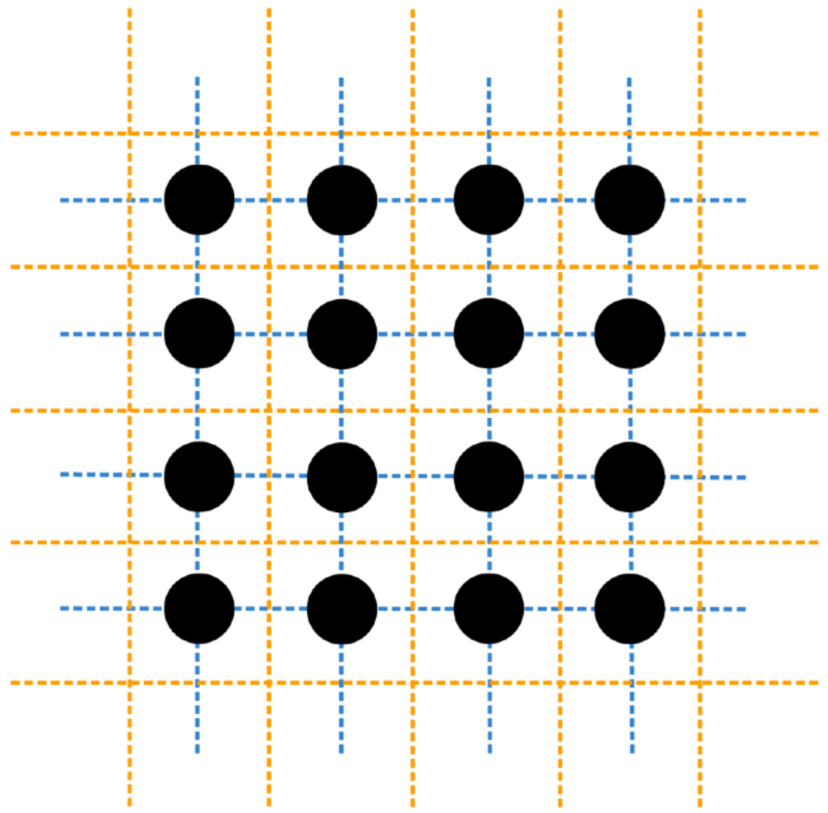


$\Omega_4 = 2\pi\{\mathbb{Z}^3/L, \mathbb{Z}/T\}$

use massive photon m_γ

M.G.Endres et al., [1507.08916]

$\mathrm{QED_{C^*}}$



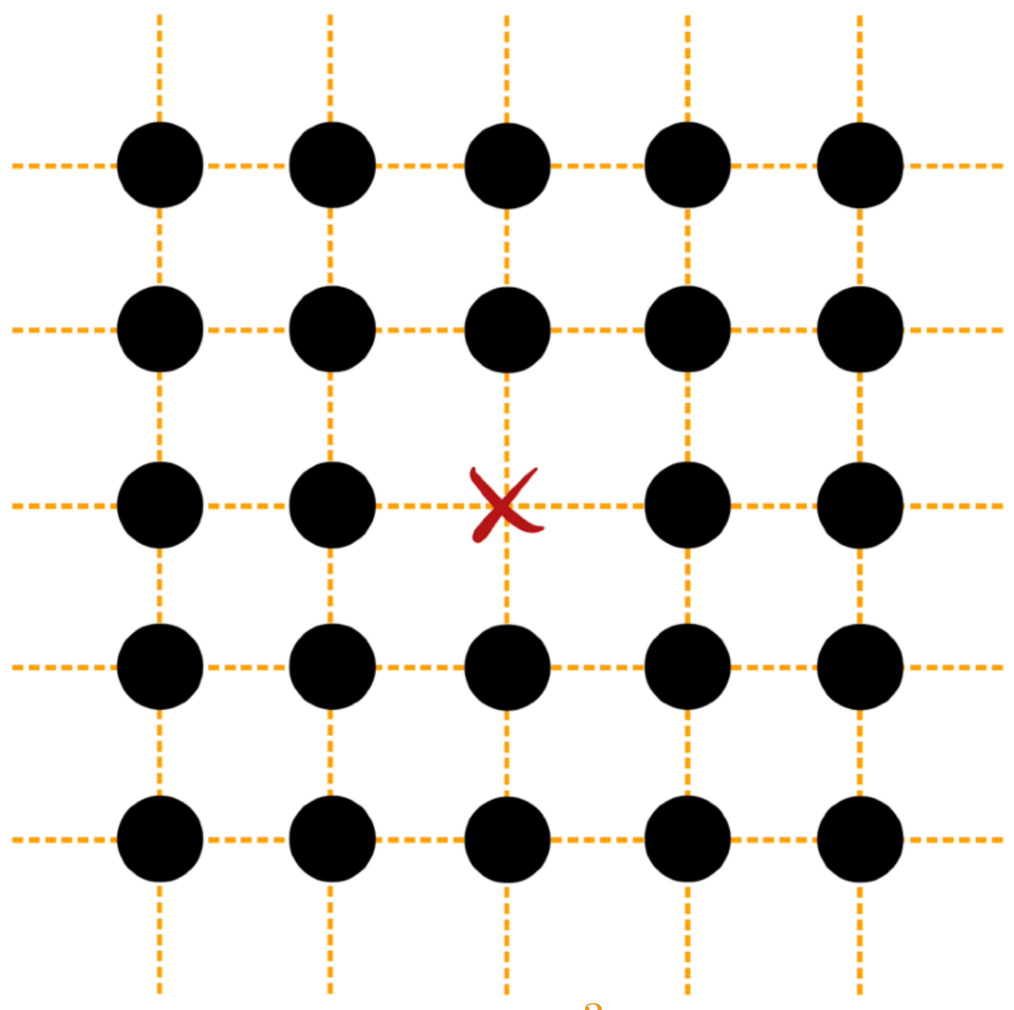
$\Omega_3 = 2\pi\mathbb{Z}^3/L$ $\quad \Omega_3' = (2\,\mathbb{Z}^3 + \bar{\mathbf{n}})\pi/L$

employ C* boundary conditions

A.S.Kronfeld & U.-J.Wiese, NPB 357 (1991)
B.Lucini et al., JHEP 1602 (2016)

Charged states in a finite box

QED$_L$



- **Spatial zero-mode** of the photon field is **removed** at each timeslice

$$\int \mathrm{d}^3\mathbf{x}\; A_\mu(t,\mathbf{x}) = 0 \quad \longrightarrow \quad \Delta^\gamma_{\mu\nu}(x) = \frac{1}{V}\sum_{k_0}\sum_{\mathbf{k}\neq 0}\Delta^\gamma_{\mu\nu}(k)\,\mathrm{e}^{\mathrm{i}k\cdot x}$$

(non-local)

- **Long-distance** translates into **power law finite-size effects**

$$\mathcal{O}(L) = \mathcal{O}(\infty) + \frac{\kappa_1}{L} + \frac{\kappa_2}{L^2} + \frac{\kappa_3}{L^3} + \dots \qquad \kappa_3 \propto \left(\sum_{\mathbf{n}\neq 0} - \int \mathrm{d}^3\mathbf{n}\right) = -1$$

- **Finite-size effects** well studied for **hadron masses** and **leptonic decays**

S.Borsanyi et al., Science 347 (2015)
Z.Davoudi & M.Savage, PRD 90 (2014)
V.Lubicz et al., PRD 95 (2017)
N.Tantalo et al., [1612.00199]
Z.Davoudi et al., PRD 99 (2019)
MDC et al., PRD 105 (2022)

Implementing QCD+QED on the lattice

- RM123 perturbative approach

G.M.de Divitiis et al. (RM123), PRD 87 (2013)

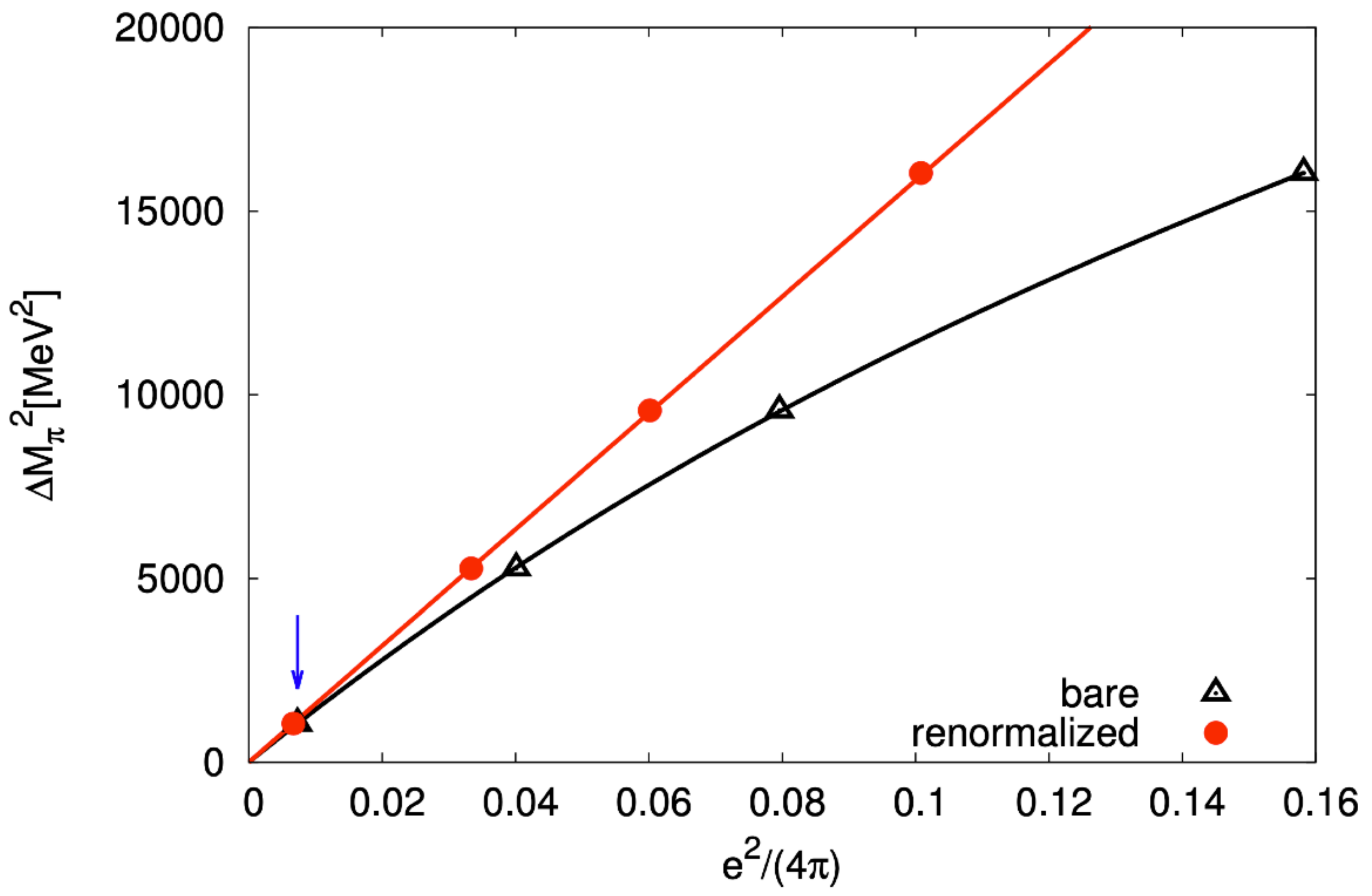
$$\langle \mathcal{O} \rangle = \int \mathcal{D}\Phi\, \mathcal{O}\, e^{-S_{\text{iso}} - \Delta S} = \langle \mathcal{O} \rangle_{\text{iso}} + \langle \Delta S\, \mathcal{O} \rangle_{\text{iso}} + \ldots$$

Pros: only evaluate QCD observables

Cons: need to compute many diagrams, also disconnected: 

- Full QCD+QED lattice simulations

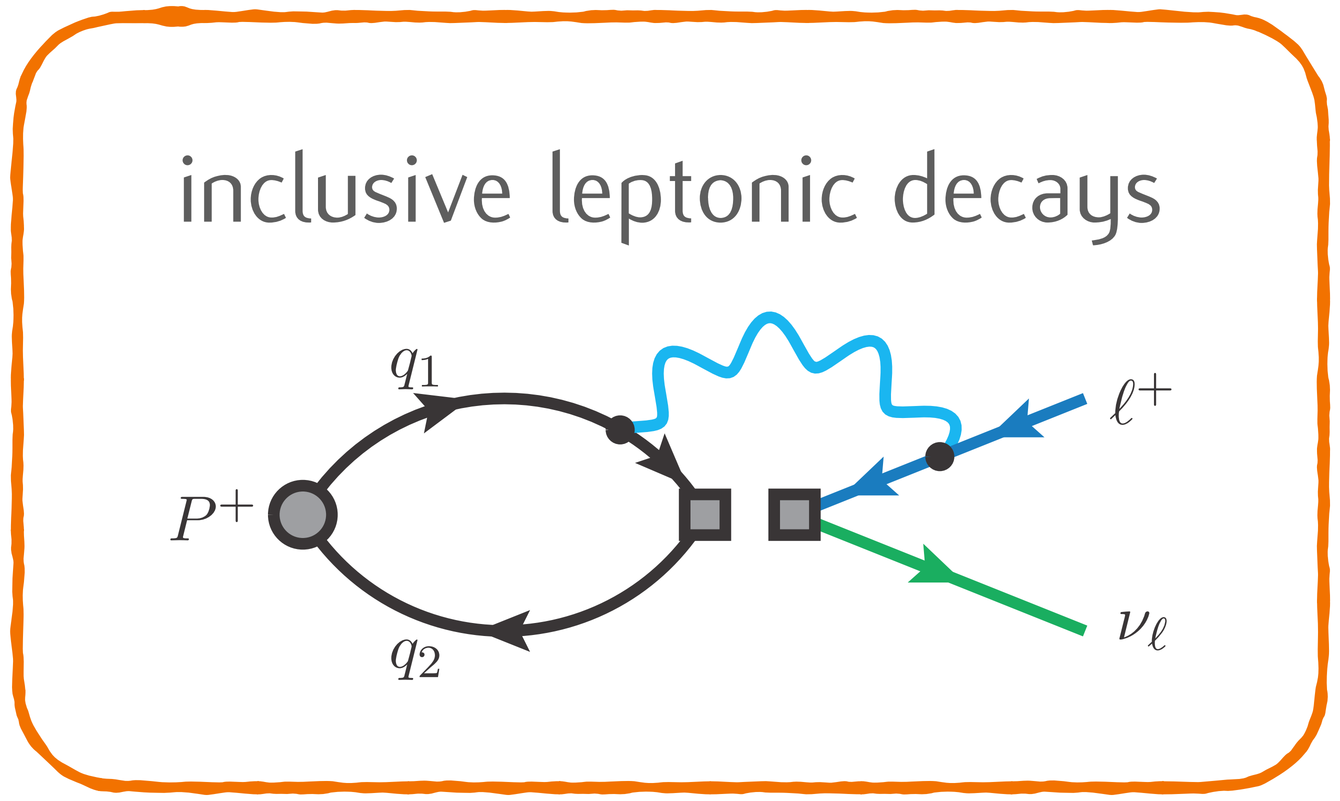
S.Borsanyi et al., Science 347 (2015)



Pros: simpler observables: 

Cons: need of dedicated gauge configurations

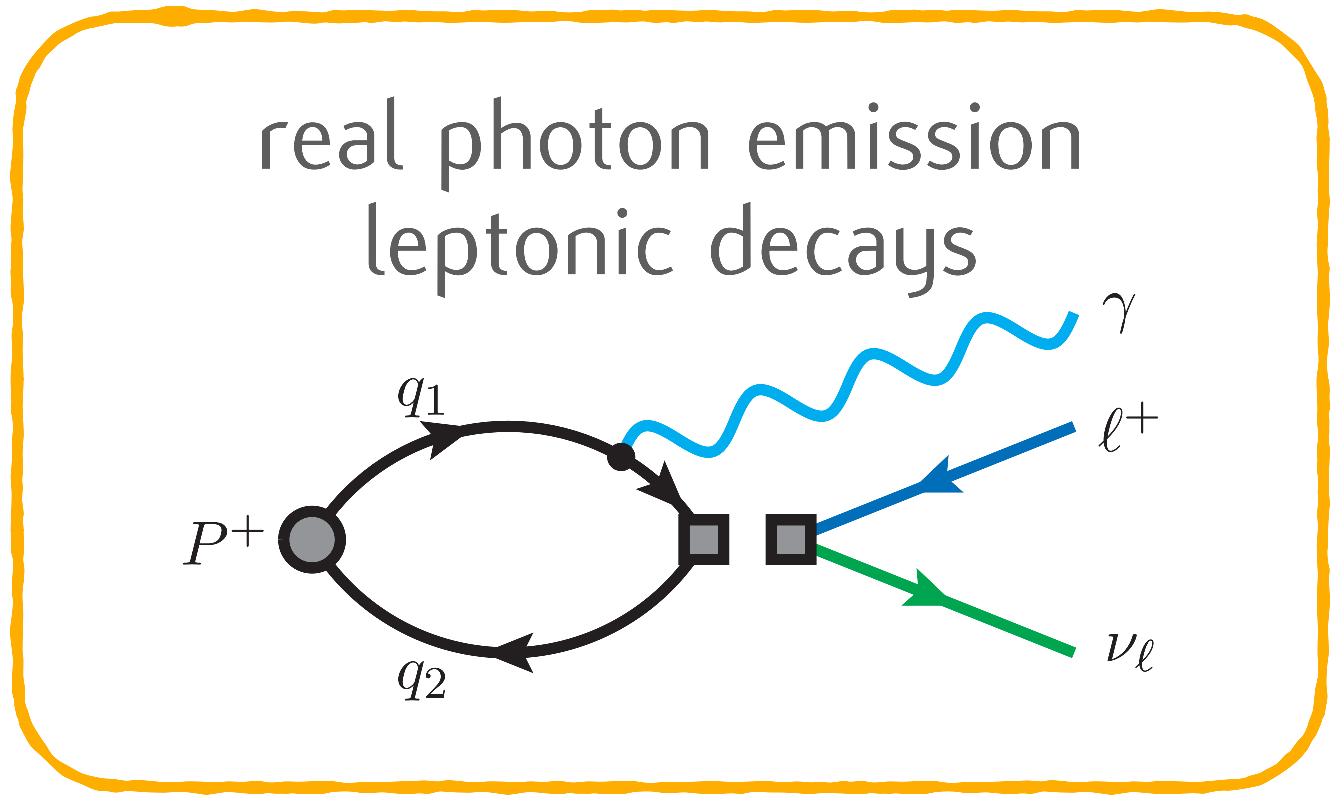
3. What

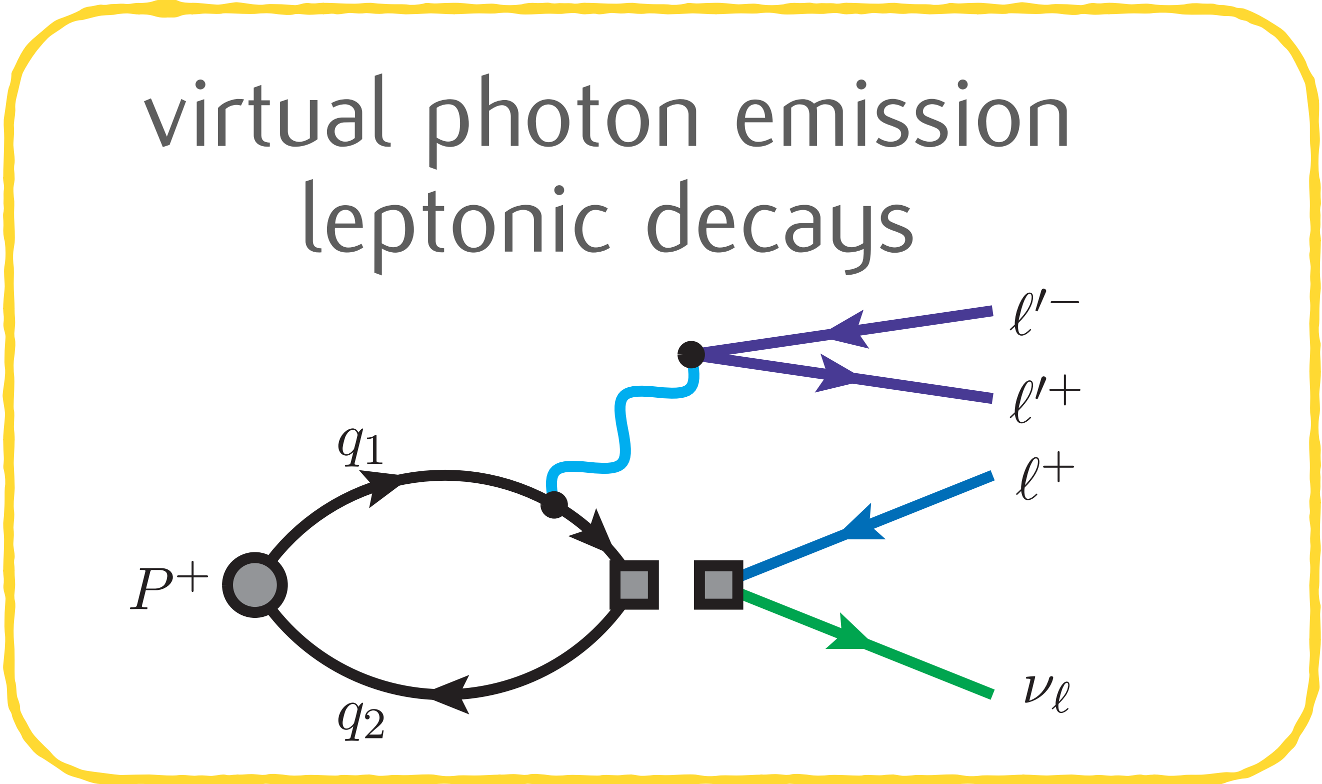


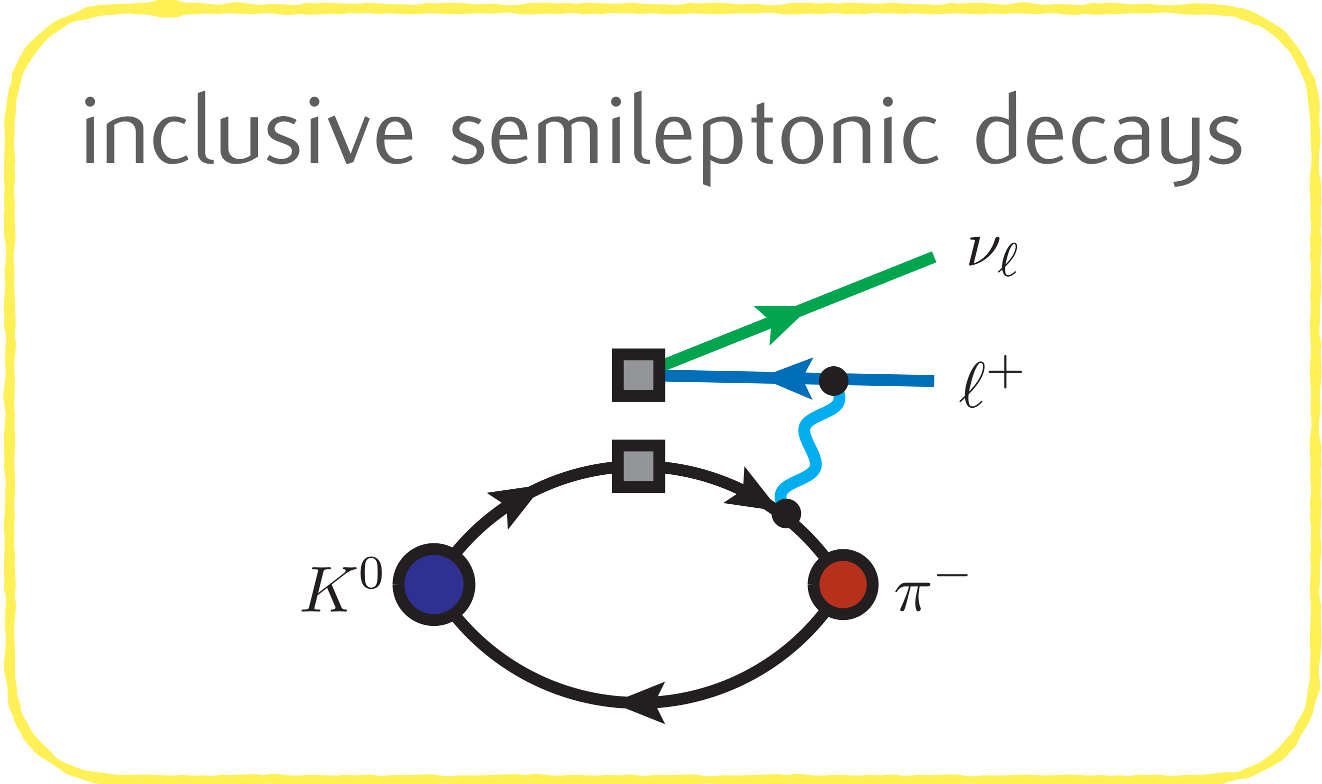
a. RM123S calculation (QED_L) — D.Giusti et al., PRL 120 (2018); MDC et al., PRD 100 (2019)

b. RBC-UKQCD calculation (QED_L) — P.Boyle et al., JHEP 02 (2023)

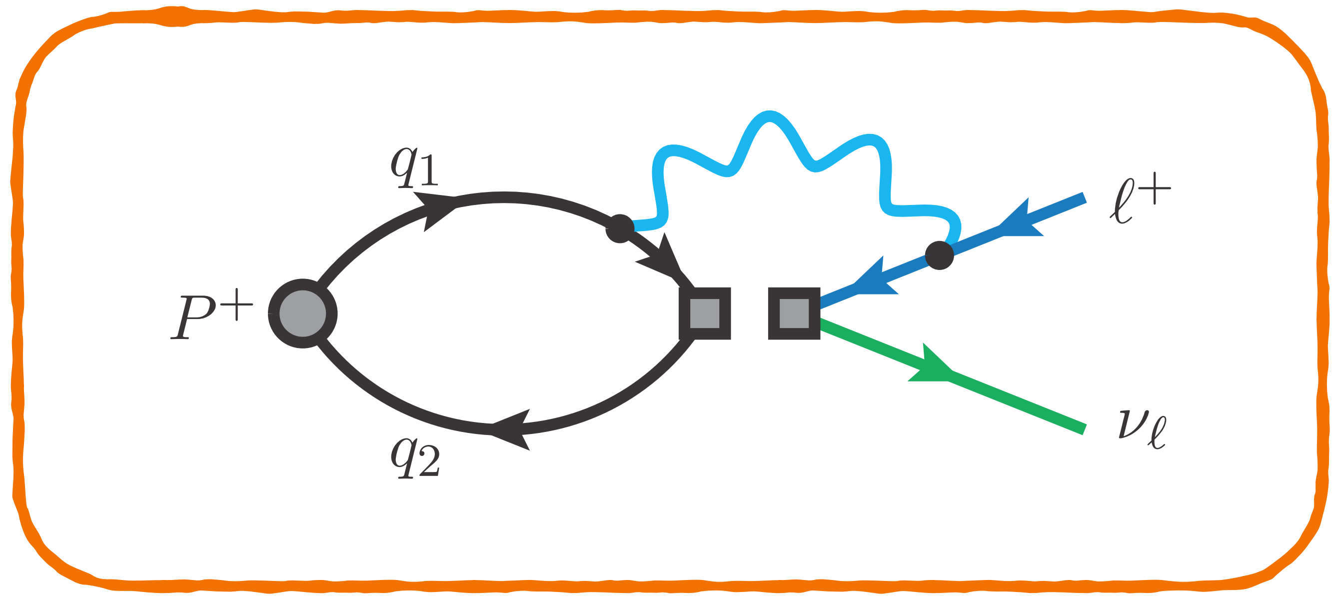
c. Recent proposal with QED_∞ — N.Christ et al., [2304.08026]







inclusive leptonic decays of light pseudoscalar mesons



$P^+ = \pi^+, K^+$ $\ell^+ = \mu^+$

- $\Gamma(K_{\mu2})$ and $\Gamma(\pi_{\mu2})$ separately
- Twisted Mass fermions
- multiple volumes and 3 lattice spacings
- unphysical pion masses ($\gtrsim$ 230 MeV)

1904.08731

PHYSICAL REVIEW D **100,** 034514 (2019)

Editors' Suggestion

Light-meson leptonic decay rates in lattice QCD + QED

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Isospin-breaking corrections to light-meson leptonic decays from lattice simulations at physical quark masses

Peter Boyle,[a,b] Matteo Di Carlo,[b] Felix Erben,[b] Vera Gülpers,[b] Maxwell T. Hansen,[b] Tim Harris,[b] Nils Hermansson-Truedsson,[c,d] Raoul Hodgson,[b] Andreas Jüttner,[e,f] Fionn Ó hÓgáin,[b] Antonin Portelli,[b] James Richings[b,e,g] and Andrew Zhen Ning Yong[b]

2211.12865

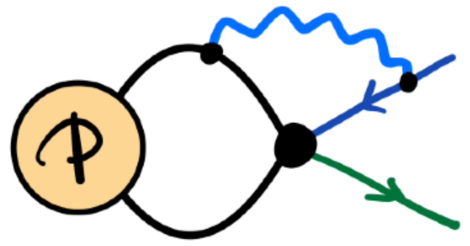
- ratio $\Gamma(K_{\mu2})$ / $\Gamma(\pi_{\mu2})$
- Domain Wall fermions
- single volume and lattice spacing
- physical quark masses

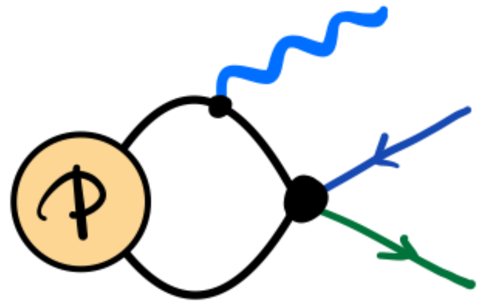
Decay rate at $\mathcal{O}(\alpha)$

When including radiative corrections many subtleties arise, for example:

- **IR divergences** appear in intermediate steps of the calculation — F. Bloch & A. Nordsieck, PR 52 (1937) 54

$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\mathrm{IR}} \to 0} \left\{ \quad + \quad \right\}$$





IR finite — IR divergent — IR divergent

- new **UV divergences**: include QED corrections to the renormalization of the weak Hamiltonian

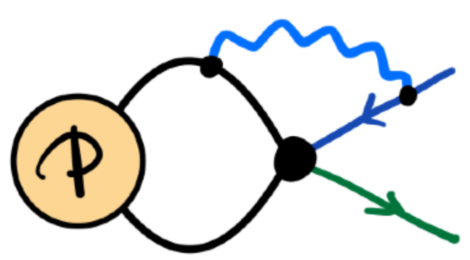
- the **decay constant** $f_{P,0}$ becomes an ambiguous and unphysical quantity: one needs to introduce a **scheme** to give a meaning to "QCD" or "iso-QCD"

Decay rate at $\mathcal{O}(\alpha)$

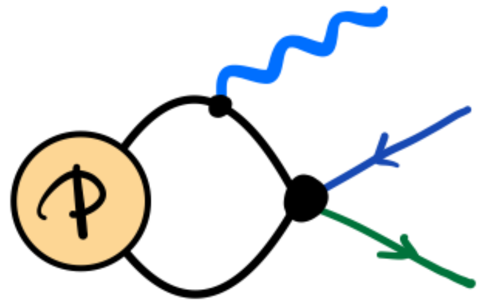
F. Bloch & A. Nordsieck, PR 52 (1937) 54

The RM123+Soton recipe

$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\mathrm{IR}}\to 0} \Big\{$$



$$+$$



$$\Big\}$$

IR finite — IR divergent — IR divergent

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F. Bloch & A. Nordsieck, PR 52 (1937) 54

N. Carrasco et al., PRD 91 (2015)
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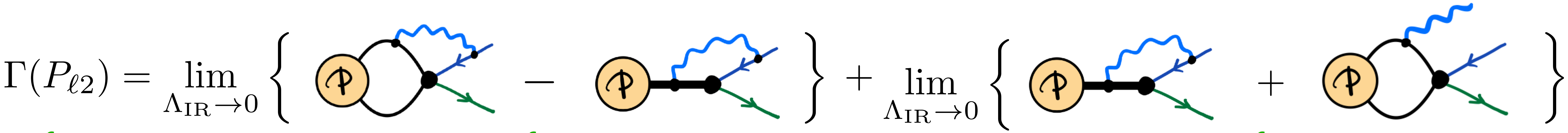

IR finite   IR finite   IR finite

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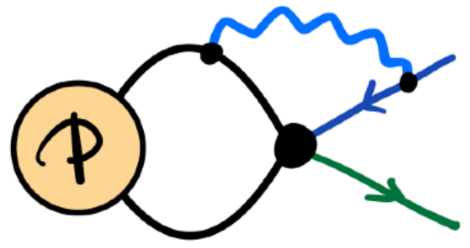
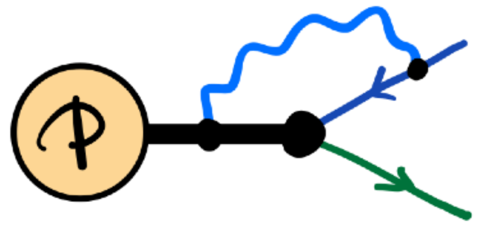
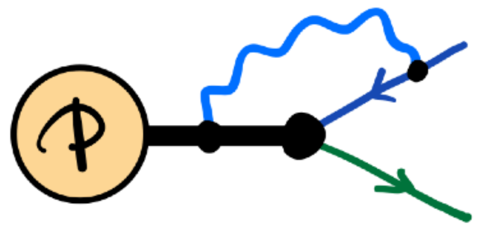
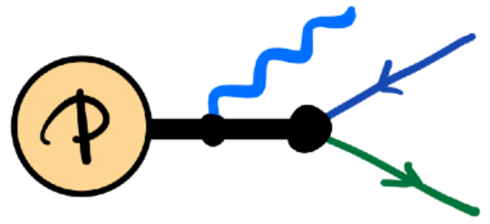
The RM123+Soton recipe (for negligible structure-dependence in real photon emission)

(point-like approximation)

$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\rm IR}\to 0} \Big\{$  $-$  $\Big\} + \lim_{\Lambda_{\rm IR}\to 0} \Big\{$  $+$  $\Big\}$

IR finite   IR finite   IR finite

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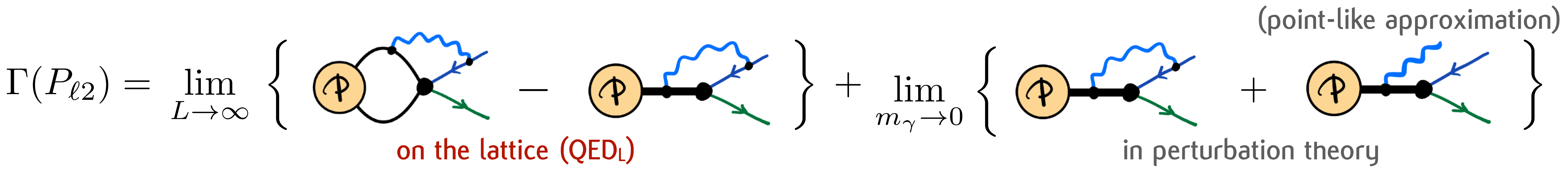

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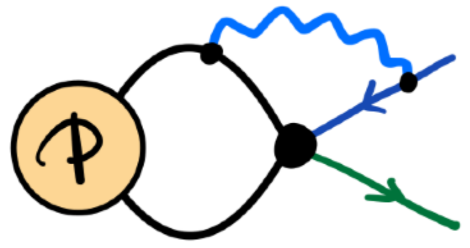
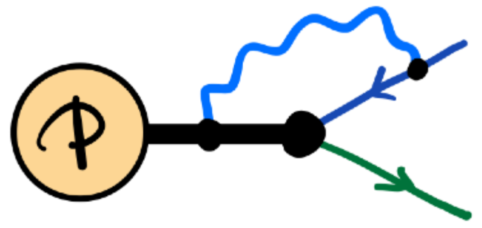
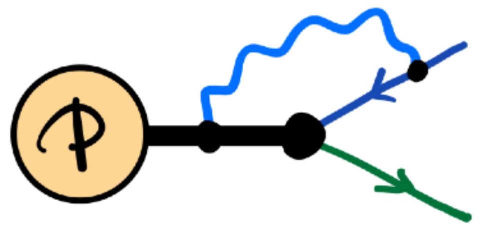
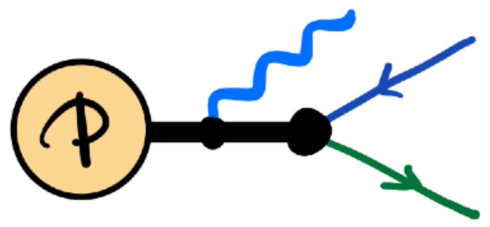
$\Gamma(P_{\ell 2}) = \lim_{L\to\infty} \{$  $-$  $\} + \lim_{m_\gamma \to 0} \{$  $+$  $\}$

on the lattice (QED$_L$) — in perturbation theory

Possible extensions:

- improving **finite-volume scaling** of the **virtual** decay rate — MDC et al., PRD 105 (2022)

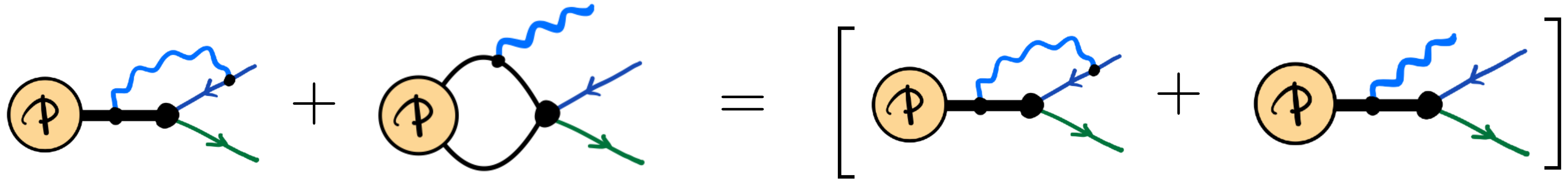
$$\Gamma_P^{\rm virt}(L) - \alpha_{\rm em}\left[y_{\log}\log(m_P L) + y_0 + y_1 \frac{1}{m_P L} + y_2 \frac{1}{(m_P L)^2} + \cdots + y_n \frac{1}{(m_P L)^n}\right] \sim \mathcal{O}\left(\frac{1}{(m_P L)^{n+1}}\right)$$

"universal" — structure-dependent

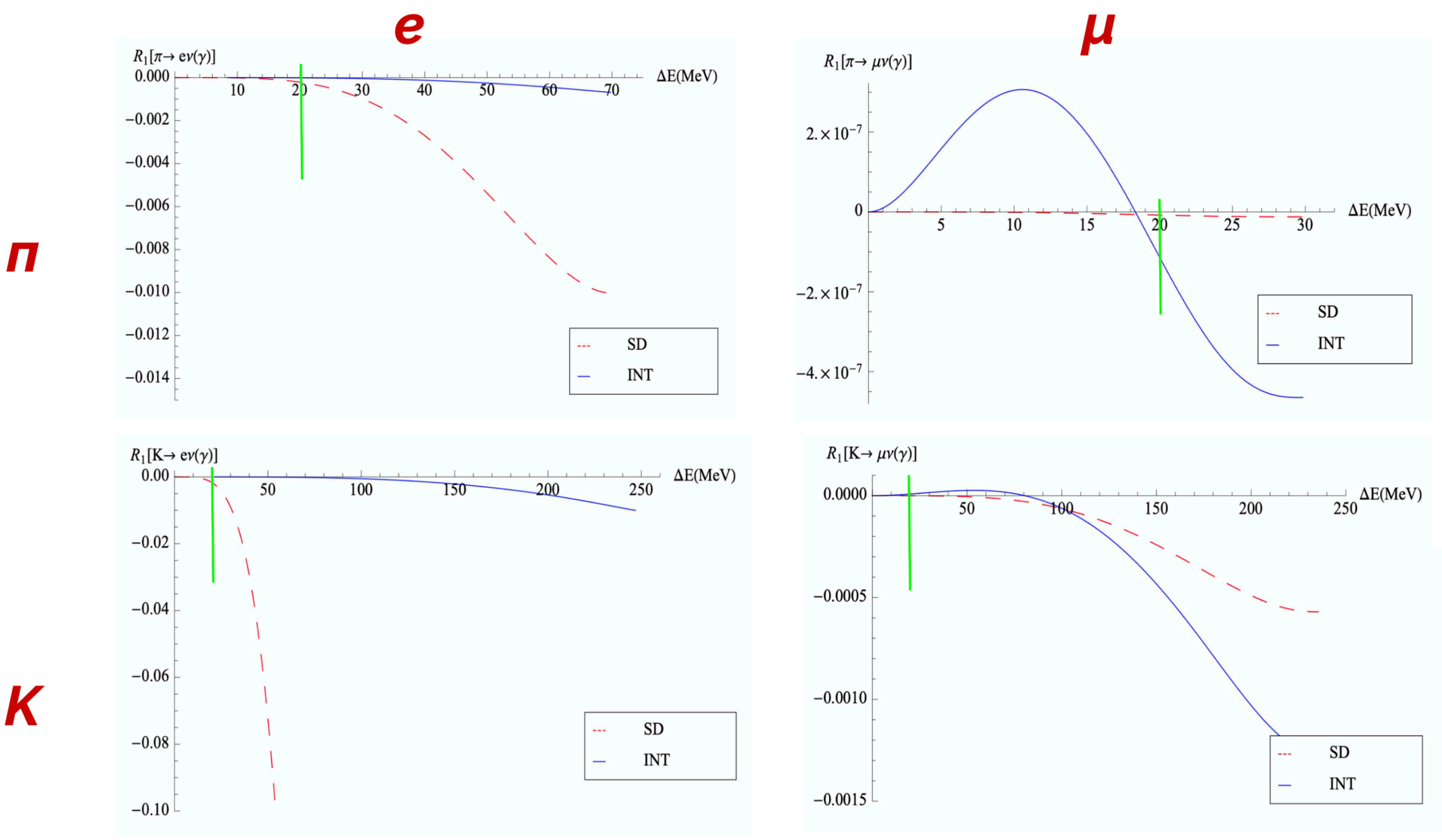
- compute **structure-dependent real photon emission** on the lattice

G.M. de Divitiis et al., [1908.10160]
R. Frezzotti et al., PRD 103 (2021)
A. Desiderio et al., PRD 102 (2021)
C. Kane et al., [1907.00279 & 2110.13196]
D. Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

Real photon emission and structure dependence



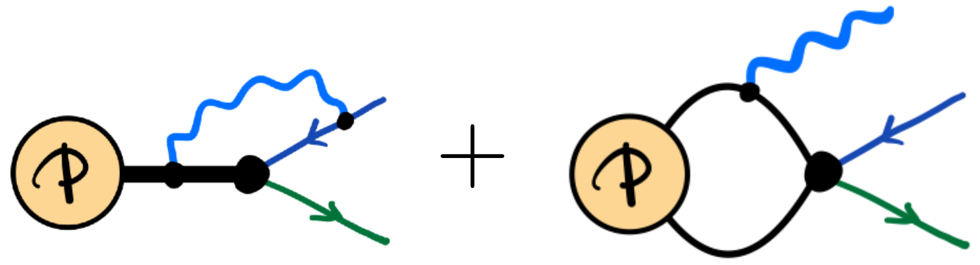
$$\left(1 + R_1^{\mathrm{SD}}(\Delta E) + R_1^{\mathrm{INT}}(\Delta E)\right)$$



Calculation at O(p⁴) in χPT

N. Carrasco et al., PRD 91 (2015)

Real photon emission and structure dependence



$$= \delta R_{\mathrm{pt}}(\Delta E) + \delta R_1^{\mathrm{SD}}(\Delta E) + \delta R_1^{\mathrm{INT}}(\Delta E)$$

	$\pi_{e2[\gamma]}$	$\pi_{\mu2[\gamma]}$	$K_{e2[\gamma]}$	$K_{\mu2[\gamma]}$
δR_0	$^{(*)}$	$0.0411\ (19)$	$^{(*)}$	$0.0341\ (10)$
$\delta R_{\mathrm{pt}}(\Delta E_\gamma^{max})$	-0.0651	-0.0258	-0.0695	-0.0317
$\delta R_1^{\mathrm{SD}}(\Delta E_\gamma^{max})$	$5.4\ (1.0) \times 10^{-4}$	$2.6\ (5) \times 10^{-10}$	$1.19\ (14)$	$2.2\ (3) \times 10^{-5}$
$\delta R_1^{\mathrm{INT}}(\Delta E_\gamma^{max})$	$-4.1\ (1.0) \times 10^{-5}$	$-1.3\ (1.5) \times 10^{-8}$	$-9.2\ (1.3) \times 10^{-4}$	$-6.1\ (1.1) \times 10^{-5}$
ΔE_γ^{max} (MeV)	69.8	29.8	246.8	235.5

$^{(*)}$ *Not yet evaluated by numerical lattice QCD+QED simulations.*

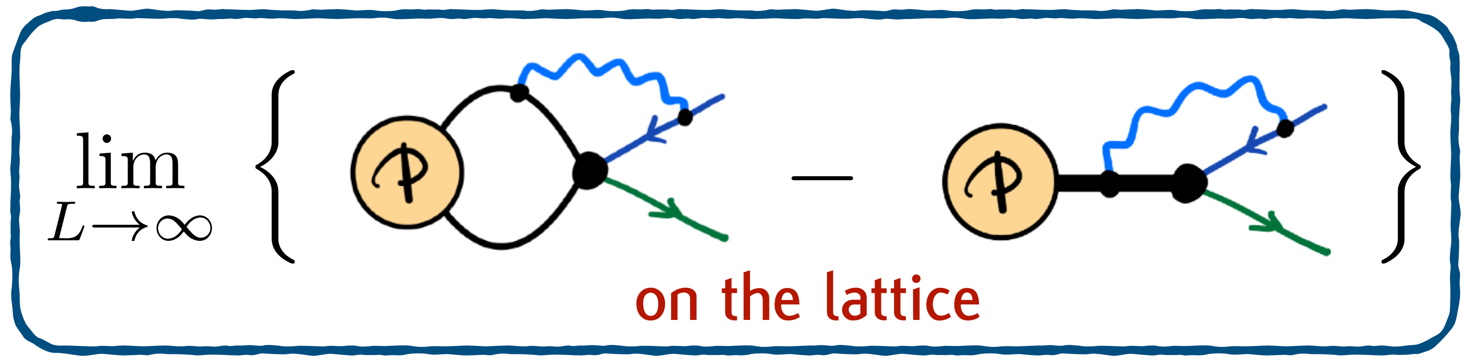
Confirmed by numerical lattice calculation

A. Desiderio et al., PRD 102 (2021)

R. Frezzotti et al., PRD 103 (2021)

Decay rate at $\mathcal{O}(\alpha)$

Virtual decay rate



$$\Gamma(P_{\ell 2}) = \Gamma_P^{\text{tree}}(1+\delta R_P) \quad \blacktriangleright \quad \Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2 \quad \blacktriangleright \quad \delta R_P = 2\left(\frac{\delta \mathcal{A}_P}{\mathcal{A}_{P,0}} - \frac{\delta m_P}{m_{P,0}} + \frac{\delta \mathcal{Z}}{\mathcal{Z}_0}\right)$$

PDG convention

- $\boldsymbol{\delta \mathcal{A}_P}$ from the correction to the (bare) matrix element $\mathcal{M}_P^{rs}(\mathbf{p}_\ell) = \langle \ell^+, r, \mathbf{p}_\ell; \nu_\ell, s, \mathbf{p}_\nu | O_W | P^+, \mathbf{0}\rangle$
- $\boldsymbol{\delta m_P}$ correction to the meson mass
- $\boldsymbol{\delta \mathcal{Z}}$ correction to the renormalization of the weak operator O_W MDC et al., PRD 100 (2019) & MDC@Lattice2019

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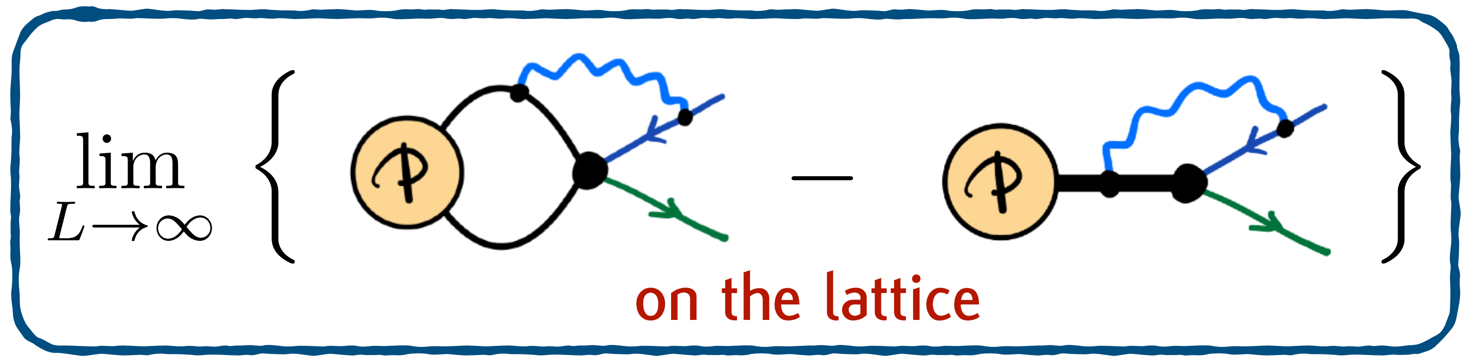

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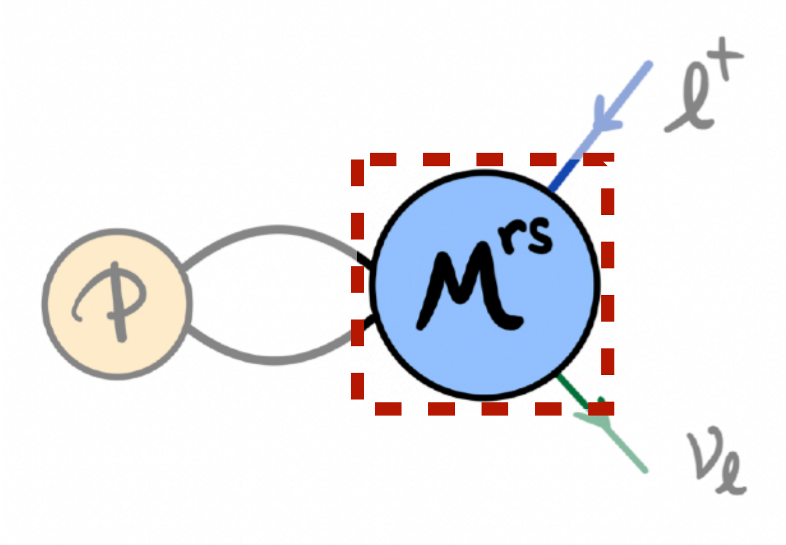
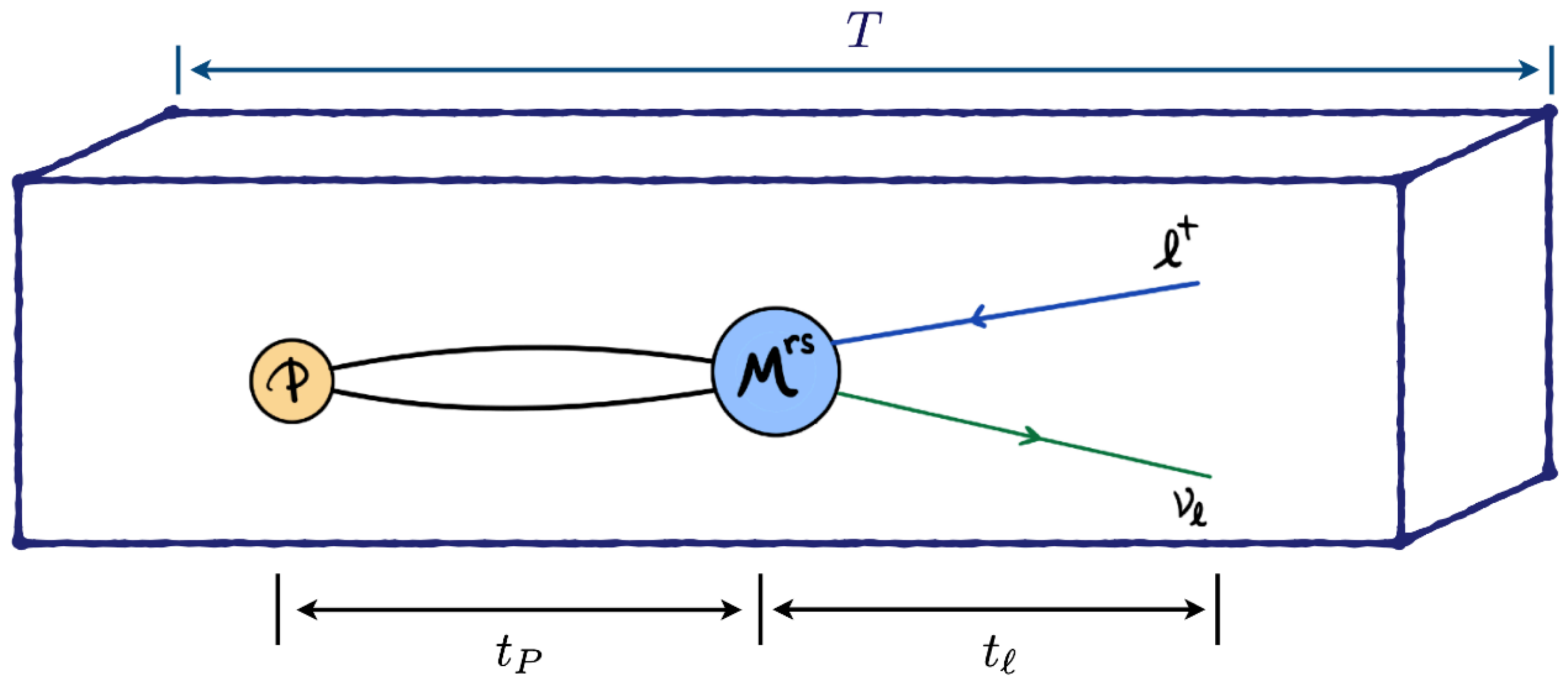
Our target: $\frac{\Gamma(K_{\ell 2})}{\Gamma(\pi_{\ell 2})} \rightarrow \delta R_{K\pi} = 2\left(\frac{\delta\mathcal{A}_K}{\mathcal{A}_{K,0}} - \frac{\delta m_K}{m_{K,0}}\right) - 2\left(\frac{\delta\mathcal{A}_\pi}{\mathcal{A}_{\pi,0}} - \frac{\delta m_\pi}{m_{\pi,0}}\right)$

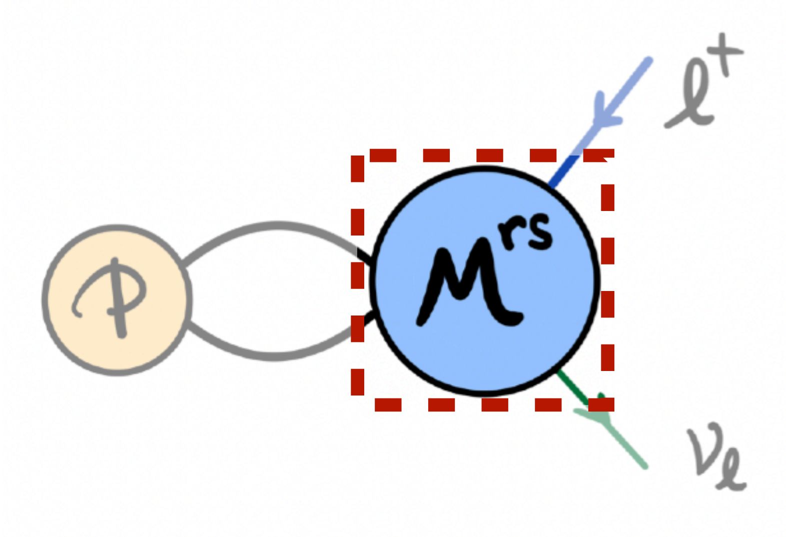
From correlators to matrix elements



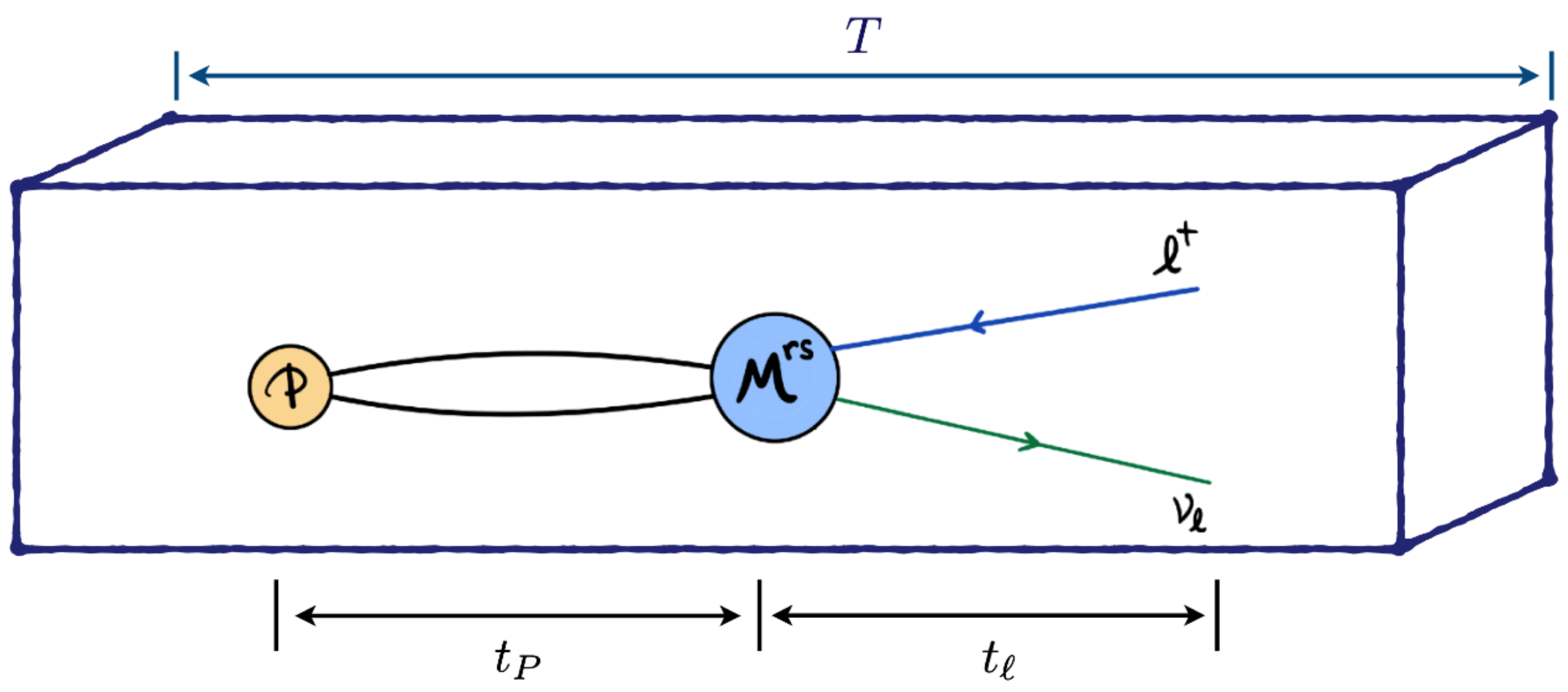
$\Longleftrightarrow$



From correlators to matrix elements



$\Longleftrightarrow$

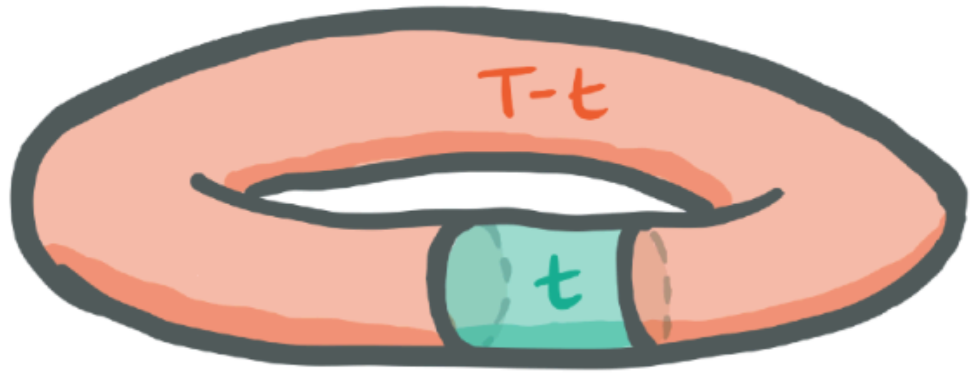


Tree-level decay amplitude:

$$= \langle 0|A^0(0)\phi^\dagger(-t)|0\rangle_T \to \frac{Z_{P,0}\mathcal{A}_{P,0}}{2m_{P,0}}\left\{e^{-m_{P,0}t} - e^{-m_{P,0}(T-t)}\right\}$$

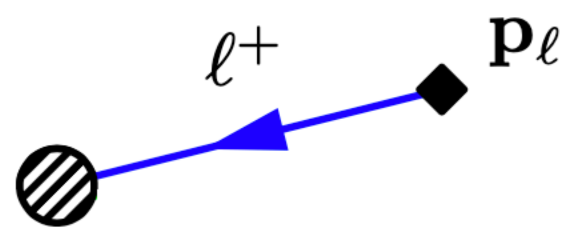
$$Z_{P,0} = \langle P, \mathbf{p}=\mathbf{0}|\phi^\dagger|0\rangle_0$$

$$= \langle 0|\phi(0)\phi^\dagger(-t)|0\rangle_T \to \frac{Z_{P,0}^2}{2m_{P,0}}\left\{e^{-m_{P,0}t} + e^{-m_{P,0}(T-t)}\right\}$$



Non-factorisable QED corrections

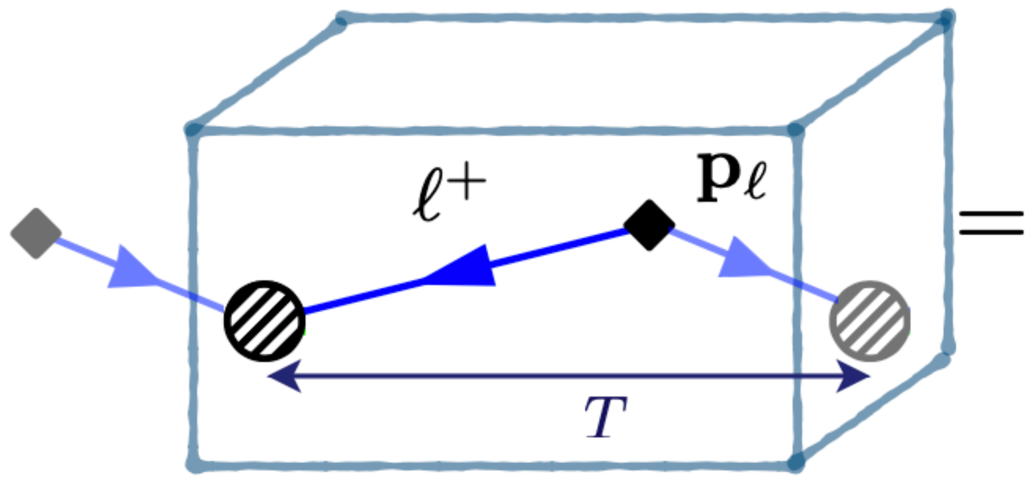
The lepton in a finite volume



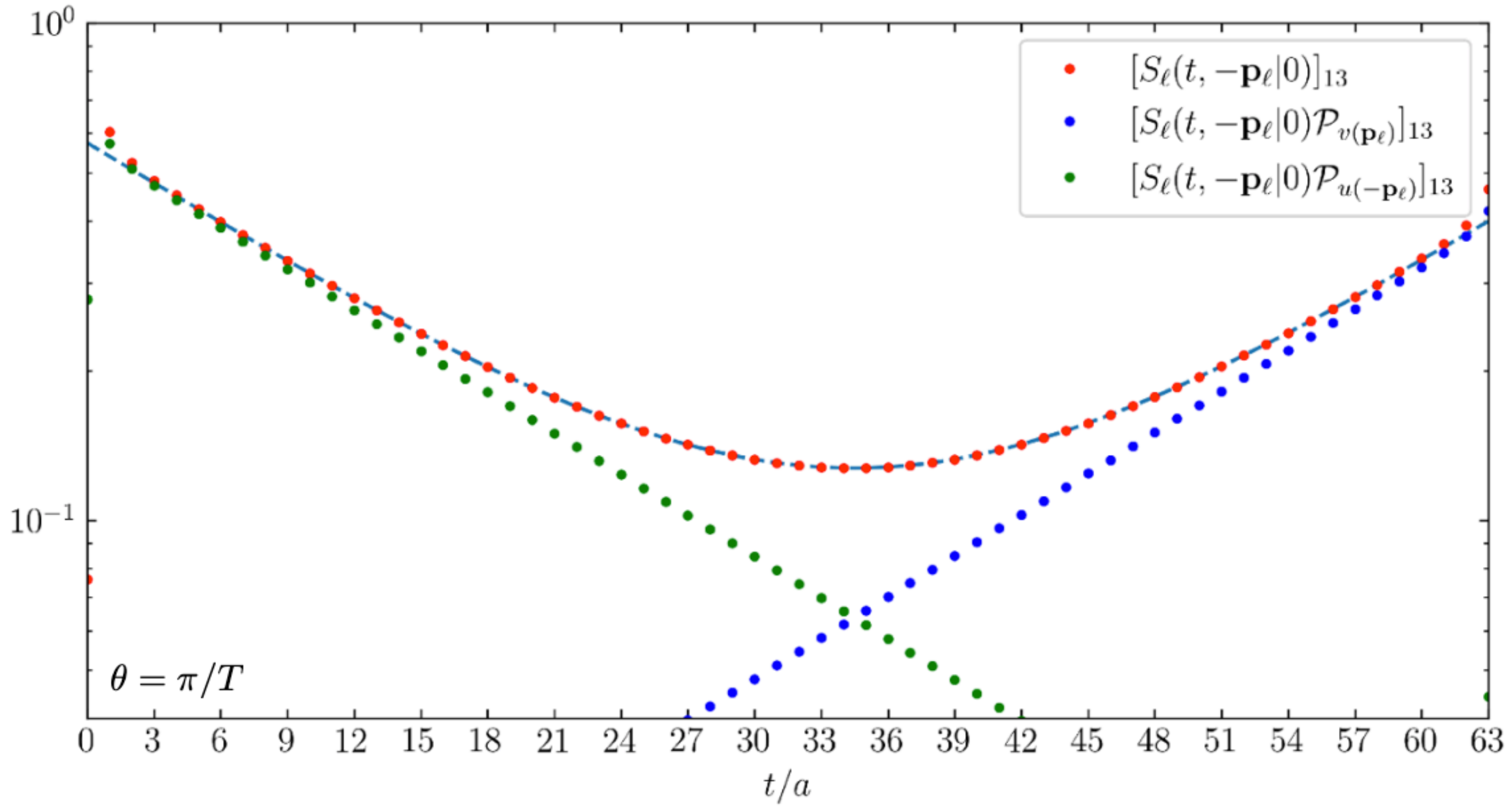
$$= \quad S(0|t,\mathbf{p}_\ell) = \sum_r \left\{ -\mathrm{e}^{-tE_\ell} \, \frac{v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)}{2\Omega_\ell} \right\}$$

Non-factorisable QED corrections

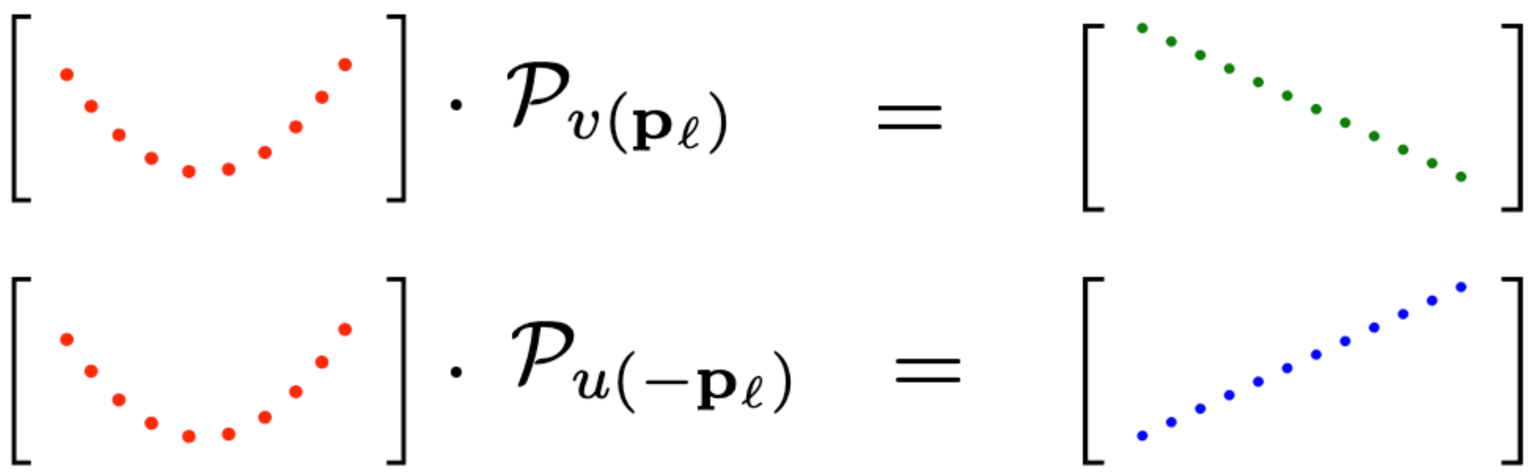
The lepton in a finite volume



$$= S(0|t,\mathbf{p}_\ell) = \sum_r \left\{ -\mathrm{e}^{-tE_\ell} \frac{v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)}{2\Omega_\ell} + \mathrm{e}^{\mathrm{i}\theta T}\mathrm{e}^{-(T-t)E_\ell} \frac{u_r(-\mathbf{p}_\ell)\bar{u}_r(-\mathbf{p}_\ell)}{2\Omega_\ell} \right\} \times \frac{1}{1-\mathrm{e}^{-TE_\ell}\mathrm{e}^{\mathrm{i}\theta T}}$$

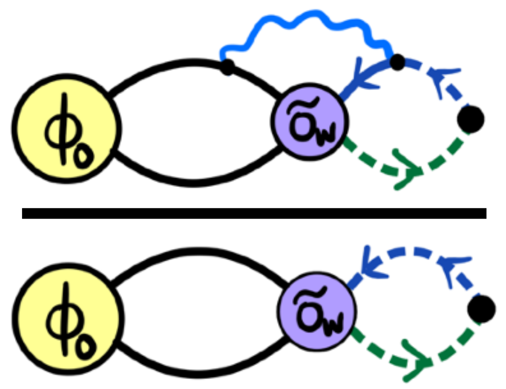


We can select specific components using projectors:

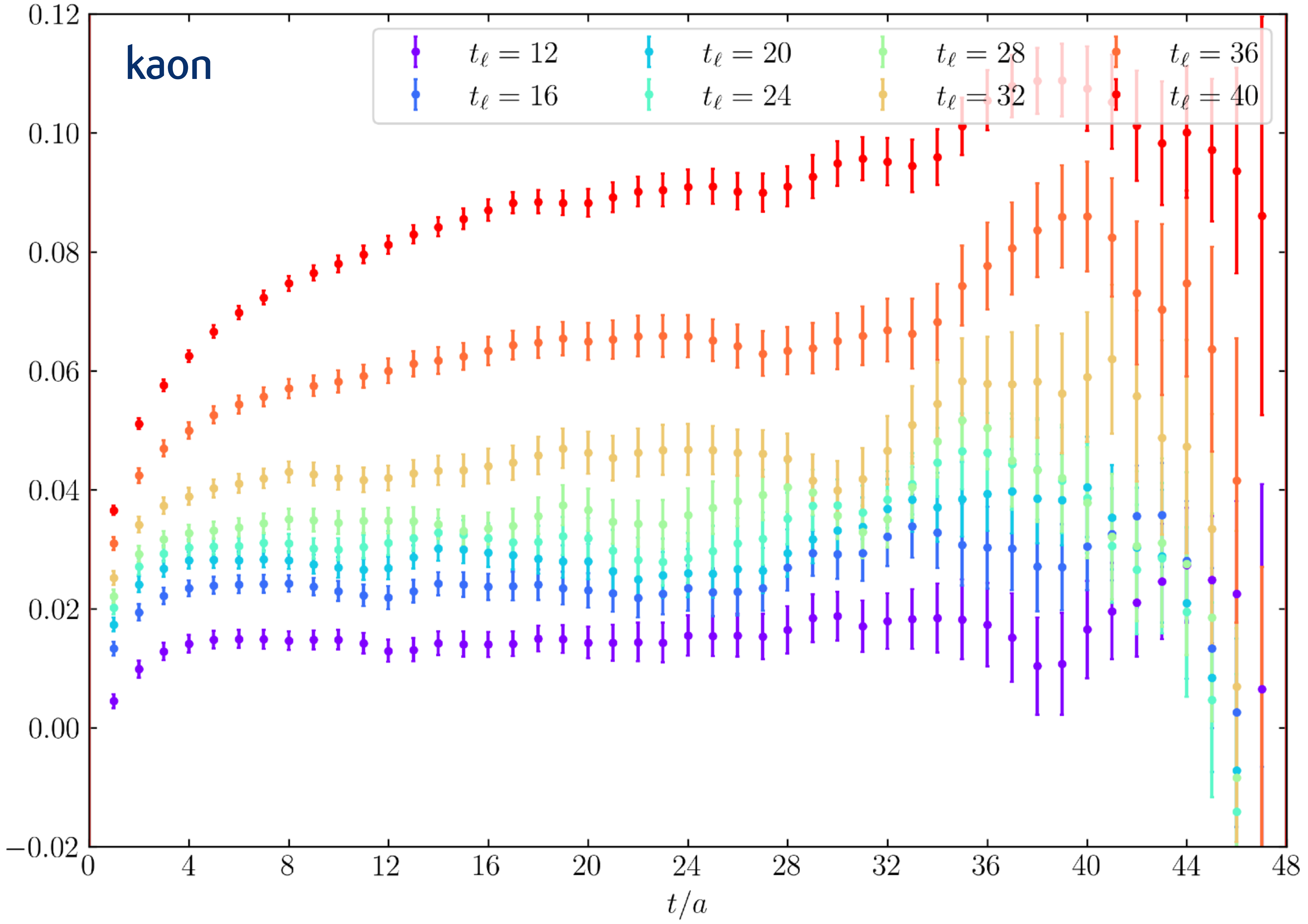


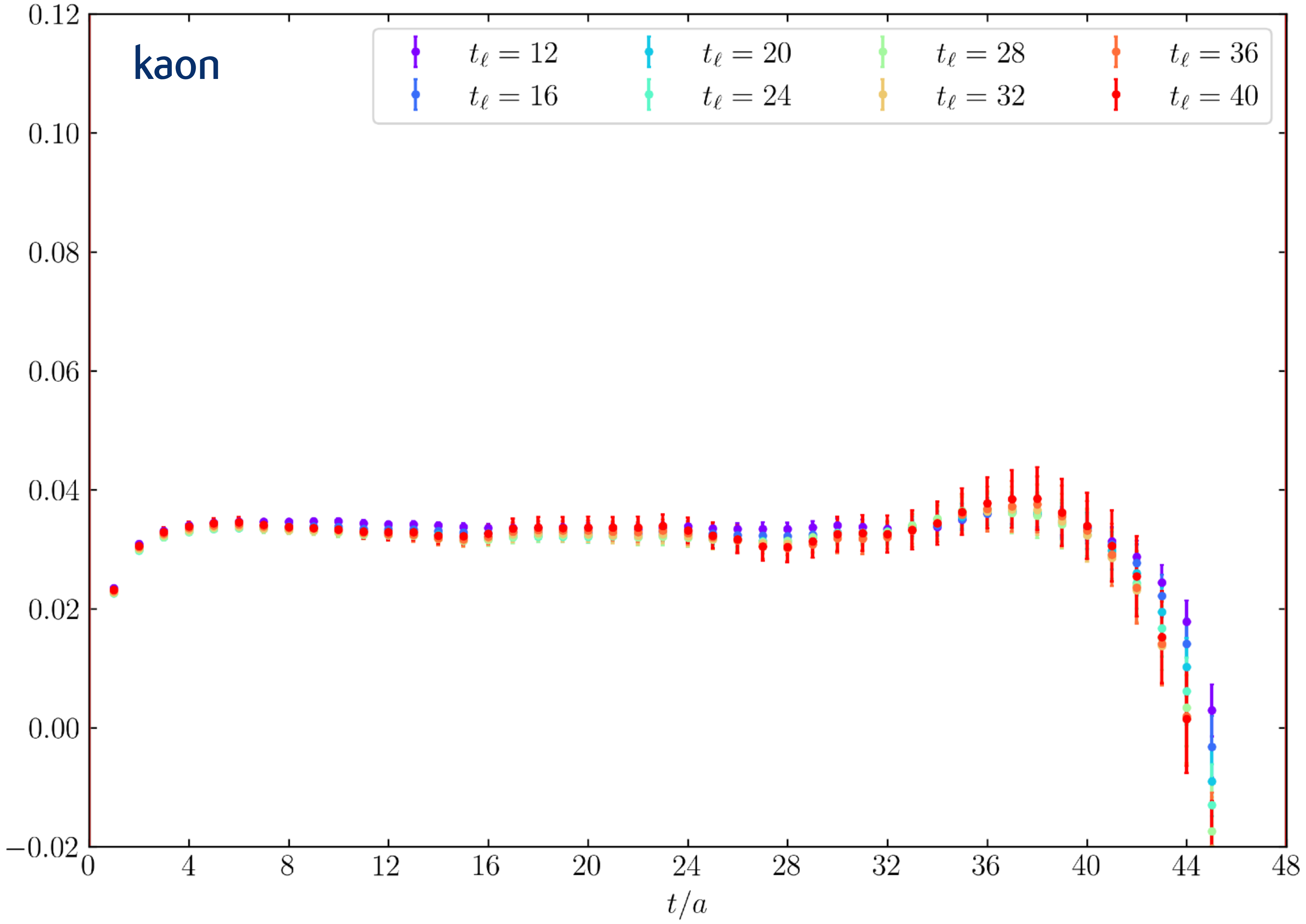
$$\mathcal{P}_{v(\mathbf{p}_\ell)} = \{u_t(-\mathbf{p}_\ell)\bar{u}_t(-\mathbf{p}_\ell) + v_s(\mathbf{p}_\ell)\bar{v}_s(\mathbf{p}_\ell)\}^{-1}\,[v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)]$$
$$\mathcal{P}_{u(-\mathbf{p}_\ell)} = \{u_t(-\mathbf{p}_\ell)\bar{u}_t(-\mathbf{p}_\ell) + v_s(\mathbf{p}_\ell)\bar{v}_s(\mathbf{p}_\ell)\}^{-1}\,[u_r(-\mathbf{p}_\ell)\bar{u}_r(-\mathbf{p}_\ell)]$$

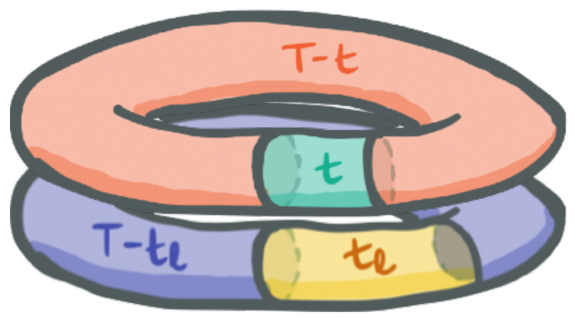
Non-factorisable QED corrections



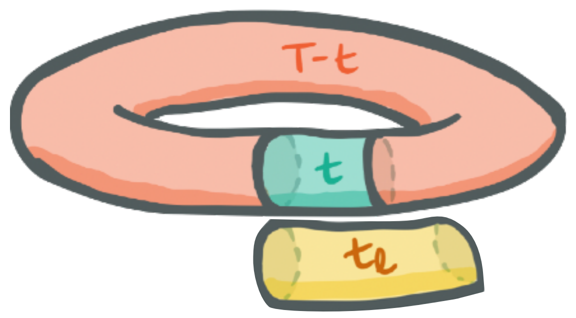
$$\rightarrow \frac{\delta_{\text{non-fact}}\mathcal{A}_P}{\mathcal{A}_{P,0}} \quad f_{\mathrm{P}\ell}(t,T)$$





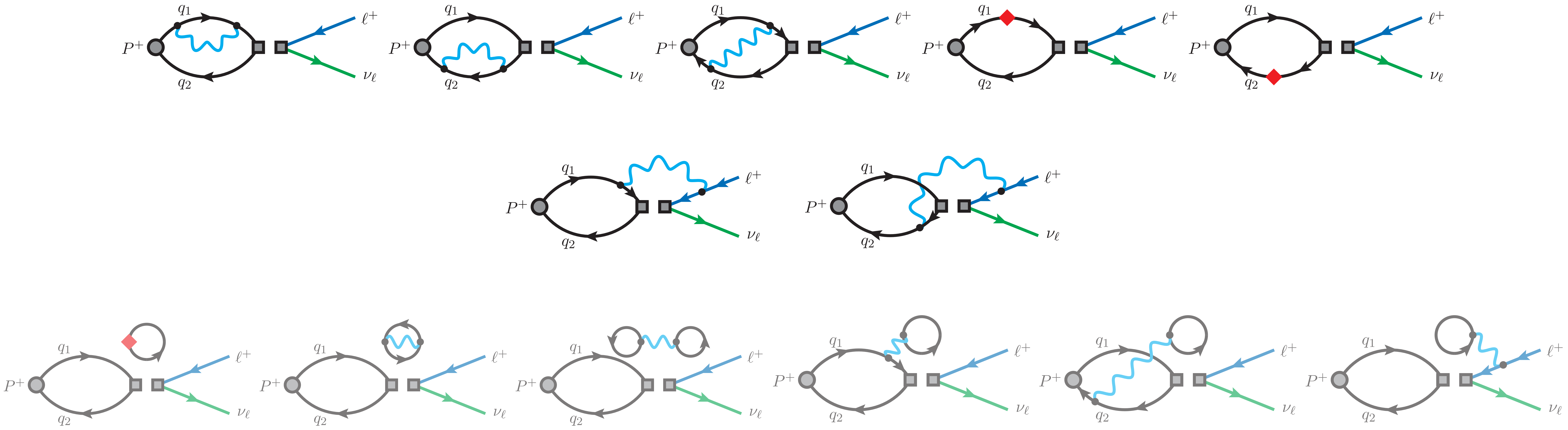


without projection



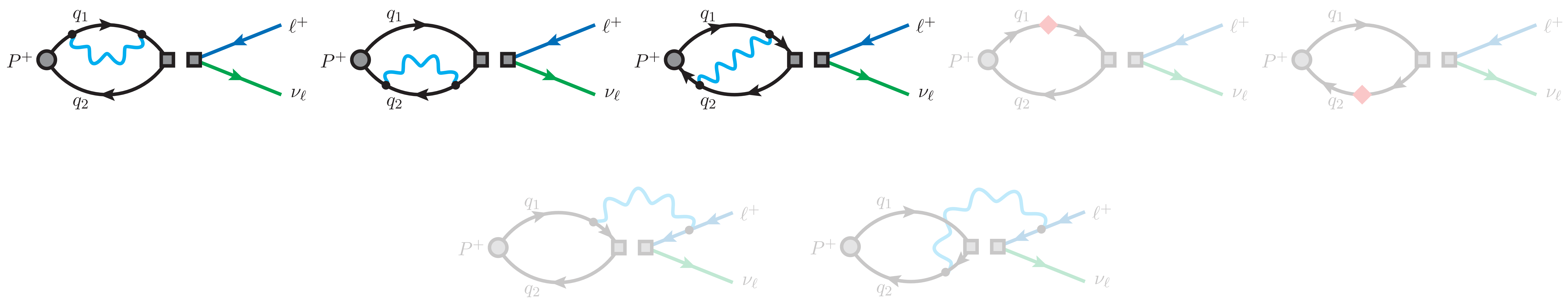
with projection

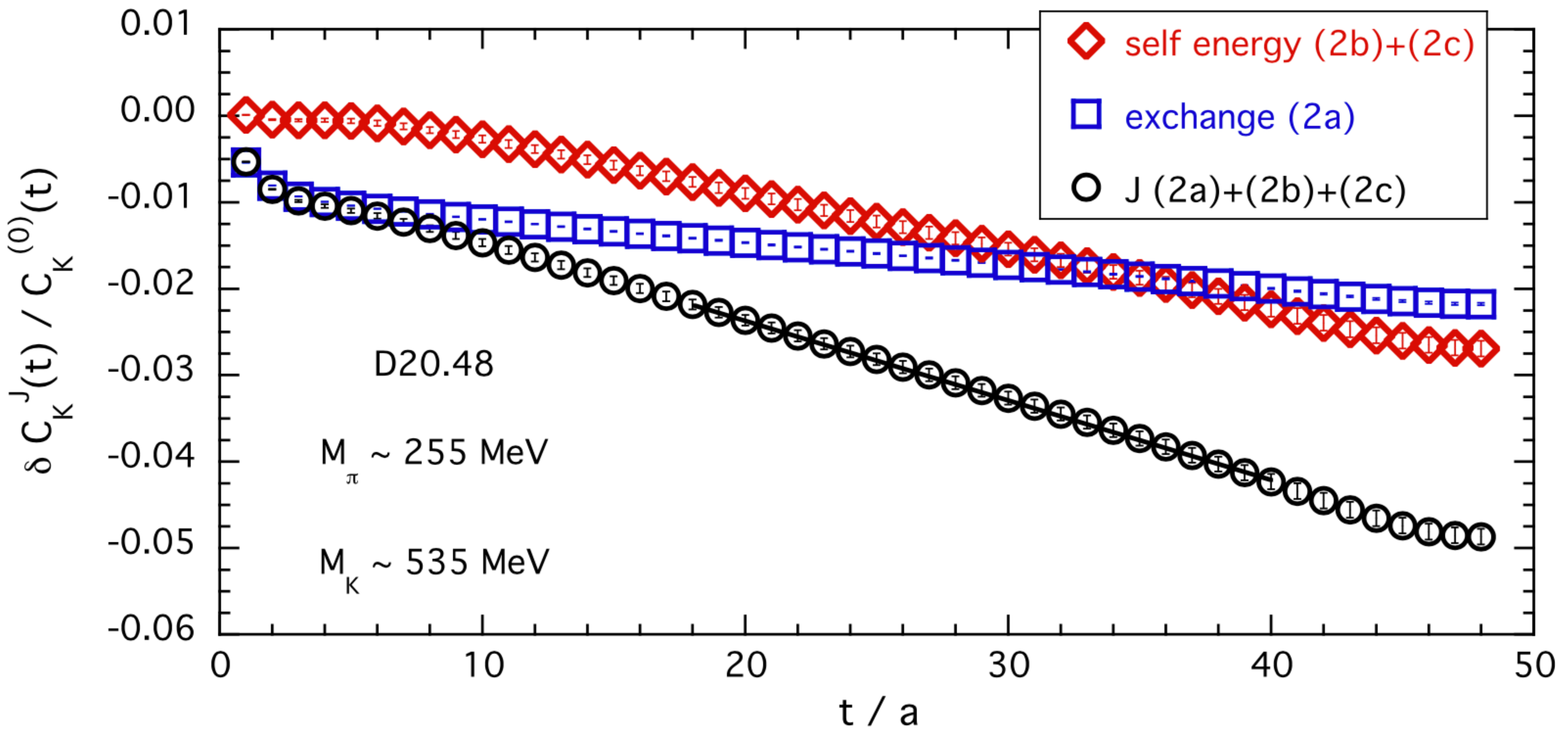
IB corrections to the decay amplitude



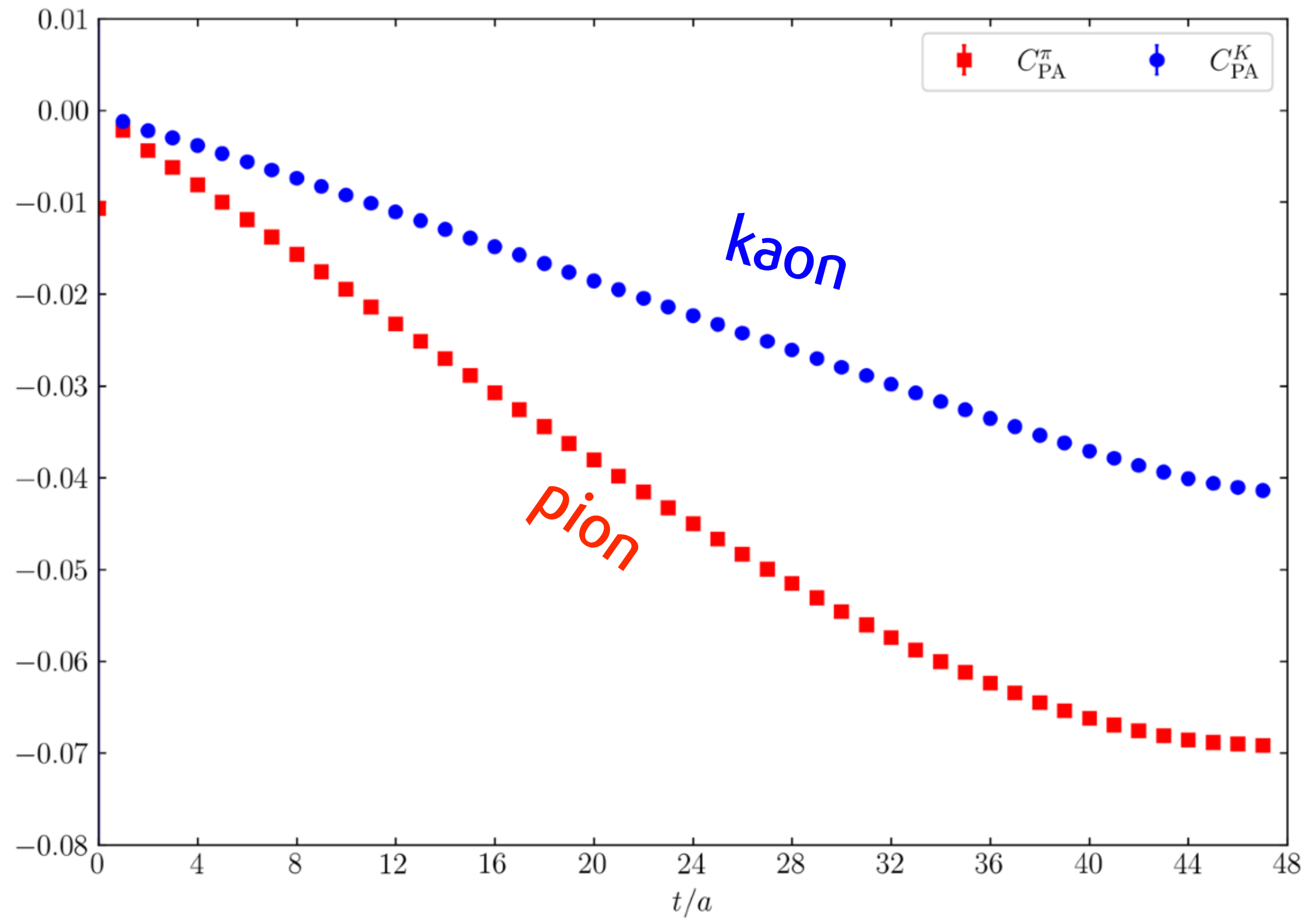
Both RM123S and RBC-UKQCD calculations are performed in the **electro-quenched approximation: sea quarks electrically neutral**

IB corrections to the decay amplitude



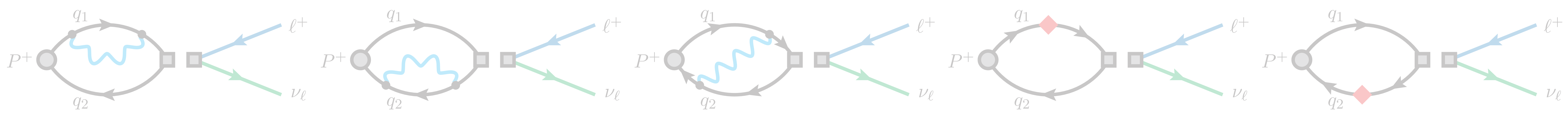


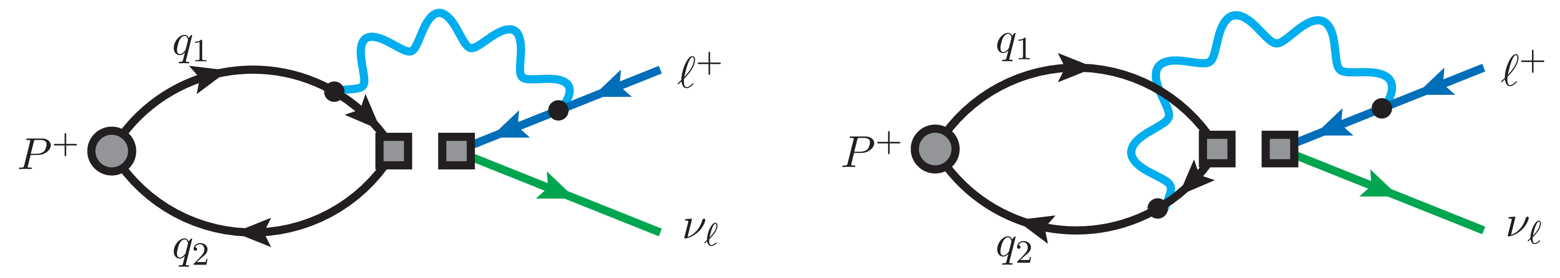
1904.08731

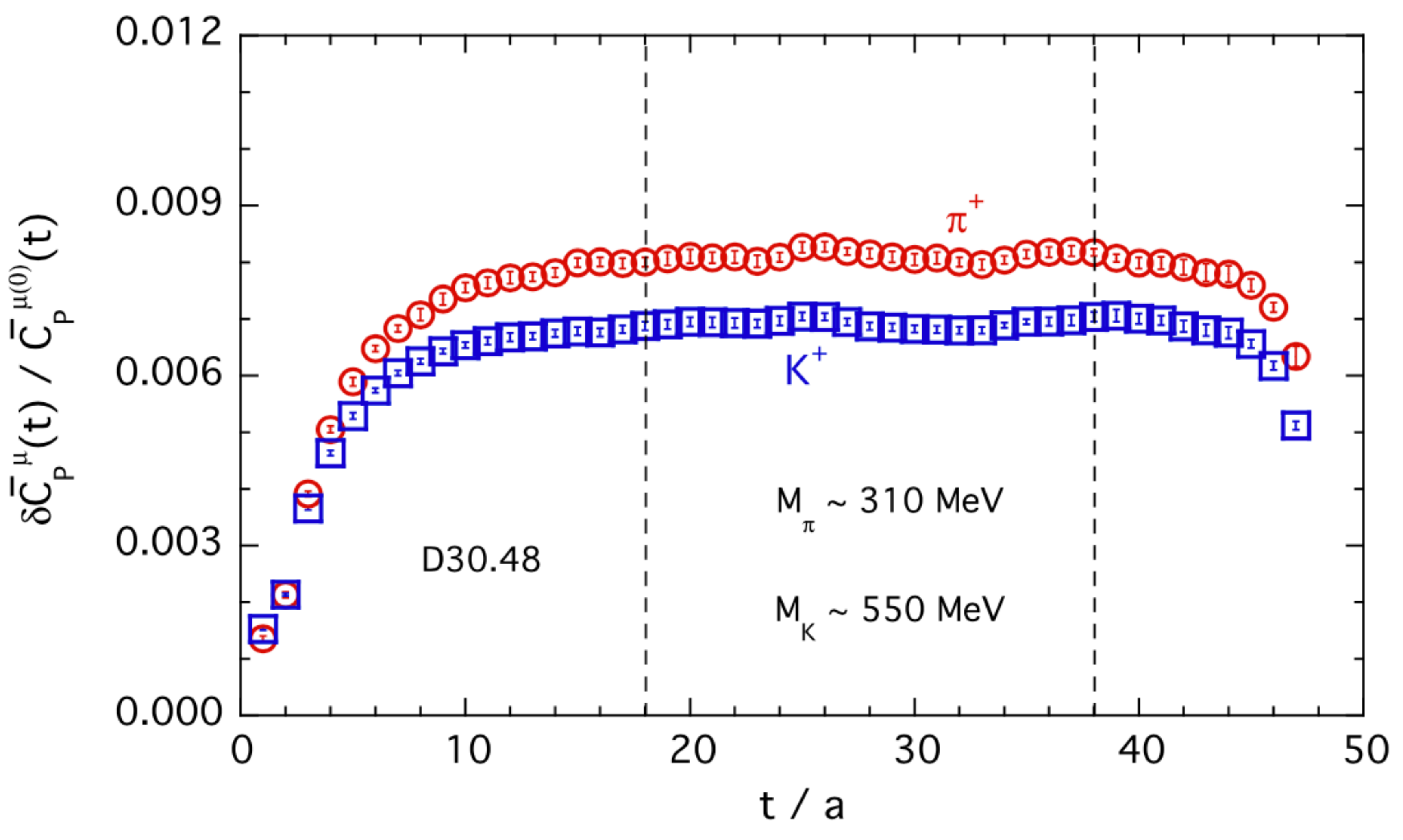


2211.12865

IB corrections to the decay amplitude







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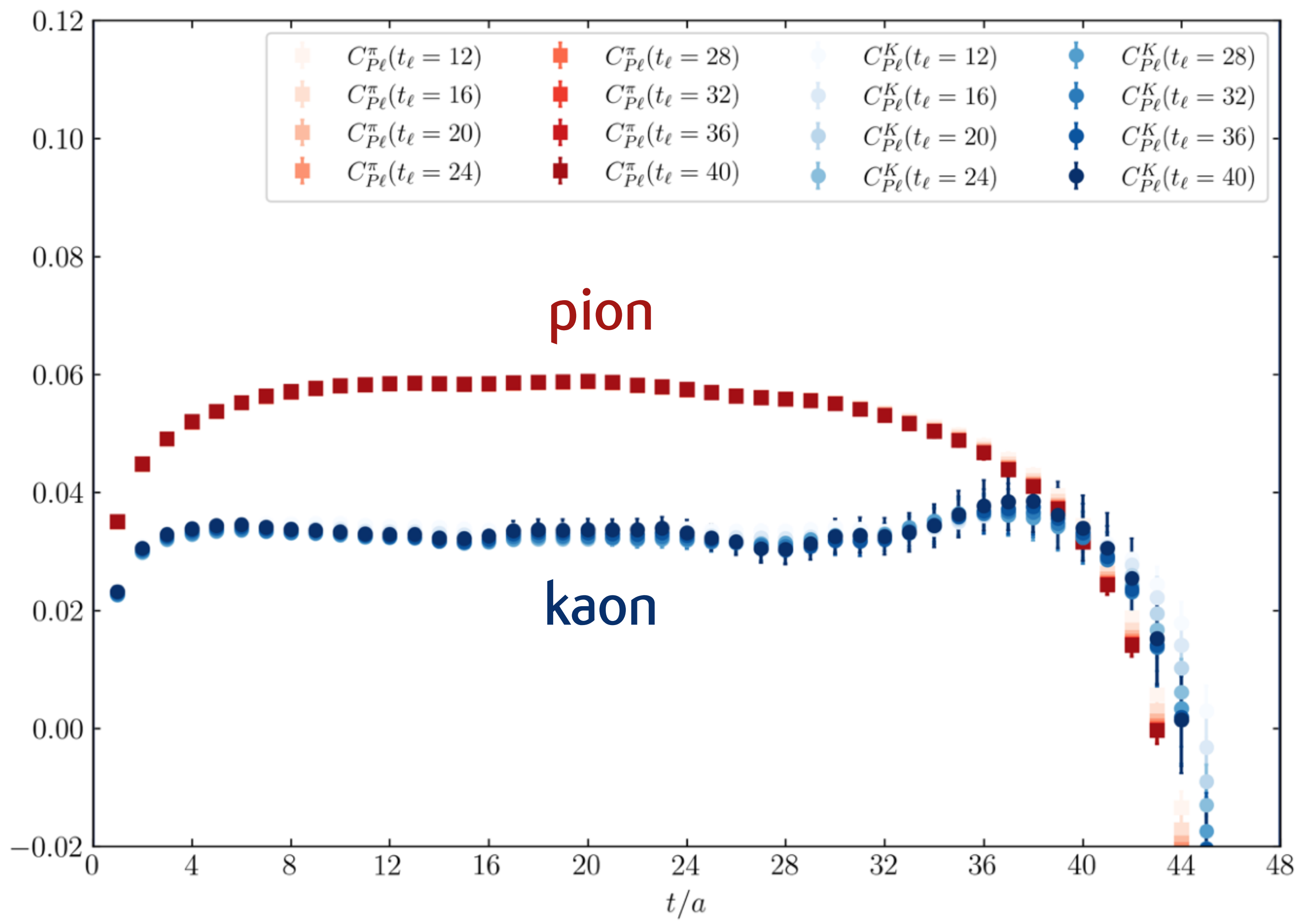
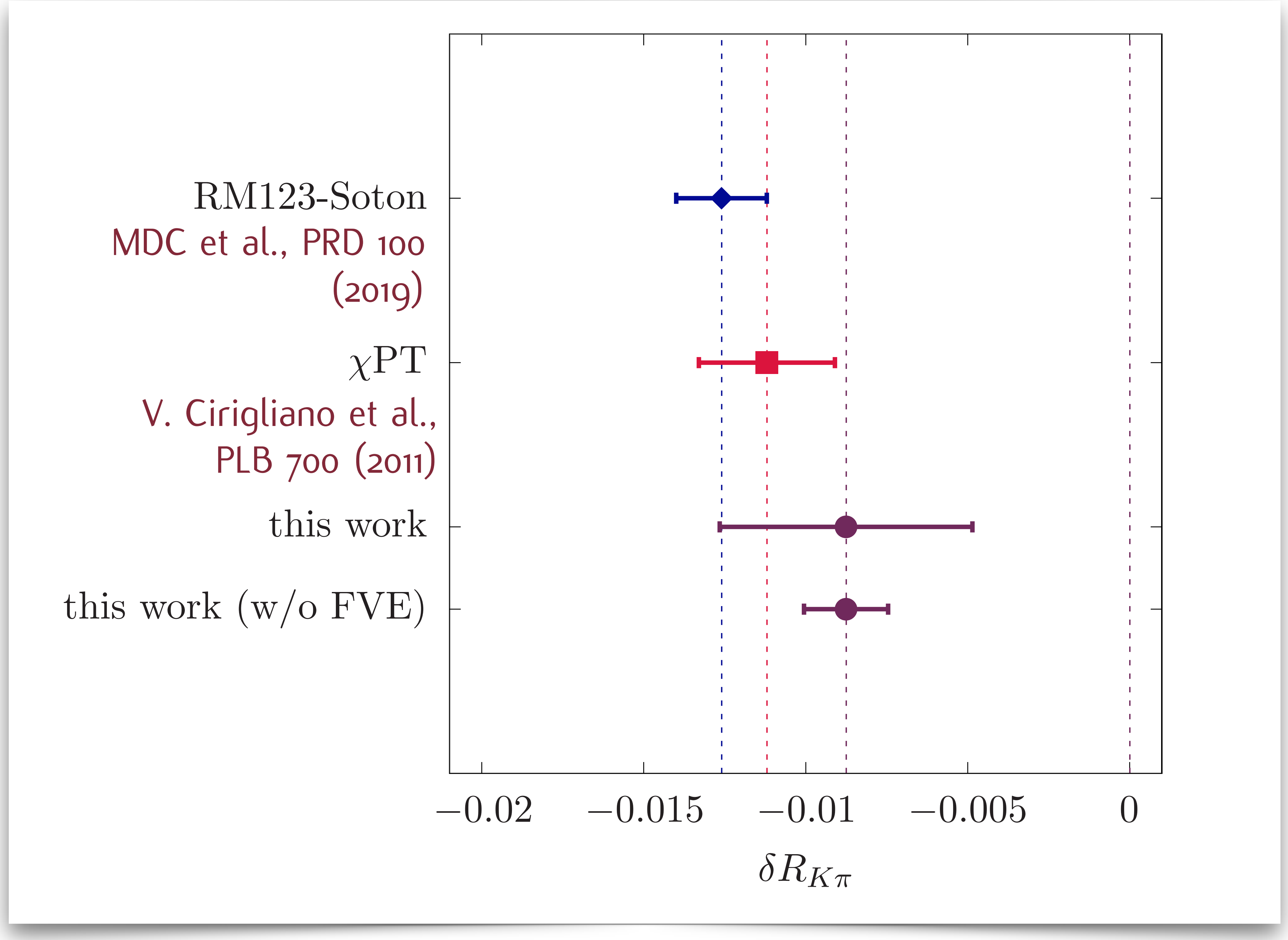


2211.12865

Results for $\delta R_{K\pi}$



$$\delta R_{K\pi} = -0.0086\,(3)_{\text{stat.}}\left({}^{+11}_{-4}\right)_{\text{fit}}(5)_{\text{disc.}}(5)_{\text{quench.}}(39)_{\text{vol.}}$$

RM123S: $\delta R_{K\pi} = -0.0126\,(14)$ **χPT:** $\delta R_{K\pi} = -0.0112\,(21)$

- Our recent result is **compatible** with previous lattice calculation (RM123S) and with χPT
- The error is dominated by a large systematic uncertainty related to **finite-volume effects**

Solid evidence that $\delta R_{K\pi}$ can be computed from first principles non-perturbatively on the lattice!

Prospects for $|V_{us}/V_{ud}|$

A speculative exercise on the error budget

$$\left|\frac{V_{us}}{V_{ud}}\right|^2 = \left[\frac{\Gamma(K_{\ell 2})}{\Gamma(\pi_{\ell 2})}\frac{M^3_{K^+}}{M^3_{\pi^+}}\frac{(M^2_{K^+}-M^2_{\mu^+})^2}{(M^2_{\pi^+}-M^2_{\mu^+})^2}\right]_{\text{exp}} \cdot \left[\frac{f_{K,0}}{f_{\pi,0}}\right]^2 (1+\delta R_{K\pi})$$

- Let us use $\delta R_{K\pi} = -0.0086\,(13)(39)_{\text{vol.}}$

$[f_{K,0}/f_{\pi,0}]$		$\|V_{us}/V_{ud}\|$
FLAG21 2+1 average	1.1930 (33)	$0.23154\ (28)_{\text{exp}}\ (15)_{\delta R}\ (45)_{\delta R,\text{vol.}}\ (65)_{f_P}$

- From RM123+Soton calculation $\delta R_{K\pi} = -0.0126\,(14)$

$[f_{K,0}/f_{\pi,0}]$		$\|V_{us}/V_{ud}\|$
FLAG19 2+1+1 average	1.1966 (18)	$0.23131\ (28)_{\text{exp}}\ (17)_{\delta R}\ (35)_{f_P}$

- the uncertainty on $[f_{K,0}/f_{\pi,0}]$ dominates in the error budget
- if improved, precision from **lattice** starts being competitive with the **experimental** one

Finite-volume QED effects

Leptonic decay rate

V. Lubicz et al., PRD 95 (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD 105 (2022)

$$\Gamma_0(L) = \Gamma_0^{\rm tree}\left\{1 + 2\,\frac{\alpha}{4\pi}\,Y(L)\right\}$$

$$Y(L) = Y_{\log}(L) + Y_0 + \frac{1}{m_P L}\,Y_1 + \frac{1}{(m_P L)^2}\,Y_2 + \frac{1}{(m_P L)^3}\,Y_3^{\rm pt} + \frac{1}{(m_P L)^3}Y_3^{\rm SD} + \mathrm{O}(1/L^4) + \mathrm{O}(\mathrm{e}^{-\alpha L})$$

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$$Y_{K\pi}^{(1)}(L/a = 48) \approx -3.96$$

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$$Y_{K\pi}^{(1)}(L/a = 48) \approx -3.96$$

57%

$$Y_{K\pi}^{(2)}(L/a = 48) \approx -6.20$$

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$Y^{(1)}_{K\pi}(L/a = 48) \approx -3.96$

57%

$Y^{(2)}_{K\pi}(L/a = 48) \approx -6.20$

−54%

$Y^{(3),\text{pt}}_{K\pi}(L/a = 48) \approx -2.83$

Finite volume scaling should be carefully studied!

Current status

Finite volume effects produce large **systematic uncertainty**

$$\delta R_{K\pi} = -0.0086\,(13)(39)_{\text{vol.}}$$

Current status

Finite volume effects produce large **systematic uncertainty**

$$\delta R_{K\pi} = -0.0086\,(13)(39)_{\text{vol.}}$$

repeat the calculation on **multiple volumes** & take **infinite volume limit**

$$\frac{1}{(m_P L)^3}\left[\textbf{structure-dependent}\right]$$

compute **missing effects** at $\mathcal{O}(1/L^3)$

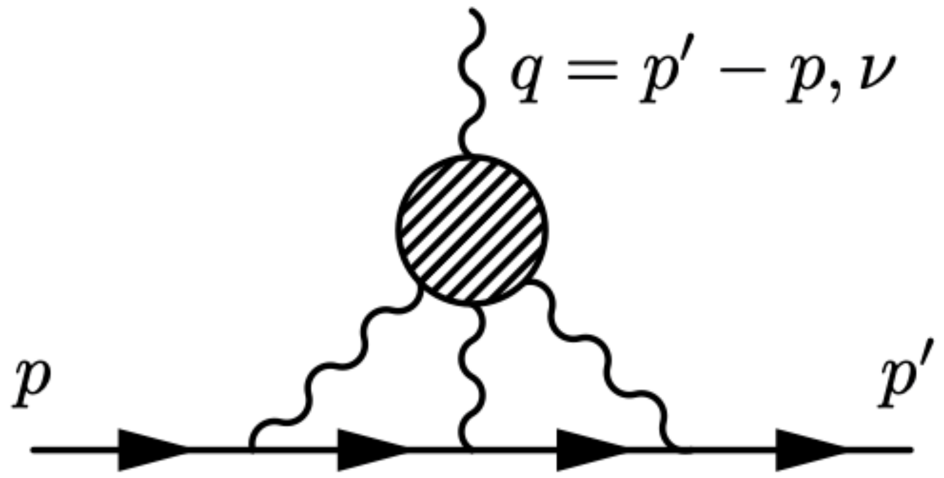
$$\left(\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\right)$$

adopt or **develop QED formulations** with **reduced** finite volume effects

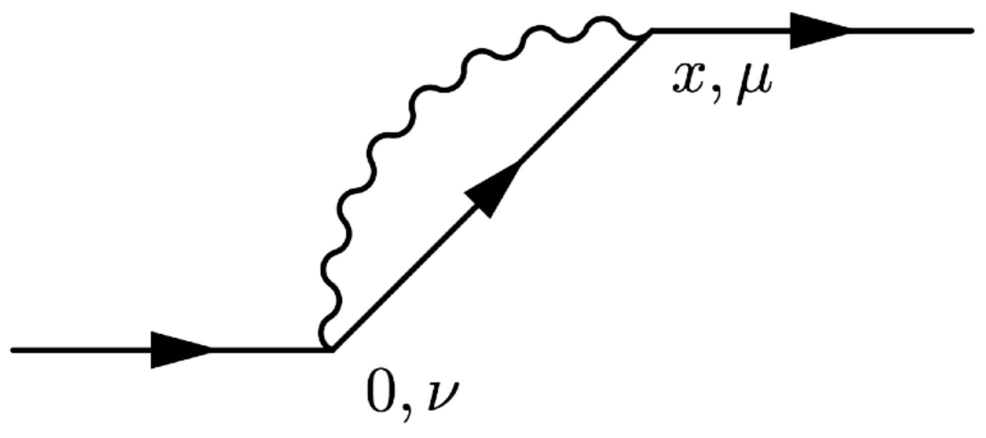
Infinite volume reconstruction

QED$_\infty$

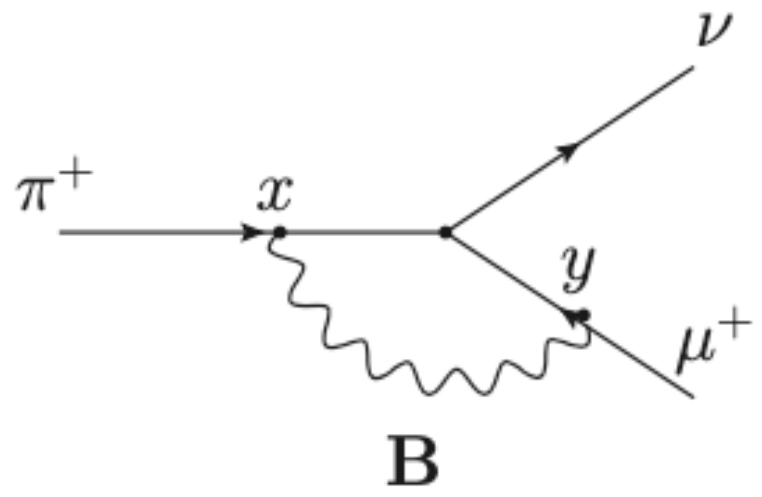
- An alternative approach is to compute radiative corrections as a **convolution** of **hadronic correlators** with **infinite-volume QED kernels**



N.Asmussen et al., [1609.08454]
T.Blum et al., PRD 96 (2017)



X.Feng & L.Jin, PRD 100 (2019)



N.Christ et al., [2304.08026]

and other quantities...

Infinite volume reconstruction

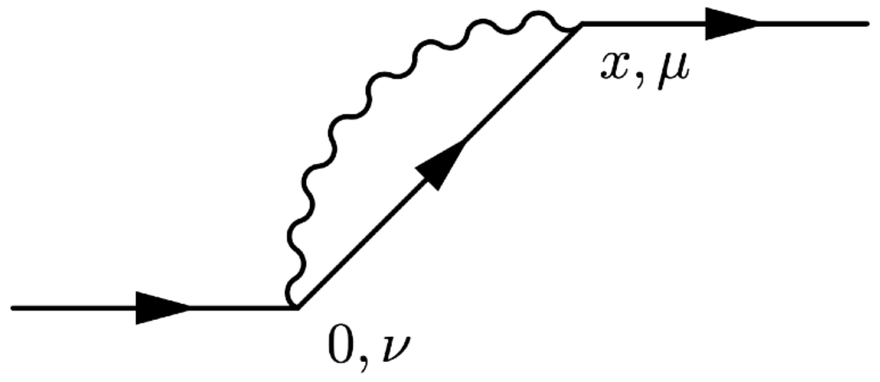
X.Feng & L.Jin, PRD 100 (2019)

QED$_\infty$

$$\Delta\mathcal{O} = \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\ \mathcal{H}(t,\mathbf{x})\, f_{\mathrm{QED}}(t,\mathbf{x})$$

single-hadron state dominance

Separate correlator into **short** and **long** distance part:



$$\Delta\mathcal{O} = \Delta\mathcal{O}^{(s)} + \Delta\mathcal{O}^{(l)}$$

$$\Delta\mathcal{O}^{(s)} = \frac{1}{2}\int_{-t_s}^{t_s} \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\ \mathcal{H}(t,\mathbf{x}) f_{\mathrm{QED}}(t,\mathbf{x}) \approx \frac{1}{2}\int_{-t_s}^{t_s} \mathrm{d}t \int_{L^3} \mathrm{d}^3\mathbf{x}\ \mathcal{H}^L(t,\mathbf{x}) f_{\mathrm{QED}}(t,\mathbf{x})$$

$$\Delta\mathcal{O}^{(l)} = \int_{t_s}^{\infty} \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\ \mathcal{H}(t,\mathbf{x}) f_{\mathrm{QED}}(t,\mathbf{x}) \approx \int_{L^3} \mathrm{d}^3\mathbf{x}\ \mathcal{H}^L(t_s,\mathbf{x})\, \mathcal{F}_{\mathrm{QED}}(t_s,\mathbf{x})$$

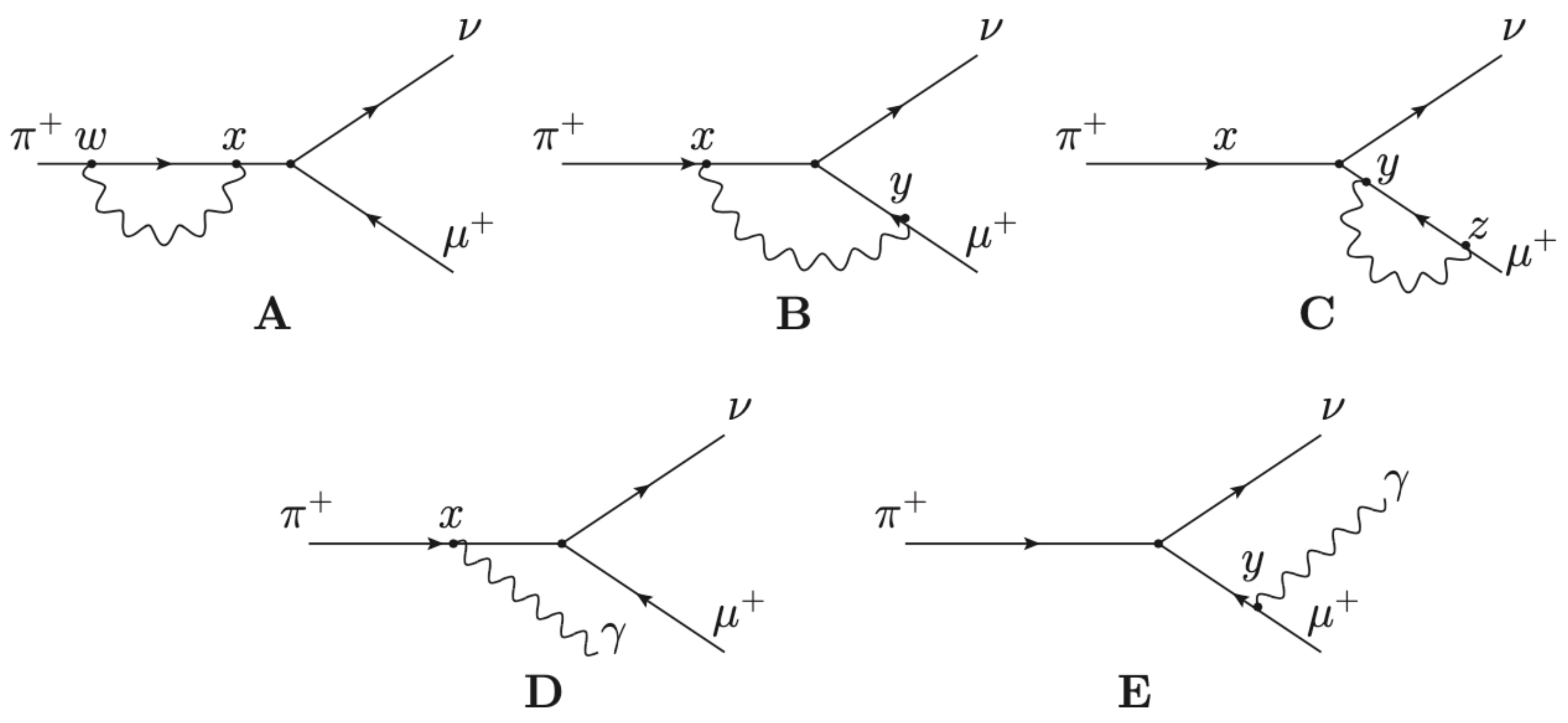
Exponentially suppressed

- finite-volume effects
- contributions of states with higher energy

Infinite volume reconstruction

N.Christ et al., [2304.08026]

QED∞



- Diagram A:

$$H^{(2)}_{\mu,\rho,\sigma}(t_1, t_2, \vec{x}) = \int d^3\vec{w}\, \langle 0|T\{J^W_\mu(0) J^{\mathrm{EM}}_\rho(t_1, \vec{w}+\vec{x}) J^{\mathrm{EM}}_\sigma(t_2, \vec{w})\}|\pi(\vec{0})\rangle$$

- Diagram B and D:

$$H^{(1)}_{\mu,\rho}(x) = H^{(1)}_{\mu,\rho}(x_t, \vec{x}) = \langle 0|T\{J^W_\mu(0) J^{\mathrm{EM}}_\rho(x)\}|\pi(\vec{0})\rangle$$

- Diagram C and E ($f_\pi \approx 130$ MeV):

$$H^{(0)}_\mu = H^{(0)}_t \delta_{\mu,t} = \langle 0|J^W_\mu(0)|\pi(\vec{0})\rangle = -i m_\pi f_\pi \delta_{\mu,t}$$

from Luchang Jin's talk @Edinburgh May 30, 2023

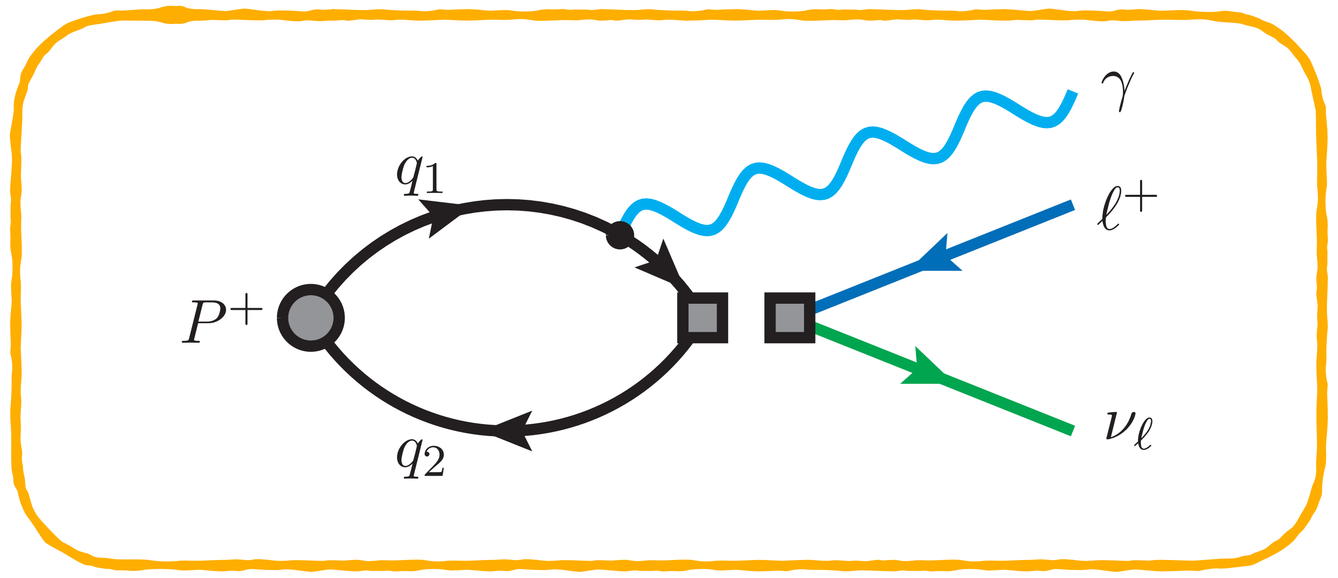
Method applied to **leptonic decay rates:**

- **Logarithmic IR divergences** appear
- **BUT they cancel analytically** between diagrams
- numerical calculation is **ongoing**...

The method is appealing given the large finite-volume effects in $\mathrm{QED_L}$ at $\mathrm{O}(1/L^3)$

... systematics under control?

real photon emission in leptonic decays



$$P^+ = \pi^+, K^+, D_s^+ \qquad \ell^+ = e^+, \mu^+$$

PRD 103, 014502 (2021)

First lattice calculation of radiative leptonic decay rates of pseudoscalar mesons

A. Desiderio,[1] R. Frezzotti,[1] M. Garofalo,[2] D. Giusti,[3,4] M. Hansen,[5] V. Lubicz,[2] G. Martinelli,[6] C. T. Sachrajda,[7] F. Sanfilippo,[4] S. Simula,[4] and N. Tantalo[1]

- first calculation of $P^+ \to \ell^+ \nu \gamma$ for pion and kaon + D_s in part of the kinematical range ($E_\gamma \lesssim 0.4$ GeV)

PRD 103, 053005 (2021)

Comparison of lattice QCD + QED predictions for radiative leptonic decays of light mesons with experimental data

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- comparison of lattice results with experimental measurements
- good agreement with KLOE on $K \to e \nu_e \gamma$
- 3-4σ tensions on $K \to \mu \nu_\mu \gamma$ (also among experiments)

PRD 107, 074507 (2023)

Methods for high-precision determinations of radiative-leptonic decay form factors using lattice QCD

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- study of $D_s^+ \to \ell^+ \nu \gamma$ with different "3d" method
- improved control of systematic uncertainties
- but single lattice spacing

arXiv:2306.05904

Lattice calculation of the D_s meson radiative form factors over the full kinematical range

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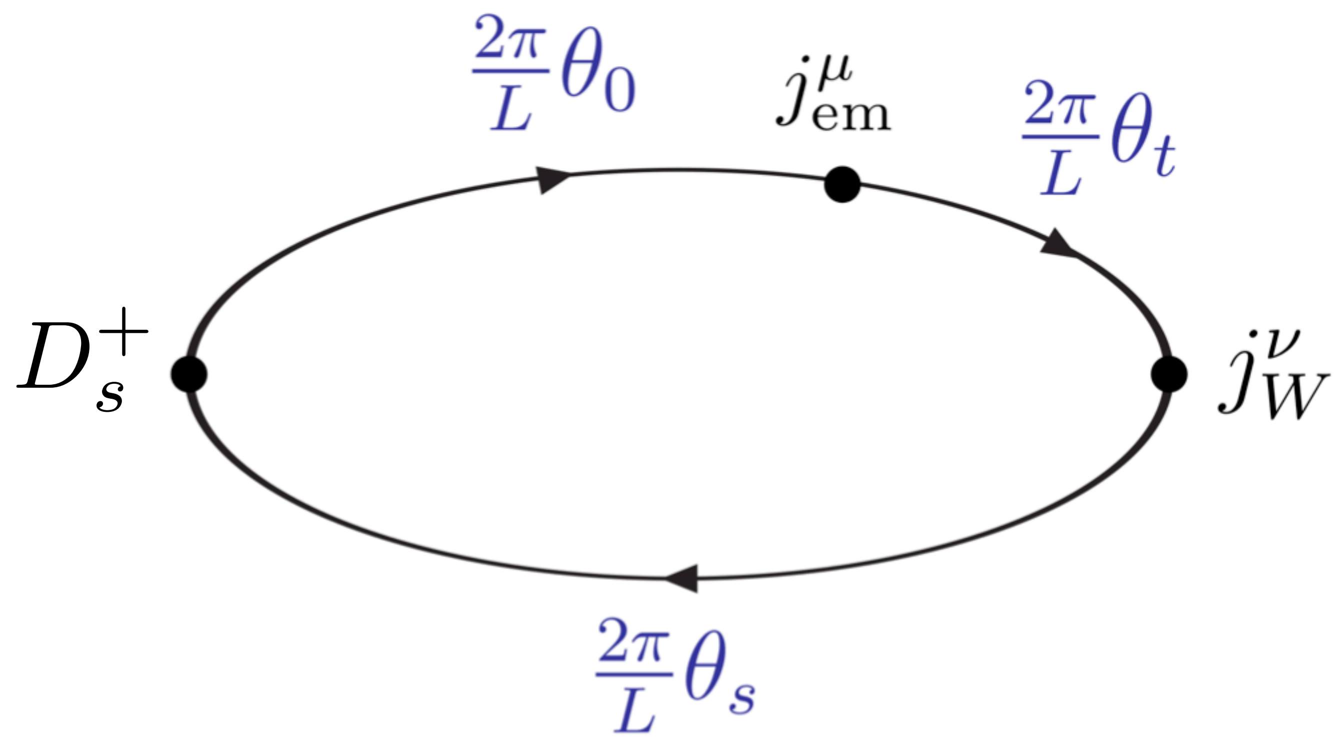
- new calculation of $D_s^+ \to \ell^+ \nu \gamma$ on full kinematical range

The hadronic matrix element

$$H_W^{r\nu}(k,\boldsymbol{p}) = \epsilon_\mu^r(k)\, H_W^{\mu\nu}(k,\boldsymbol{p}) = \epsilon_\mu^r(k) \int d^4y\, e^{ik\cdot y}\, \langle 0|\, \hat{\mathrm{T}}[\, j_W^\nu(0) j_{\mathrm{em}}^\mu(y)] \,|D_s^+(\boldsymbol{p})\rangle$$

$$H_W^{\mu\nu}(k,\boldsymbol{p}) \; = \; H_{\mathrm{SD}}^{\mu\nu}(k,\boldsymbol{p}) + H_{\mathrm{pt}}^{\mu\nu}(k,\boldsymbol{p})$$



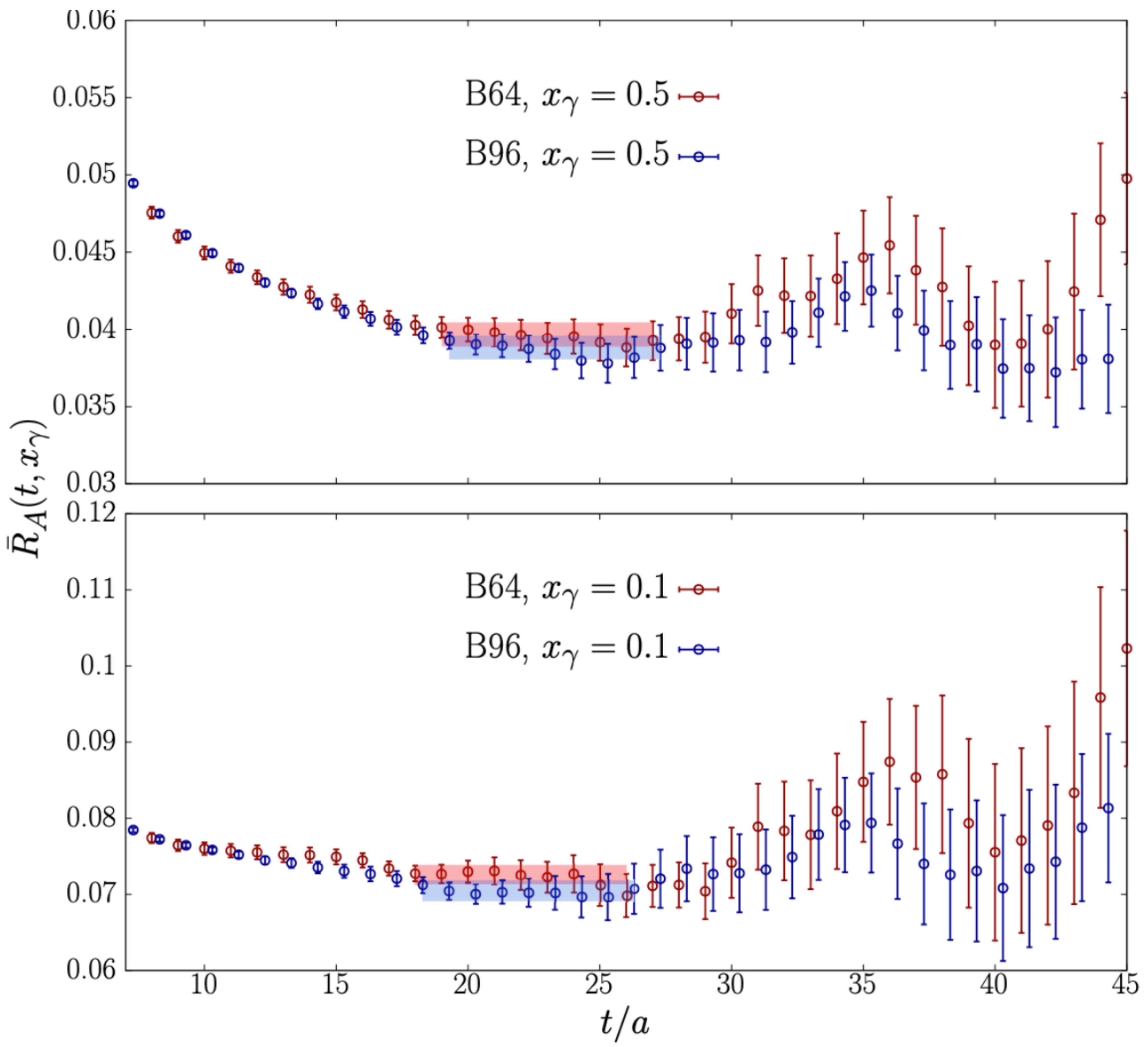
$$H_{\mathrm{SD}}^{\mu\nu}(k,\boldsymbol{p}) \; = \; \frac{H_1(p\cdot k, k^2)}{M_{D_s}} \left[k^2 g^{\mu\nu} - k^\mu k^\nu\right] + \frac{H_2(p\cdot k, k^2)}{M_{D_s}} \frac{\left[(p\cdot k - k^2)k^\mu - k^2(p-k)^\mu\right]}{(p-k)^2 - M_{D_s}^2}(p-k)^\nu$$
$$-i\frac{F_V(p\cdot k, k^2)}{M_{D_s}}\varepsilon^{\mu\nu\gamma\beta}k_\gamma p_\beta + \frac{F_A(p\cdot k, k^2)}{M_{D_s}} \left[(p\cdot k - k^2)g^{\mu\nu} - (p-k)^\mu k^\nu\right]$$

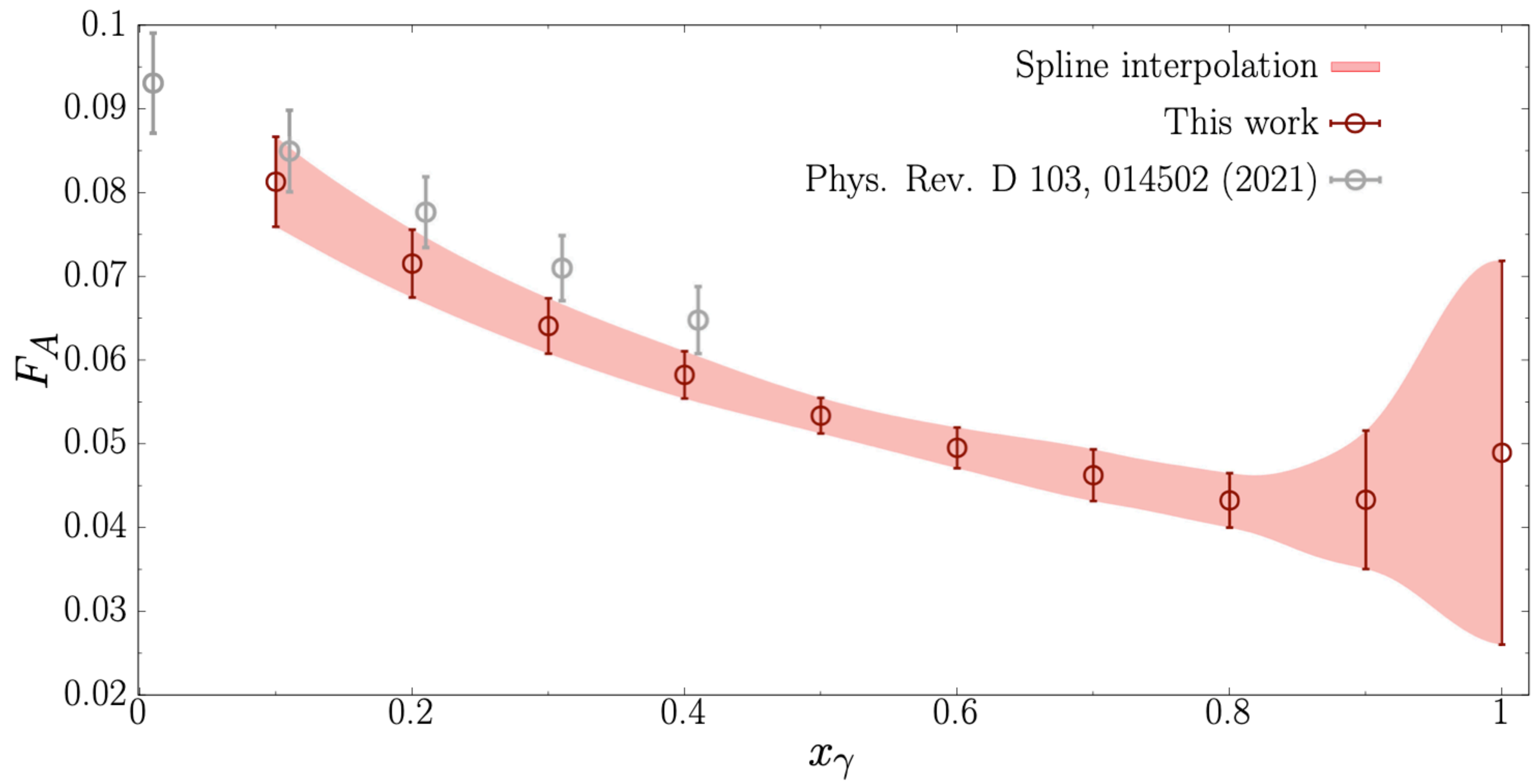
$$H_{\mathrm{pt}}^{\mu\nu}(k,\boldsymbol{p}) \; = \; f_{D_s}\left[g^{\mu\nu} + \frac{(2p-k)^\mu(p-k)^\nu}{2p\cdot k - k^2}\right] \; ,$$

Extraction of the form factors

Axial form factor F_A

$$R_A(t,\boldsymbol{k}) \equiv \frac{1}{2E_\gamma}\left[\left(R_A^{11}(t,\boldsymbol{k},\mathbf{0}) - R_A^{11}(t,\mathbf{0},\mathbf{0})\right) + \left(R_A^{22}(t,\boldsymbol{k},\mathbf{0}) - R_A^{22}(t,\mathbf{0},\mathbf{0})\right)\right] \xrightarrow[0 \ll t \ll T/2]{} F_A(x_\gamma)$$



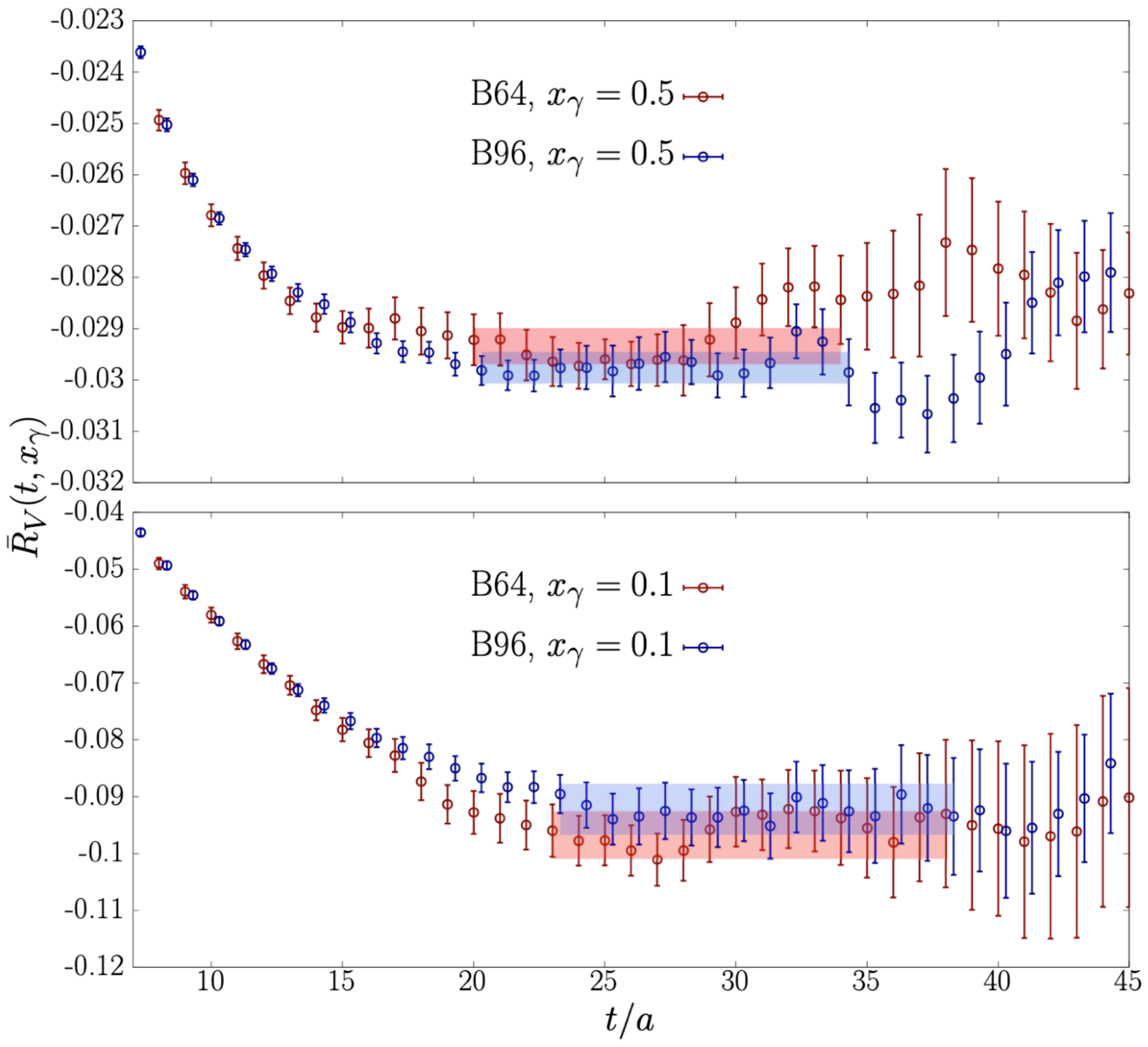


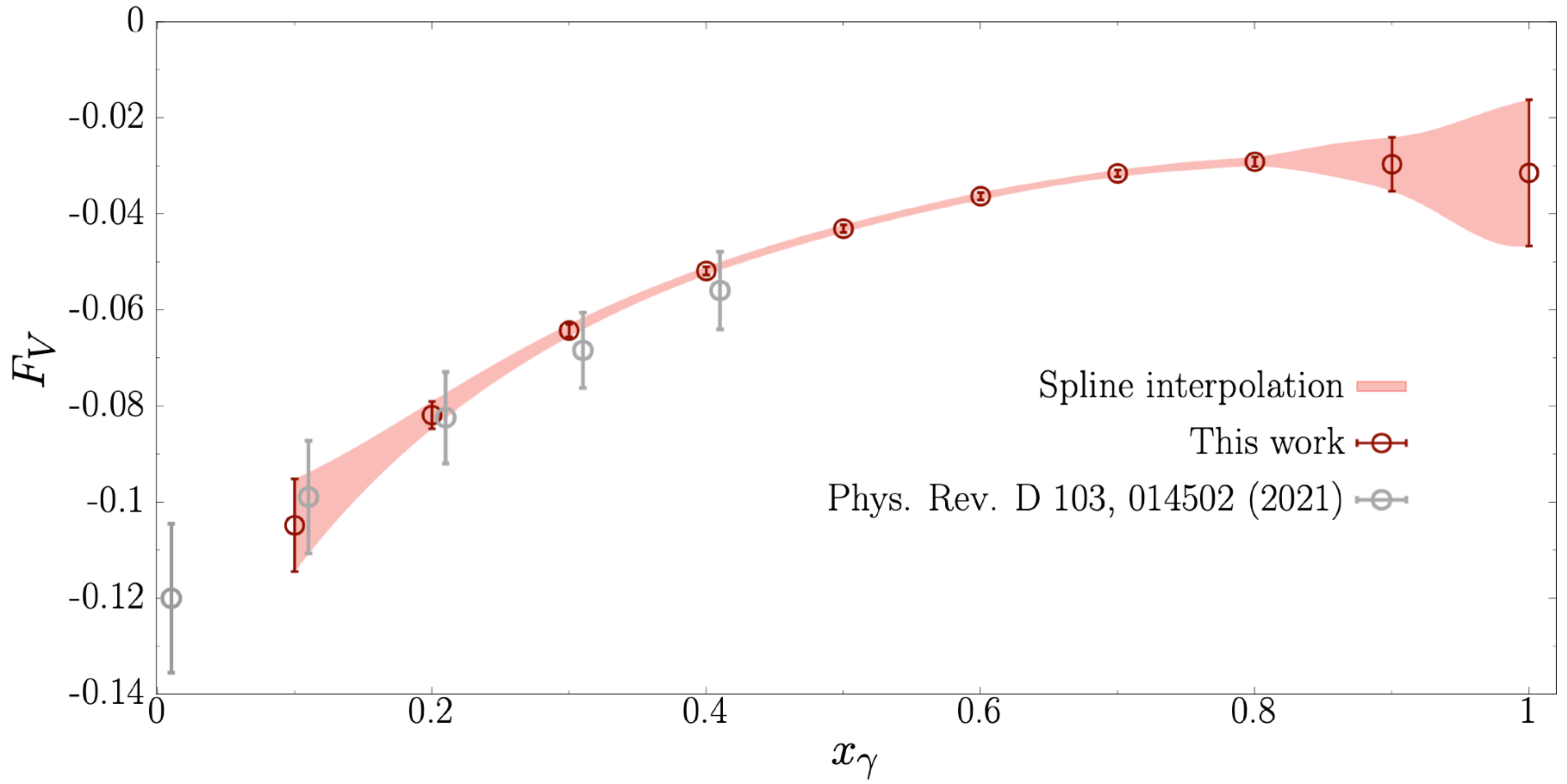
R.Frezzotti et al., [2306.05904]

Extraction of the form factors

Vector form factor F_V

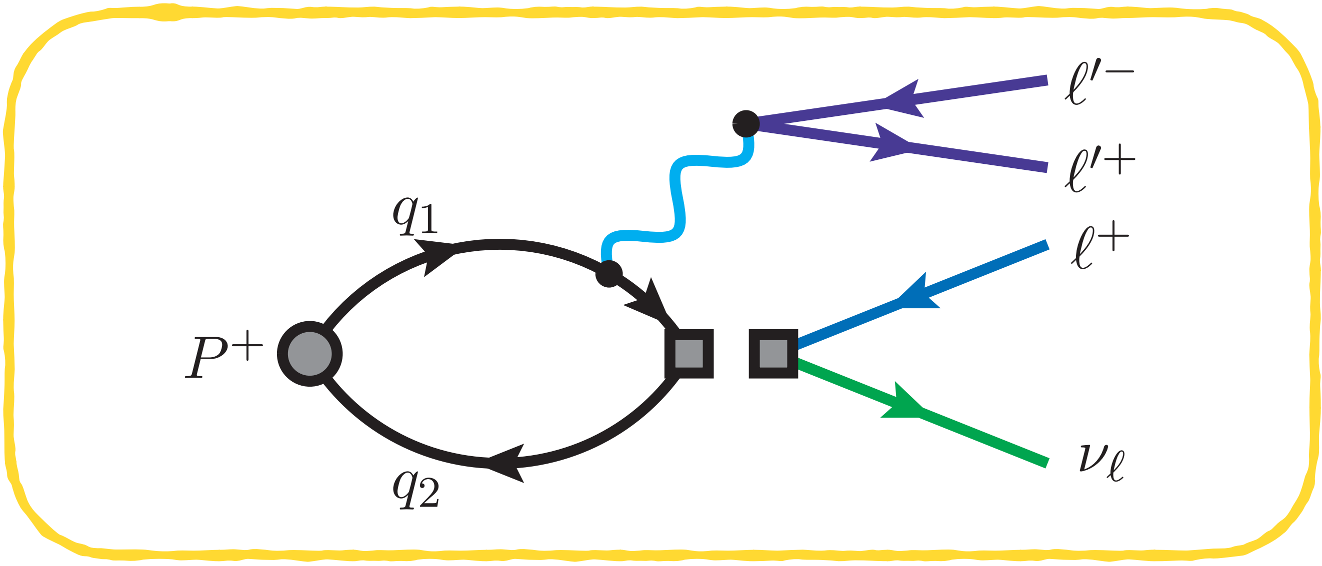
$$R_V(t,\boldsymbol{k}) \equiv \frac{1}{2k_z}\left(R_V^{12}(t,\boldsymbol{k},\boldsymbol{0}) - R_V^{21}(t,\boldsymbol{k},\boldsymbol{0})\right) \xrightarrow[0\ll t\ll T/2]{} F_V(x_\gamma)$$





R.Frezzotti et al., [2306.05904]

virtual photon emission in leptonic decays



$P^+ = D_s^+$ $\quad\quad$ $\ell^+ = e^+, \mu^+$

material from Giuseppe Gagliardi's talk @Nepsi23 (Pisa Feb 16, 2023)

yesterday on arXiv: R.Frezzotti et al., [2306.07228]

Analytical continuation to Euclidean spacetime

from G.Gagliardi@Nepsi23

$$H_W^{\mu\nu}(k,0) = \underbrace{\int_{-\infty}^{0} dt\, e^{iE_\gamma t}\big\langle 0\big| J_W^\nu(0) J_{\rm em}^\mu(t,\boldsymbol{k})\big|P(\boldsymbol{0})\big\rangle}_{H_{W,1}^{\mu\nu}(k)} + \underbrace{\int_{0}^{\infty} dt\, e^{iE_\gamma t}\big\langle 0\big| J_{\rm em}^\mu(t,\boldsymbol{k}) J_W^\nu(0)\big|P(\boldsymbol{0})\big\rangle}_{H_{W,2}^{\mu\nu}(k)}$$

Inserting a complete set of states between the two currents:

$$H_{W,1}^{\mu\nu}(k) = -i\sum_r \frac{\langle 0|J_W^\nu(0)|r\rangle\langle r|J_{\rm em}^\mu(\boldsymbol{k})|P(\boldsymbol{0})\rangle}{E_r + E_\gamma - M_P - i\epsilon}, \quad \boldsymbol{p}_r = -\boldsymbol{k}, \quad |r\rangle = \bar{D}\gamma^\nu U,\ \bar{D}\gamma^\nu\gamma^5 U$$

$$H_{W,2}^{\mu\nu}(k) = -i\sum_n \frac{\langle 0|J_{\rm em}^\mu(\boldsymbol{k})|n\rangle\langle n|J_W^\nu(0)|P(\boldsymbol{0})\rangle}{E_n - E_\gamma - i\epsilon}, \quad \boldsymbol{p}_n = +\boldsymbol{k}, \quad |n\rangle = \bar{D}\gamma^\mu D,\ \bar{U}\gamma^\mu U$$

- 1st TO: $E_r \geq \sqrt{M_P^2 + |\boldsymbol{k}|^2} \implies E_r + E_\gamma - M_P \geq 0$ ✓.
- 2nd TO: $E_n - E_\gamma < 0$ if $\sqrt{k^2} > M_n$ [mass of the vector state $|n\rangle$] ✗.

Threshold at: $\sqrt{k_{th}^2} = \min(M_{V_U}, M_{V_D}) \implies E_{\gamma,th} = \sqrt{k_{th}^2 + |\boldsymbol{k}|^2}$.

Reconstruction of smeared hadronic amplitudes

First calculation done in G.Gagliardi et al., Phys. Rev. D **105** (2022) for kaon decay with unphysical setup: $m_K < 2m_\pi$ such that two-pion internal states are always heavier than the external state

Proposal: use $i\epsilon$-prescription as smearing parameter

J.Bulava & M.T.Hansen, PRD **100** (2019)
R.Briceño et al., PRD **101** (2020)

$$\mathrm{Re/Im}\,\left[iH^{\mu\nu;s}_{W,2}(E_\gamma,\boldsymbol{k},\epsilon)\right] = \int_0^\infty dE'\,\rho^{\mu\nu;s}_{W,2}(E',\boldsymbol{k})\,K_{\mathrm{Re/Im}}(E'-E_\gamma,\epsilon)$$

$$C^{\mu\nu;s}_{W,2}(t,\boldsymbol{k}) = \int_0^\infty dE'\,\rho^{\mu\nu;s}_{W,2}(E',\boldsymbol{k})\,e^{-E't}$$

$$C^{\mu\nu}_{W,2}(t,\boldsymbol{k}) \equiv \langle 0|\,J^\mu_{\mathrm{em}}(t,\boldsymbol{k})\;J^\nu_W(0)\,|P(\boldsymbol{0})\rangle$$

$$K_{\mathrm{Re}}(x,\epsilon) = \frac{x}{x^2+\epsilon^2}$$

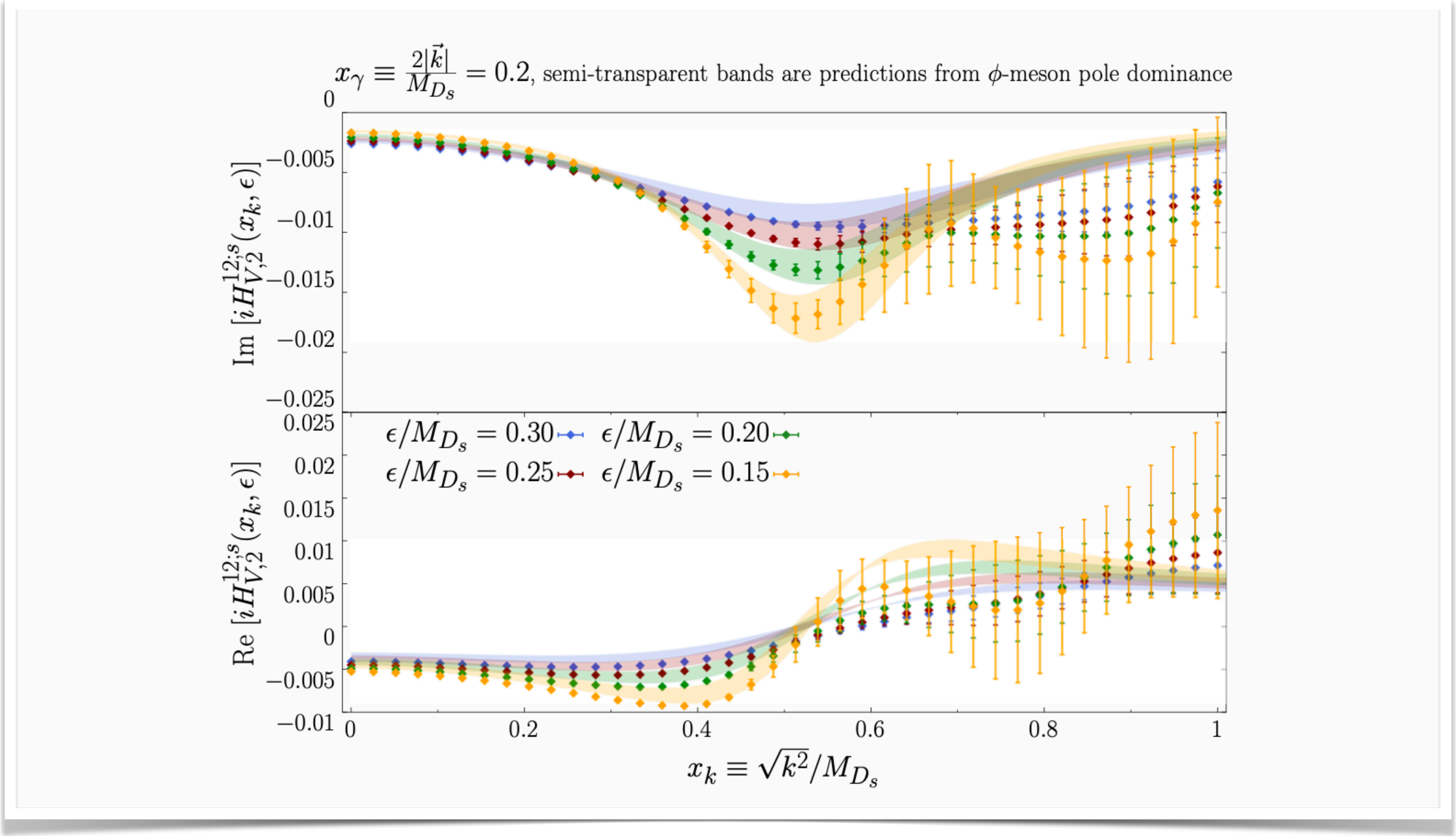
$$K_{\mathrm{Im}}(x,\epsilon) = \frac{\epsilon}{x^2+\epsilon^2}$$

- HTL method to reconstruct $H(E_\gamma,\boldsymbol{k},\epsilon)$ from $C(t,\boldsymbol{k})$ M.Hansen, N.Tantalo & A.Lupo PRD **99** (2019)

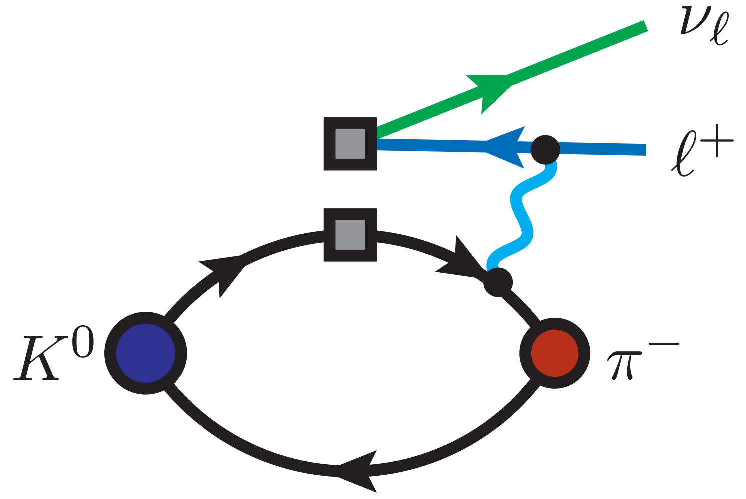
Reconstruction of smeared hadronic amplitudes

from G.Gagliardi@Nepsi23

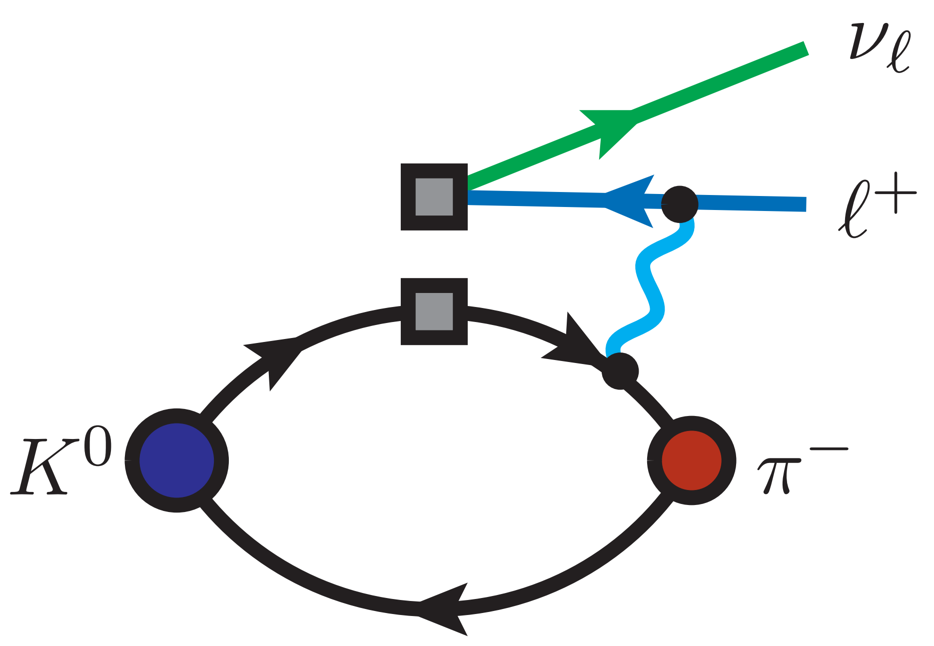
Vector part of the hadronic tensor

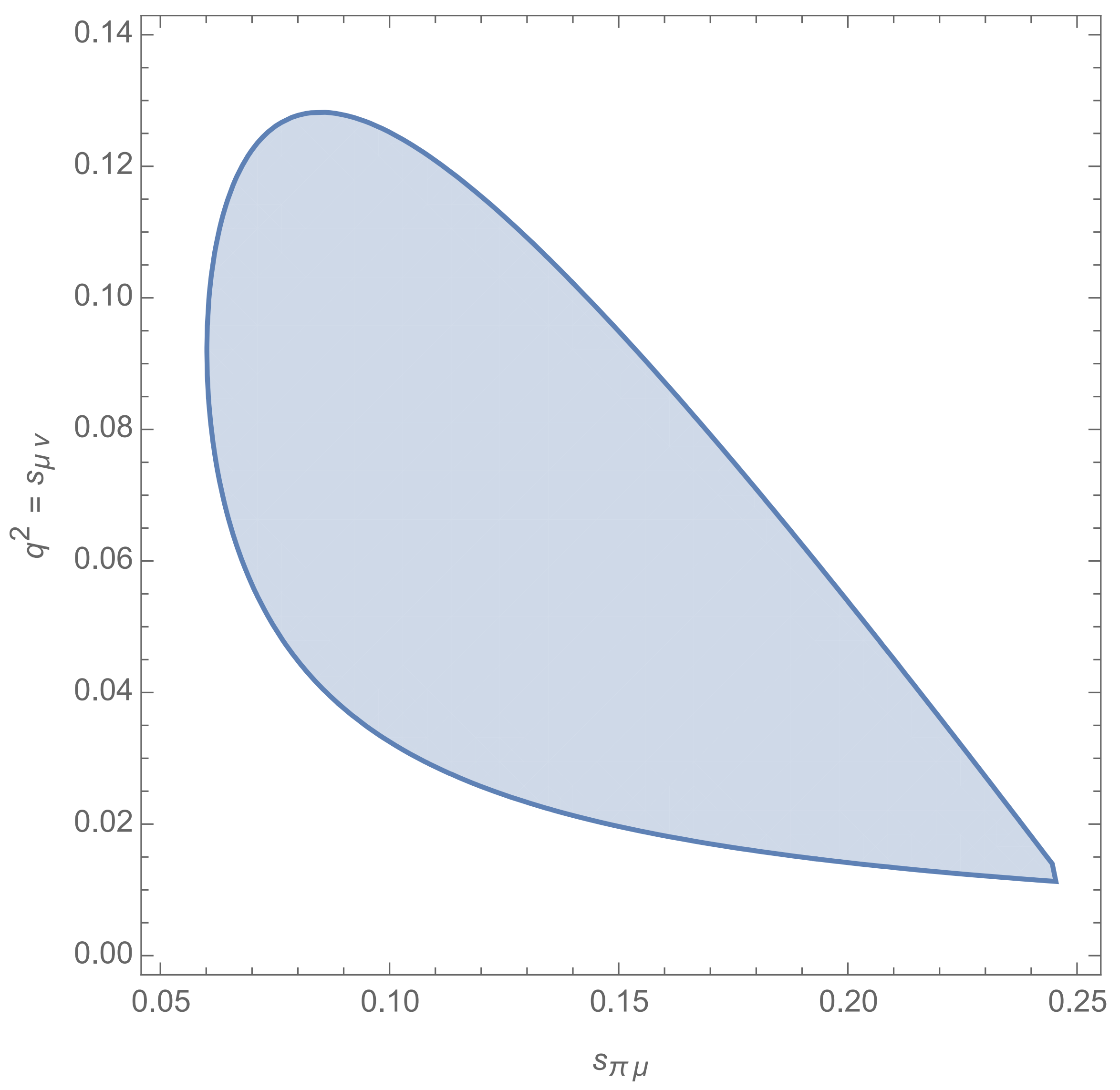


inclusive semi-leptonic decays



QED corrections to semileptonic decays





Additional **difficulties** arise compared to **leptonic decays**:

- integration over **three-body phase-space**
- problems of **analytical continuation** when intermediate states lighter than external ones go on shell:

$$e^{-(\omega^{\text{int}}_{\pi\ell} - \omega^{\text{ext}}_{\pi\ell})(t_{\pi\ell} - t_{\text{H}})}$$

growing exponentials if $\omega^{\text{int}}_{\pi\ell} < \omega^{\text{ext}}_{\pi\ell}$

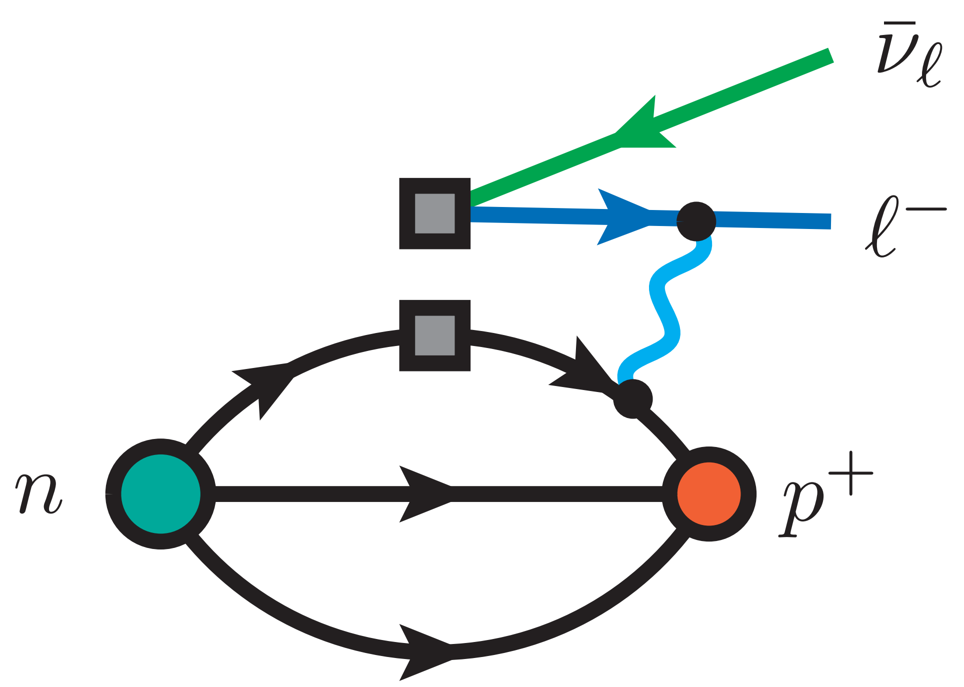
These states should be **identified** and **subtracted**.

All becomes more problematic for decays of heavy mesons!

... spectral reconstruction?

... infinite-volume QED?
N.Christ et al., [2304.08026]

QED corrections to semileptonic decays



beta decay?

the **"Holy grail"**

cit. G.Martinelli

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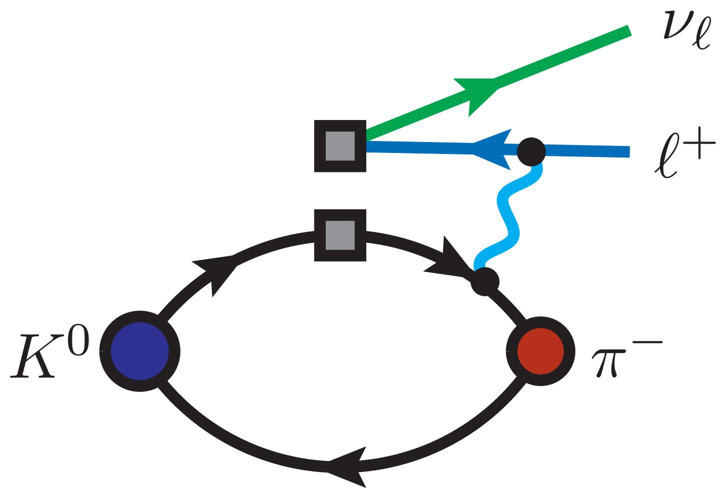
4. Where are we ...

- Current **tensions** in CKM unitarity require a combined effort of **theory** and **experiments**
- **New lattice results** for $\delta R_{K\pi}$ from calculation with DWF at the physical point
- **Finite volume effects** have to be carefully studied, including order $1/L^3$
(looking forward to seeing results with different QED prescriptions: $\mathrm{QED_C}$, $\mathrm{QED_m}$, $\mathrm{QED_\infty}$)
- Many **processes** can be studied and many **techniques** are being developed

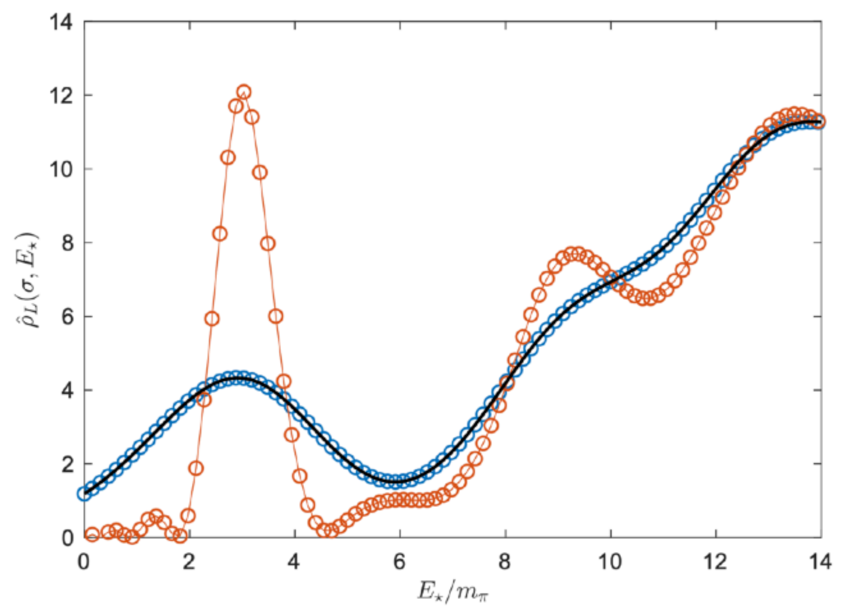
... and where to go?

$$\left(\frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\right)$$

investigate & tame finite-volume effects



study different weak processes



further explore spectral reconstruction techniques

Thank you