Sum rules and other tools for comparing the dispersive and lattice HVP contributions to g - 2

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in coll. with: Genessa Benton, Maarten Golterman, Alex Keshavarzi, Kim Maltman, Santi Peris

DB, Golterman, Maltman, and Peris, 2203.05070, Phys. Rev. D **105** 9 (2022)
DB, Golterman, Maltman, and Peris, 2211.11055, Phys. Rev. D **107** 7 (2023)
DB, Golterman, Maltman, and Peris, 2210.13677, Phys. Rev. D **107** 3 (2023)
Benton, DB, Golterman, Keshavarzi, Maltman, and Peris, in preparation

June 13, 2023

Instituto de Física Teórica UAM-CSIC

Lattice Gauge Theory Contributions to New Physics Searches

1. Notation and definitions

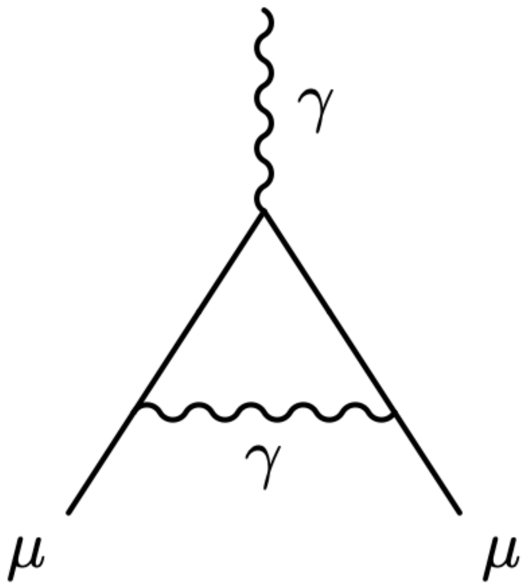
2. Light-quark connected and strange plus light-quark disconnected results from data

3. Sum rules

Muon g - 2: Standard Model

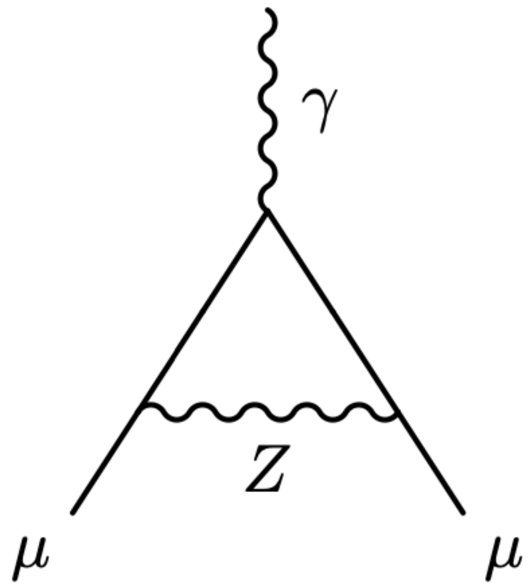
dispersive result

g-2 Theory Initiative
white paper 2006.04822



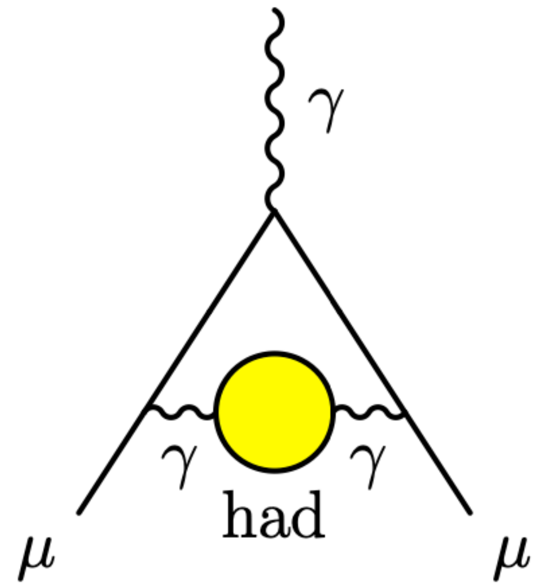
$116\,584\,718.931(104)$

$a_\mu^{\rm QED}$



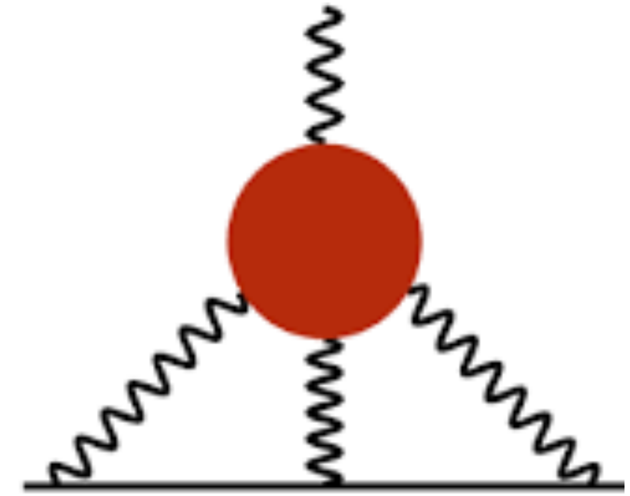
$153.6(1.0)$

$a_\mu^{\rm EW}$



$6931(40)$

$a_\mu^{\rm HVP,LO}$



$92(19)$

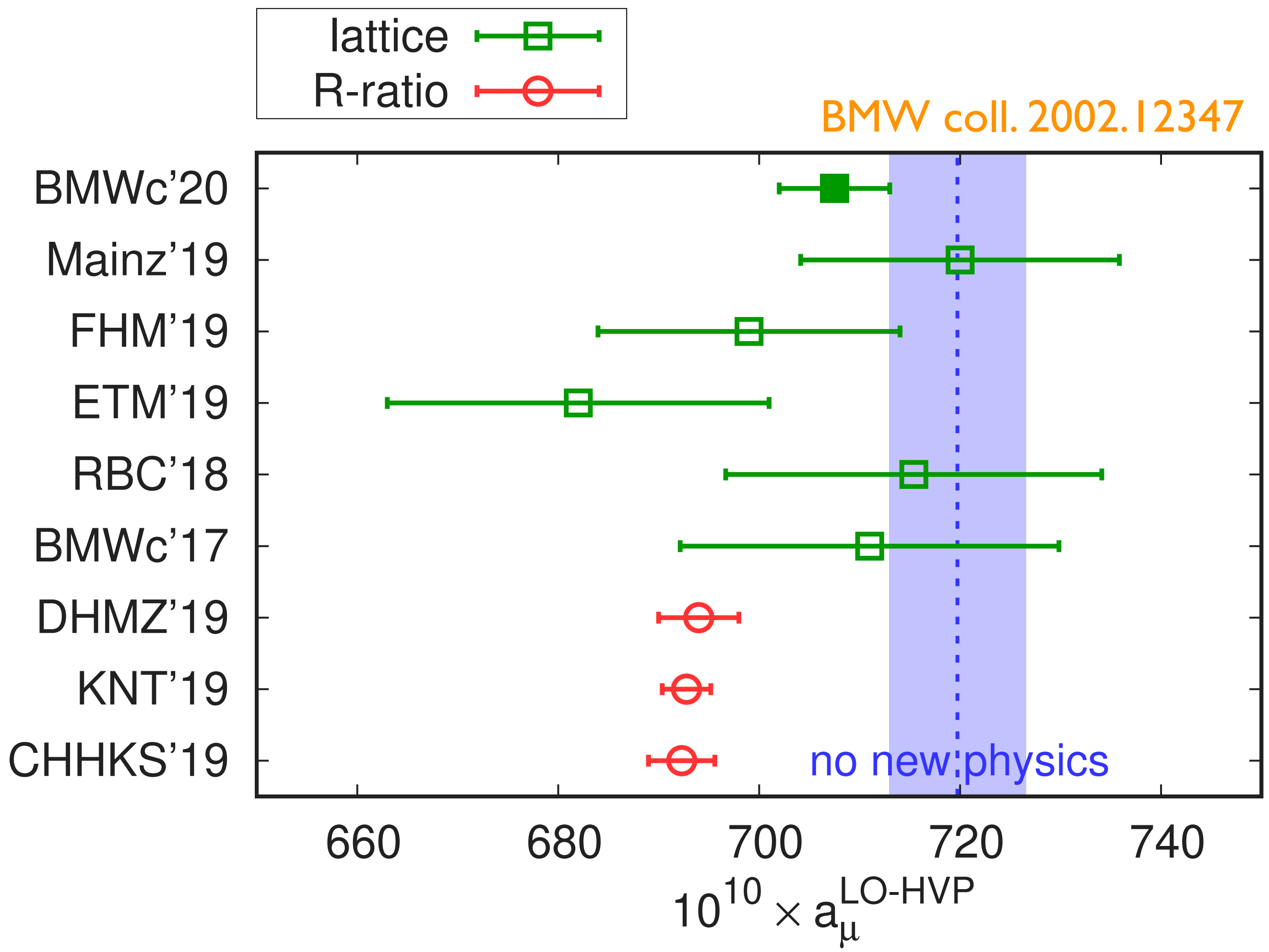
$a_\mu^{\rm HLbL,preno.}$

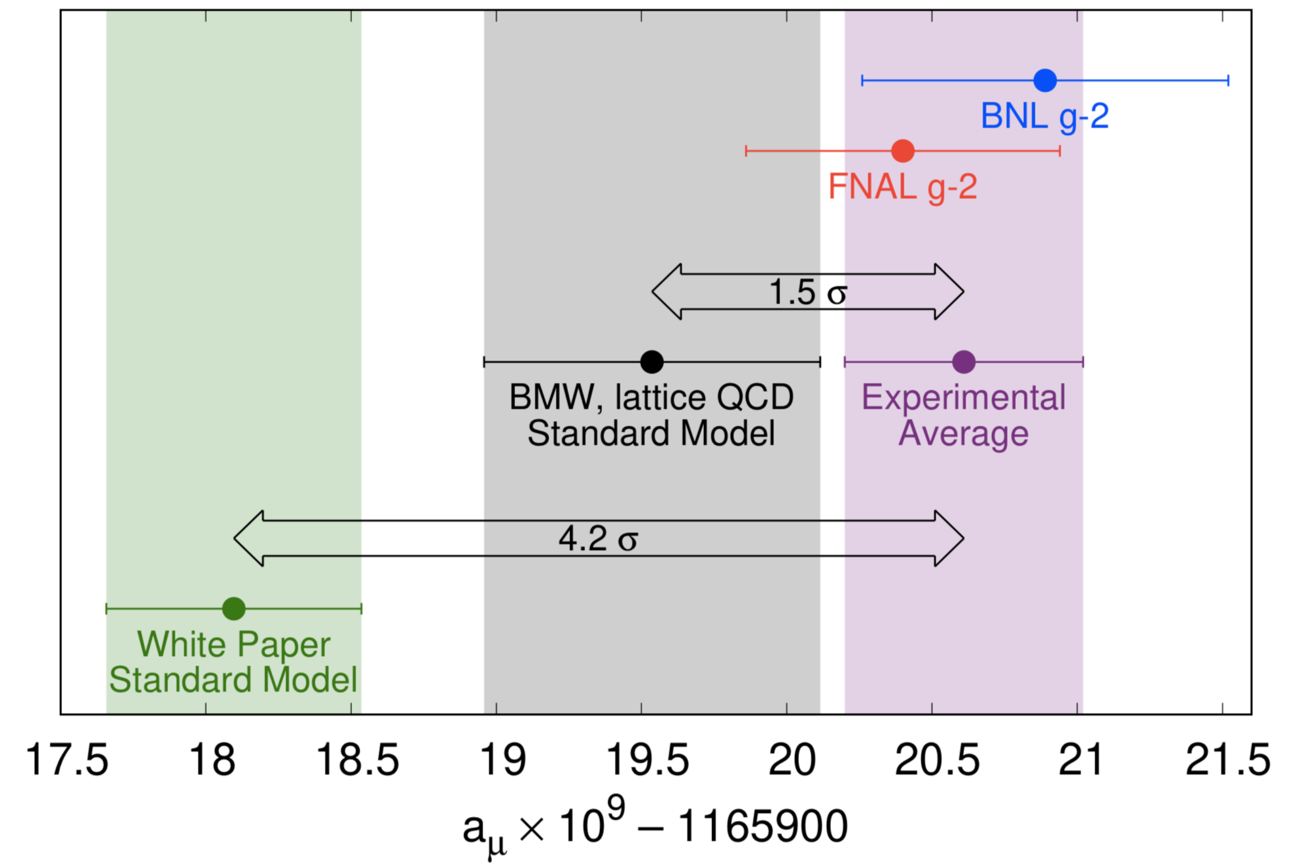
$$a_\mu^{\rm SM} = a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm HVP,\,LO} + a_\mu^{\rm HVP,\,NLO} + a_\mu^{\rm HVP,\,NNLO} + a_\mu^{\rm HLbL} + a_\mu^{\rm HLbL,\,NLO}$$
$$= 116\,591\,810(43)$$

$$a_\mu^{\rm exp.\ avg} \times 10^{11} = 116592061(41)$$

HVP contributions dominate the theory uncertainty.
Most important piece in controlling the SM g-2 result.

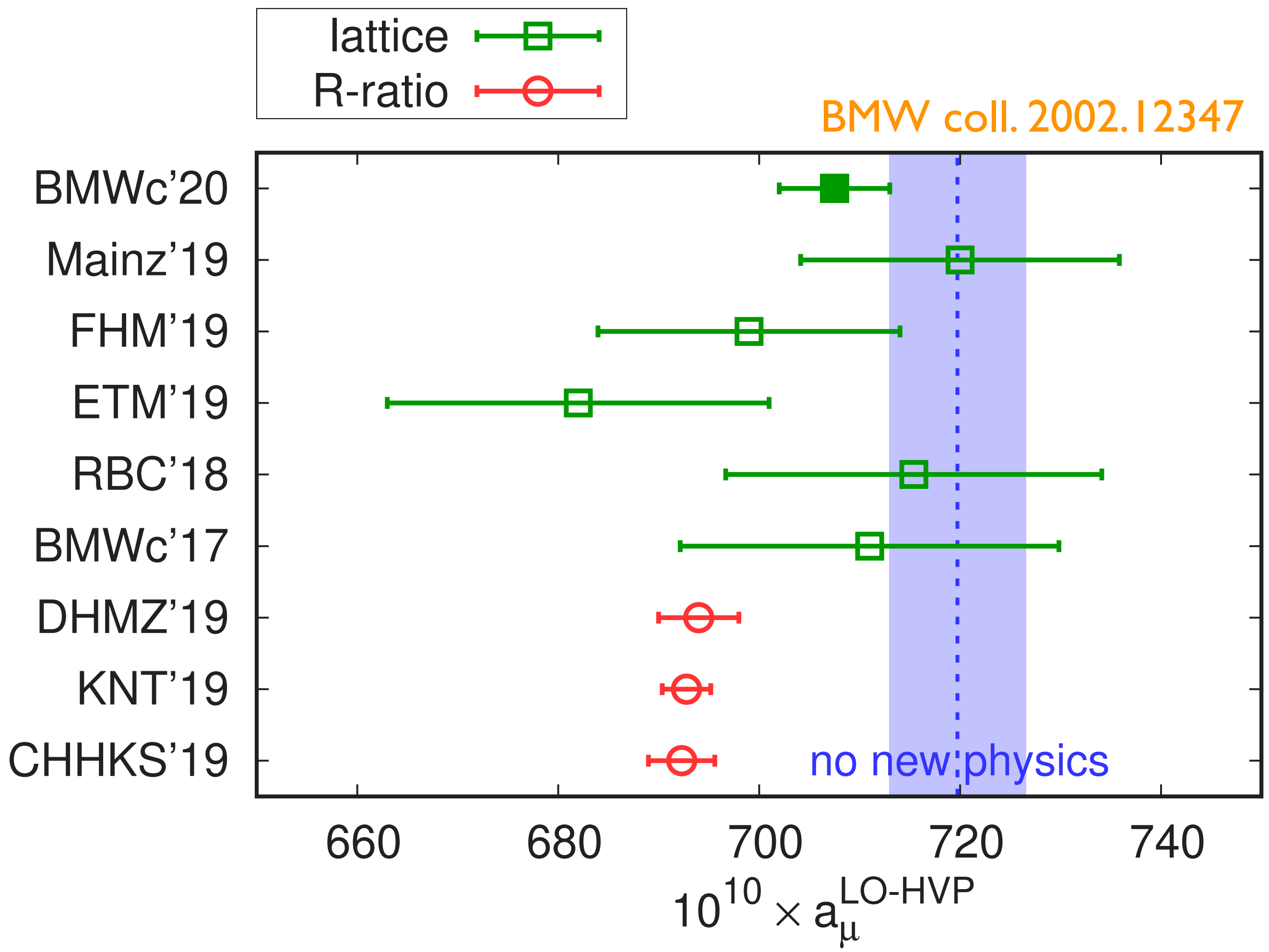
Muon g - 2: lattice vs dispersive HVP

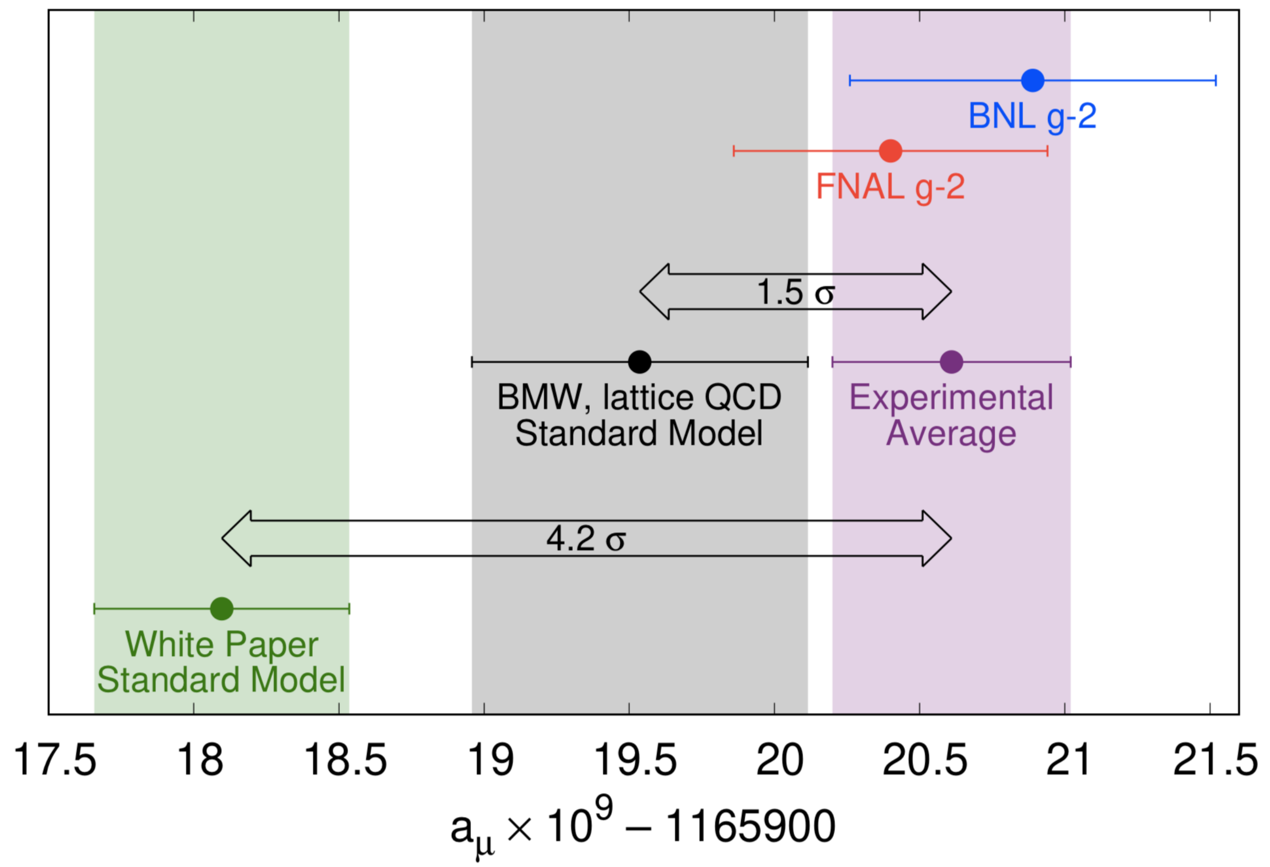




Crucial issue: ***discrepancy between lattice and dispersive results for HVP***

Muon *g* - 2: lattice vs dispersive HVP





Crucial issue: ***discrepancy between lattice and dispersive results for HVP***

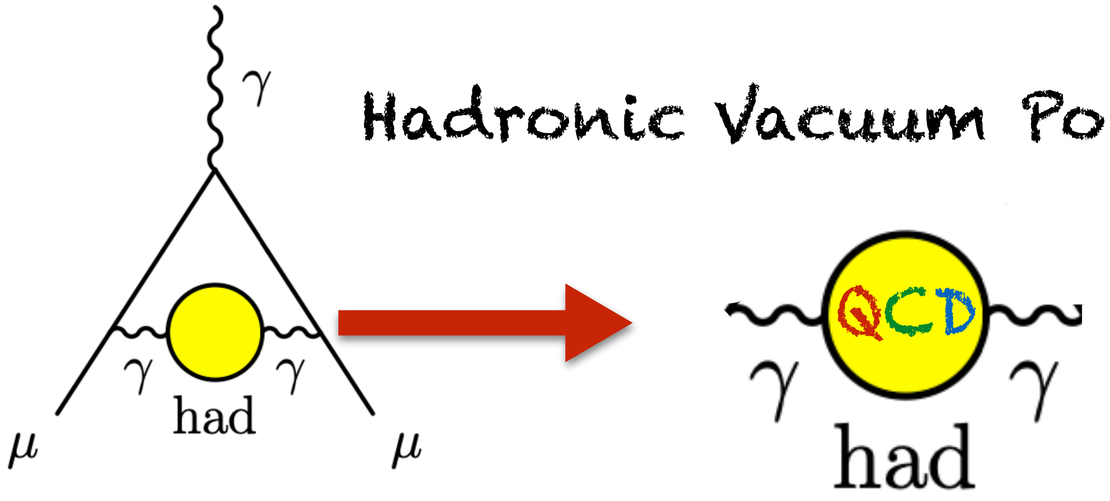
Problem: understand the origin of the discrepancy in detail.

Specific contributions (light-quark connected, disconnected, strange quark…)?

Specific energy regions or channels on the dispersive side?

Complicated comparison: Euclidean time correlators (lattice) versus time-like region data

Notation and conventions: dispersive approach



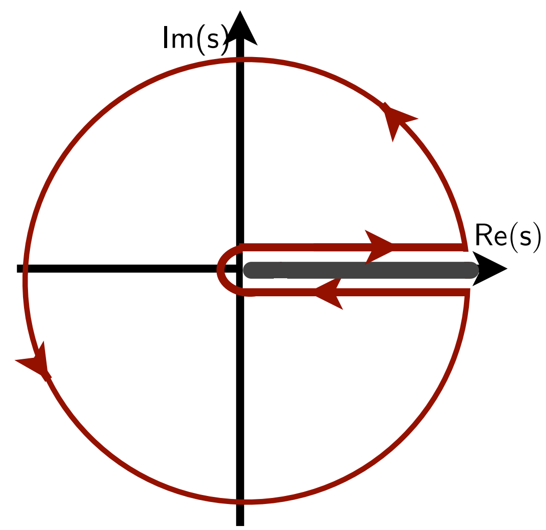
Hadronic Vacuum Polarization (HVP)

$$(q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2) = i\int d^4x\,\langle 0|T(j_\mu^{\rm EM}(x)j_\nu^{\rm EM}(0)|0\rangle$$

$$j_\mu^{\rm EM} = \frac{2}{3}\bar u\gamma_\mu u - \frac{1}{3}\bar d\gamma_\mu d - \frac{1}{3}\bar s\gamma_\mu s + \frac{2}{3}\bar c\gamma_\mu c + \cdots$$

Usual dispersive representation

$$\Pi(q^2) - \Pi(0) = q^2\int_{4m_\pi^2}^{\infty} ds\,\frac{\frac{1}{\pi}{\rm Im}\Pi(s)}{s(s-q^2+i\epsilon)}$$



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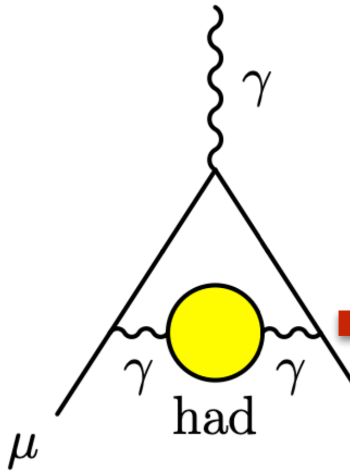

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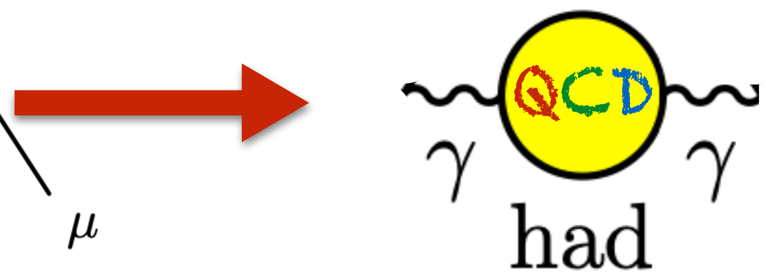

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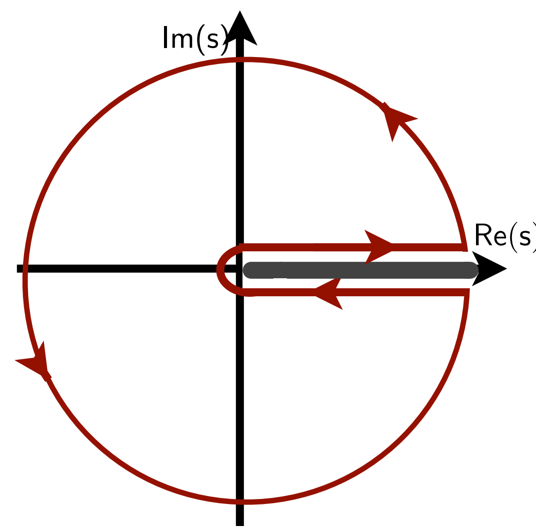

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Optical theorem relates the imaginary part to the cross section for $e^+e^- \to \mathrm{hadrons}(+\gamma)$

$$R(s) = \frac{3s}{4\pi\alpha}\sigma^{(0)}[e^+e^- \to \mathrm{hadrons}(+\gamma)]$$

Leading order contribution to a_μ^{HVP}

$$a_\mu^{\mathrm{HVP}} = \frac{4\alpha^2 m_\mu^2}{3} \int_{m_\pi^2}^{\infty} ds \, \frac{\hat{K}(s)}{s^2} \rho_{\mathrm{EM}}(s)$$

$$\rho_{\mathrm{EM}}(s) = \frac{1}{12\pi^2} R(s)$$

Brodsky & de Rafael, '68
Lautrup & de Rafael '68

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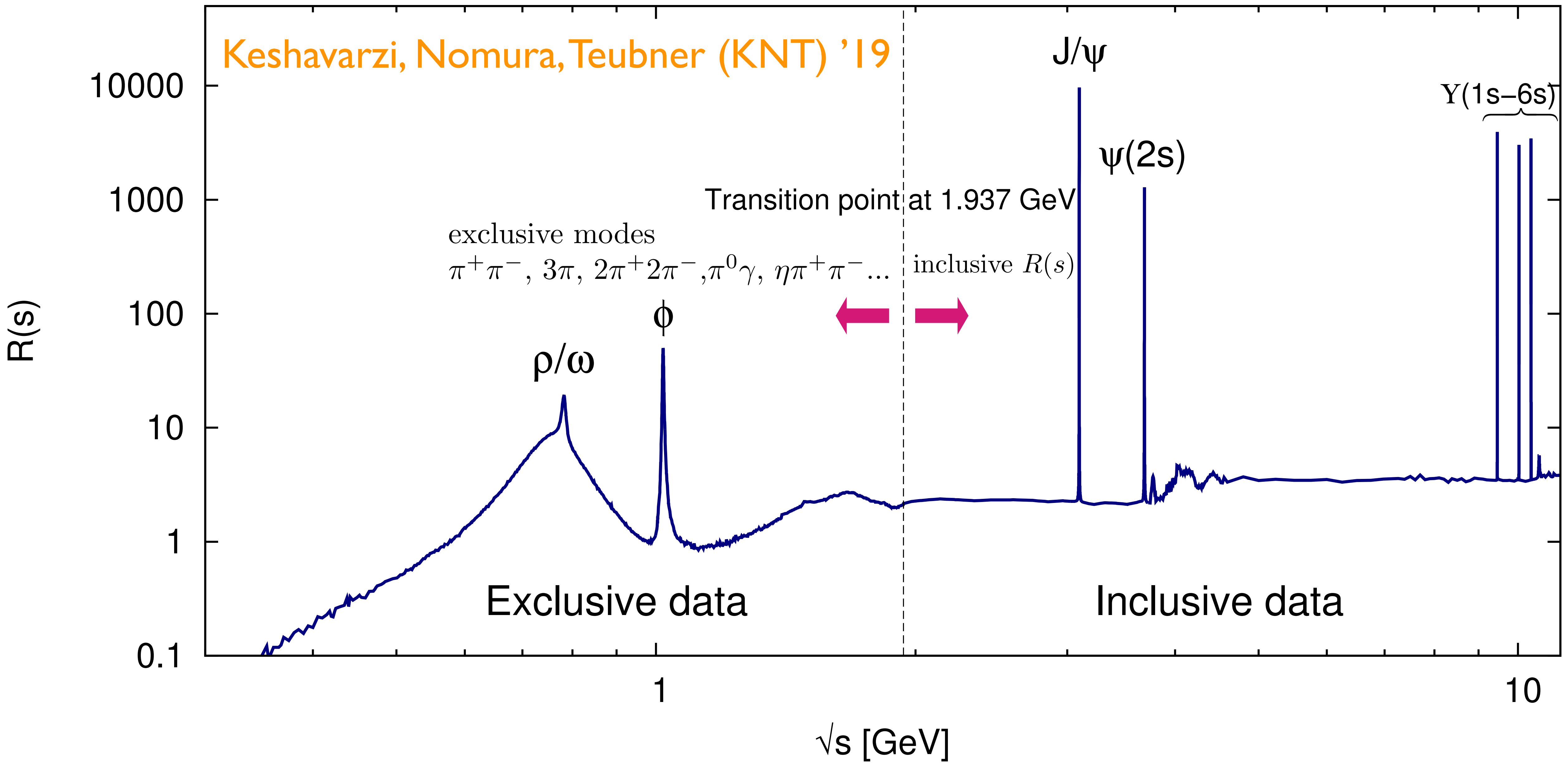
Compilation of *R*-ratio data

(see also Davier et al '19, Jegerlehner '16)



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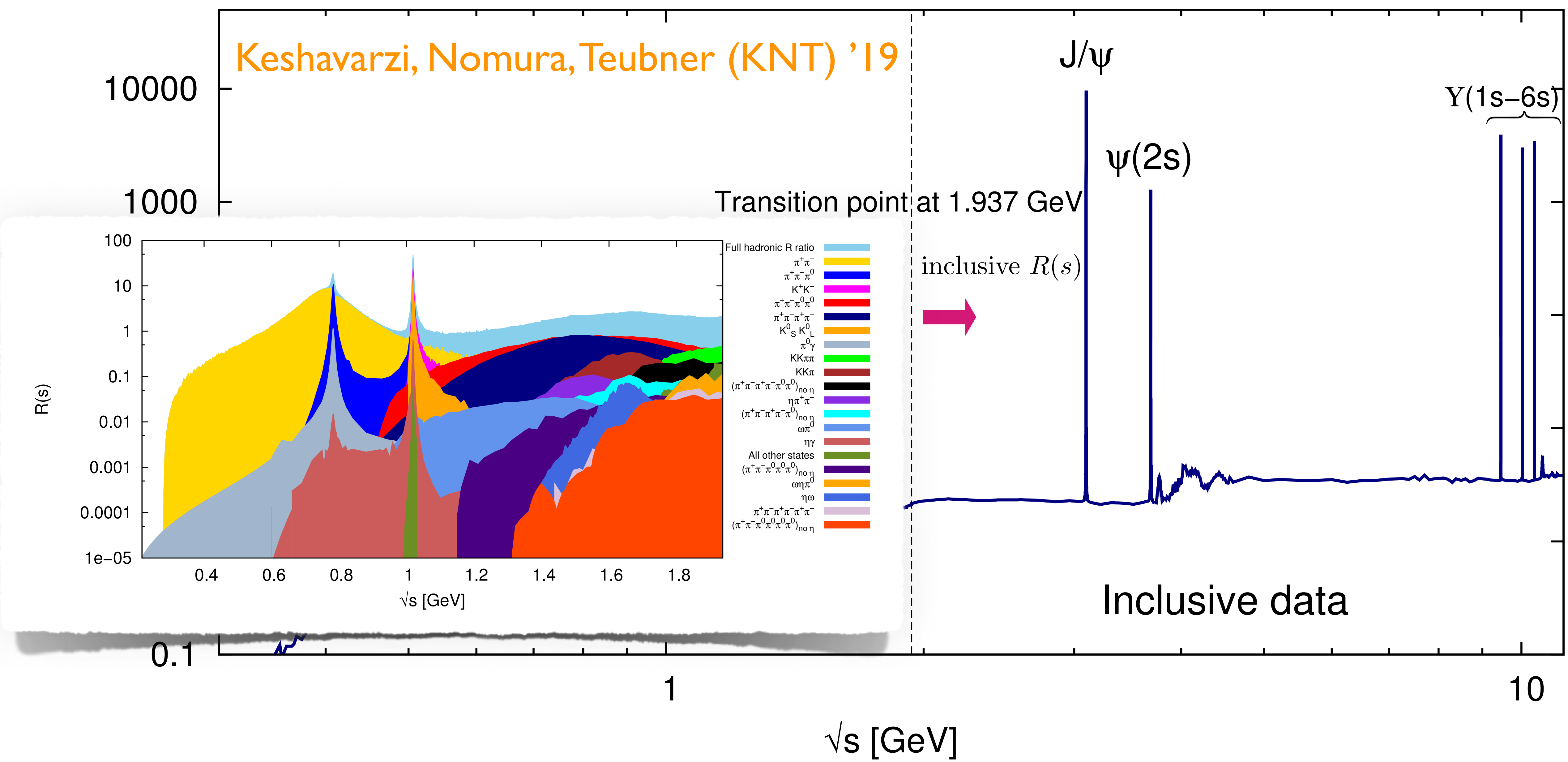

Notation and conventions: lattice

$$C(t) = \frac{1}{3}\sum_{i=1}^{3}\int d^3x \langle j_i^{\rm EM}(\vec{x},t) j_i^{\rm EM}(0)\rangle = \frac{1}{2}\int_{m_\pi^2}^{\infty} ds\,\sqrt{s}\,e^{-\sqrt{s}t}\,\rho_{\rm EM}(s) \quad (t>0)$$

Bernecker and Meyer '11

Leading order contribution to $a_\mu^{\rm HVP}$

$$a_\mu^{\rm HVP} = 2\int_0^\infty dt\,w(t)C(t)$$

$$\frac{\hat{K}(s)}{s^2} = \frac{3\sqrt{s}}{4\alpha^2 m_\mu^2}\int_0^\infty dt\,w(t)\,e^{-\sqrt{s}t}$$

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How can we best compare the lattice and dispersive approaches?

Not so simple, but the lattice splits the calculation in **different contributions**

light-quark connected gives about 90% of the total

Isospin symmetric

connected light	connected strange	connected charm	disconnected
633.7(2.1)(4.2)	53.393(89)(68)	14.6(0)(1)	-13.36(1.18)(1.36)

isospin symmetric contributions (before finite volume corrections) from BMW coll. 2002.12347

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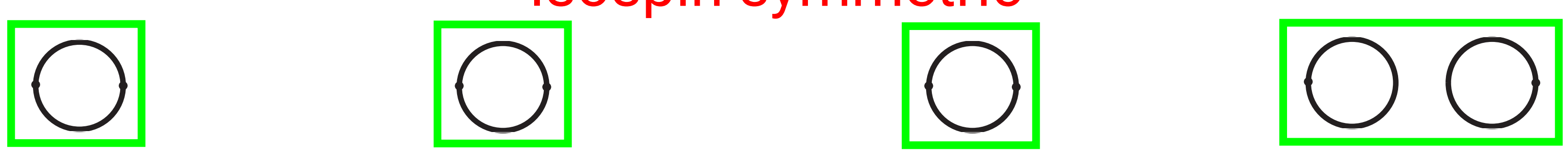

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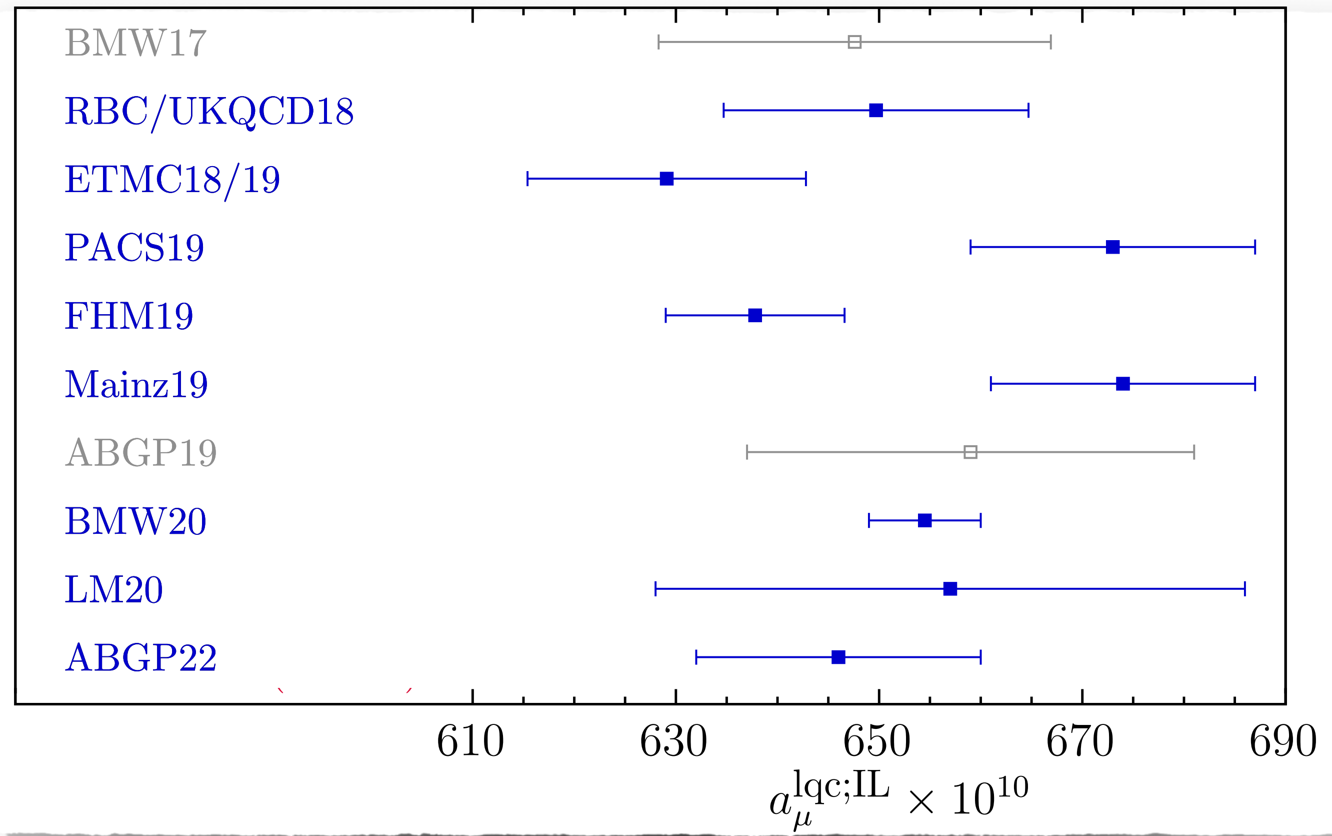
isospin symmetric contributions (before finite volume corrections) from BMW coll. 2002.12347

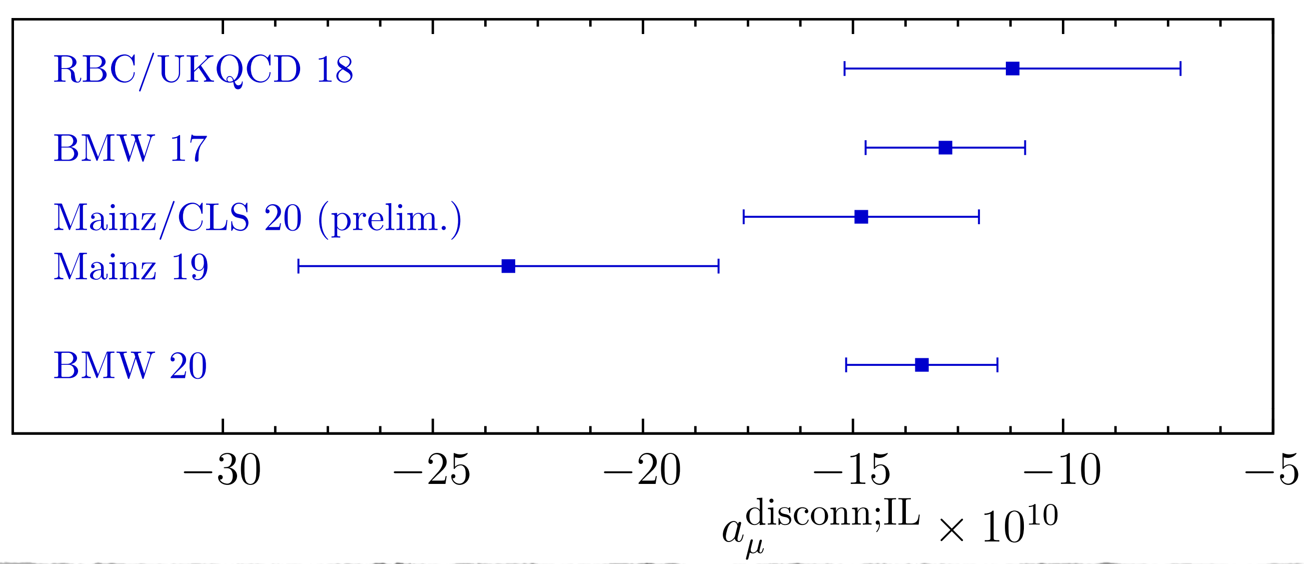
Can we get these quantities from data?

Notation and conventions: lattice

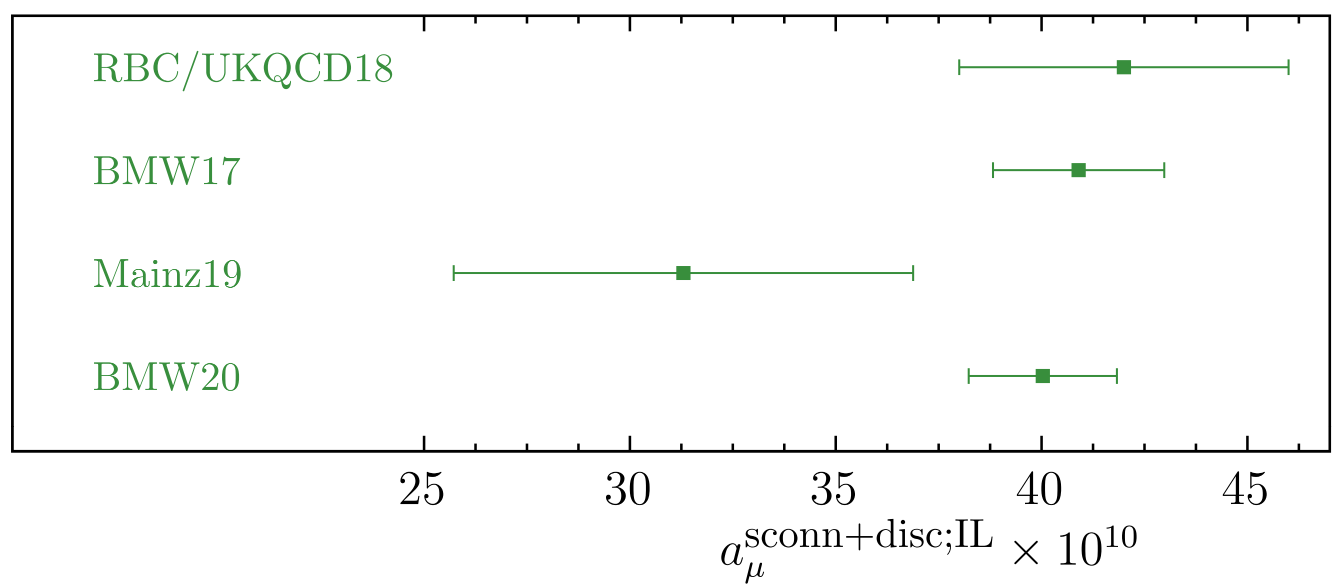
light-quark connected (lqc) contribution



disconnected contribution



sconn + disconnected contribution



Can we get these quantities from data?

1. Notation and definitions

2. Light-quark connected and strange plus light-quark disconnected results from data

3. Sum rules

the basic idea

EM current: *u*, *d*, and *s* quarks

$$j_\mu^{\mathrm{EM}} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s = \frac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d) + \frac{1}{6}(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d) - \frac{1}{3}\bar{s}\gamma_\mu s$$

I=1 I=0

the basic idea

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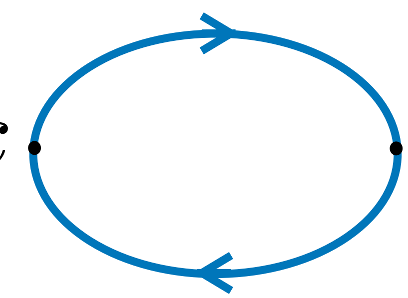
I=1 I=0

Considering the *I* =1 quark current in the isospin limit, only **connected** contributions

$$\frac{1}{4}\langle(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)(x)(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)(y)\rangle = \frac{1}{2}\ x \quad y$$



isospin 1 is purely light-quark connected

the basic idea

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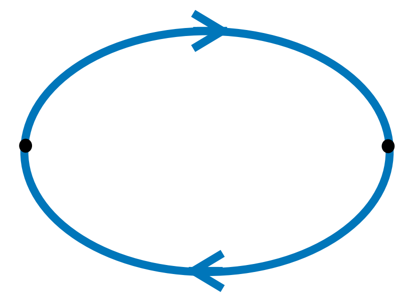
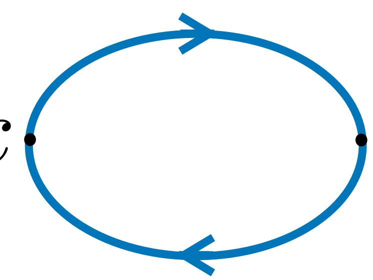
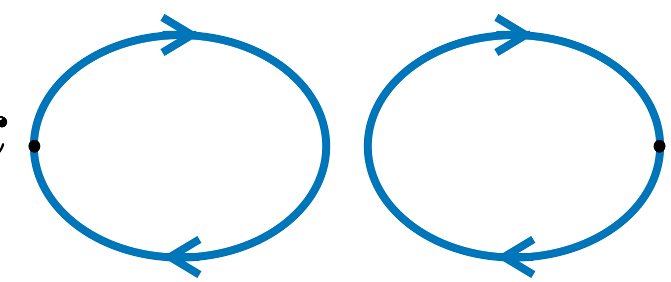
$$\frac{1}{4}\langle(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)(x)(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)(y)\rangle = \frac{1}{2}\ x$$  y

isospin 1 is purely light-quark connected

The *I* = 0 light-quark current in the isospin limit contains connected and disconnected terms

$$\frac{1}{36}\langle(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d)(x)(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d)(y)\rangle = \frac{1}{18}\ x$$  $y + \frac{1}{9}\ x$  y

$$\hat{\Pi}_{\rm EM}^{\rm sconn+disc} \equiv \hat{\Pi}_{\rm EM}^{I=0} - \frac{1}{9}\hat{\Pi}_{\rm EM}^{I=1}$$

$$a_\mu^{\rm sconn+disc} = a_\mu^{I=0} - \frac{1}{9}a_\mu^{I=1}$$

s-quark +light-quark disconnected (s+lqd)

the basic idea

EM current: *u*, *d*, and *s* quarks

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I=1 I=0

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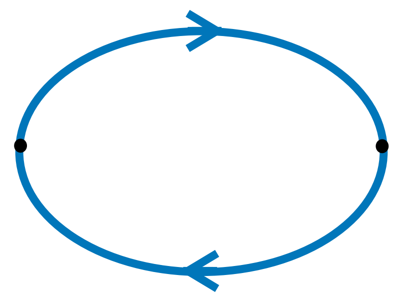
$$\frac{1}{4}\langle(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)(x)(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)(y)\rangle = \frac{1}{2}\; x$$  y

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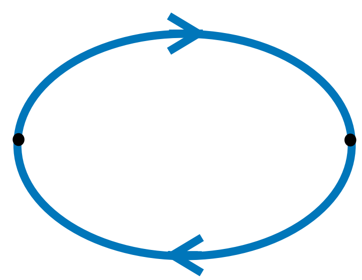
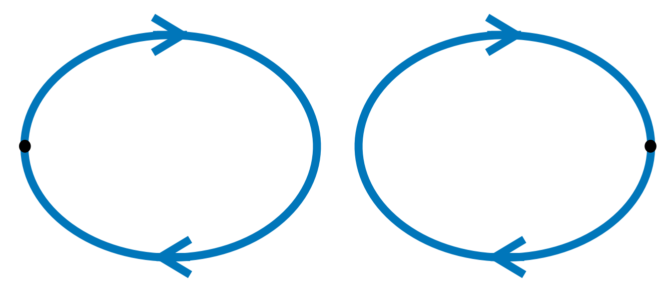
$$\frac{1}{36}\langle(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d)(x)(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d)(y)\rangle = \frac{1}{18}\; x$$  $y + \frac{1}{9}\; x$  y

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s-quark +light-quark disconnected (s+lqd)

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$$a_\mu^{\rm lqc} = \frac{10}{9}a_\mu^{I=1}$$

light-quark connected (lqc)

lqc and s+lqd cont. from data

Modes with well defined *G*-parity (unambiguous modes) give the dominant contribution

$$G = (-1)^{I+1}$$

TABLE I. *G*-parity-unambiguous exclusive-mode contributions to $a_\mu^{\rm LO,HVP}$ for $\sqrt{s} \leq 1.937$ GeV from KNT2019. Entries are in units of 10^{-10}. The notation "npp" is KNT2019's shorthand for "non purely pionic".

$I = 1$ modes X	$[a_\mu^{\rm LO,HVP}]_X \times 10^{10}$	$I = 0$ modes X	$[a_\mu^{\rm LO,HVP}]_X \times 10^{10}$
Low-s $\pi^+\pi^-$	0.87(02)	Low-s 3π	0.01(00)
$\pi^+\pi^-$	503.46(1.91)	$\pi^0\gamma$ (ω, ϕ dominated)	4.46(10)
$2\pi^+2\pi^-$	14.87(20)	3π	46.73(94)
$\pi^+\pi^-2\pi^0$	19.39(78)	$2\pi^+2\pi^-\pi^0$ (no ω, η)	0.98(09)
$3\pi^+3\pi^-$ (no ω)	0.23(01)	$\pi^+\pi^-3\pi^0$ (no η)	0.62(11)
$2\pi^+2\pi^-2\pi^0$ (no η)	1.35(17)	$3\pi^+3\pi^-\pi^0$ (no ω, η)	0.00(01)
$\pi^+\pi^-4\pi^0$ (no η)	0.21(21)	$\eta\gamma$ (ω, ϕ dominated)	0.70(02)
$\eta\pi^+\pi^-$	1.34(05)	$\eta\pi^+\pi^-\pi^0$ (no ω)	0.71(08)
$\eta 2\pi^+2\pi^-$	0.08(01)	$\eta\omega$	0.30(02)
$\eta\pi^+\pi^-2\pi^0$	0.12(02)	$\omega(\to npp)2\pi$	0.13(01)
$\omega(\to \pi^0\gamma)\pi^0$	0.88(02)	$\omega 2\pi^+2\pi^-$	0.01(00)
$\omega(\to npp)3\pi$	0.17(03)	$\eta\phi$	0.41(02)
$\omega\eta\pi^0$	0.24(05)	$\phi \to$ unaccounted	0.04(04)
Total:	543.21(2.09)	Total:	55.10(96)

lqc and s+lqd cont. from data

Modes with well defined *G*-parity (unambiguous modes) give the dominant contribution

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$\omega\eta\pi^0$	0.24(05)	$\phi \to$ unaccounted	0.04(04)
Total:	543.21(2.09)	Total:	55.10(96)

+ several ambiguous modes. External information can help (tau decays, Dalitz plot analyses…)

- Modes for which external information can help (often significantly) in reducing the separation uncertainty:

  $K\bar{K},\ K\bar{K}\pi, K\bar{K}2\pi$

- Other ambiguous modes (maximally conservative separation): 50/50 with 100% error

  $(K\bar{K}3\pi, n\bar{n}, p\bar{p}...)$

- $\sqrt{s} > 1.937$ GeV: QCD perturbation theory + duality violations

- Small isospin-breaking contributions have to be subtracted to compare with lattice isospin-symmetric results

DB, Golterman, Maltman, and Peris, 2203.05070 (PRD's editor suggestion)

ambiguous channels

- $K\bar{K}$: **example of treatment** of an ambiguous mode (expected to be dominated by *I* = 0)

From the data combination of KNT19 we have the following total $K\bar{K}$ contribution

$$a_\mu^{K\bar{K}}\Big|_{\rm tot} = 36.07 \pm 0.29$$

conservative
50/50 separation

$$a_\mu^{K\bar{K}}\Big|_{I=1,0} = 18 \pm 18$$ ✘

maximally conservative separation is not good enough!

all results in units of 10^{-10}

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all results in units of 10^{-10}

- BaBar has measured the (purely *I* = *1*) spectrum of $\tau \to K\bar{K}\nu_\tau$

With CVC we have a determination of *I* =1 $e^+e^- \to K\bar{K}$ up to $s = 2.76\ \mathrm{GeV}^2$

$$[a_\mu^{I=1}]_{K\bar{K}}\,(s < 2.76\,\mathrm{GeV}^2) = 0.764(33)$$

We then find, using KNT19 results for s > 2.76 GeV²

DB, Golterman, Maltman, and Peris, 2203.05070

$$a_\mu^{K\bar{K}}\Big|_{I=1} = 0.852(94)$$

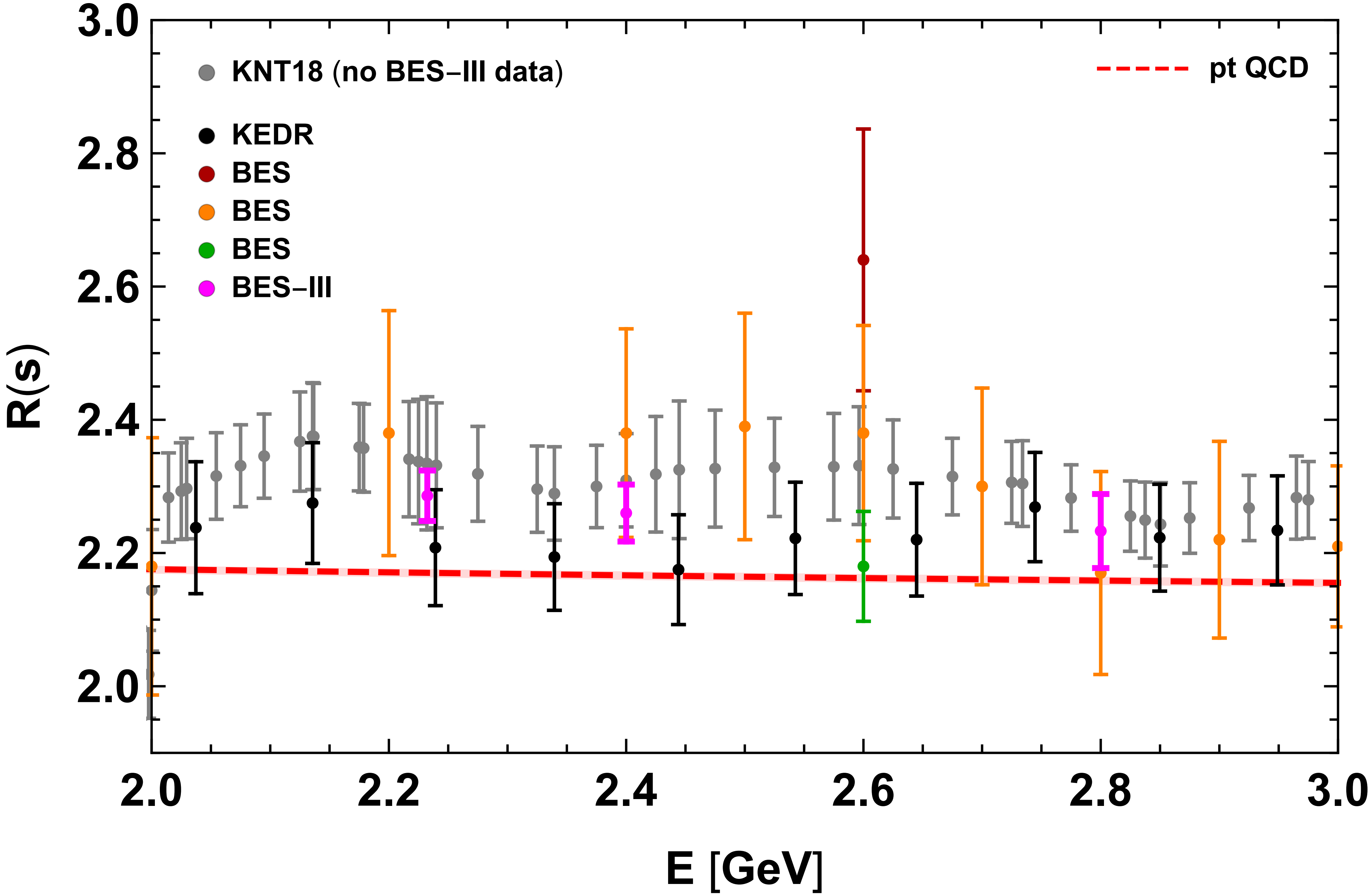
Subtracting from the total we find the *I* = 0 contribution

$$a_\mu^{K\bar{K}}\Big|_{I=0} = 35.22(30)$$

enormous reduction in the uncertainty, from 18 to 0.3 - 0.9 units.

inclusive region: perturbative QCD

QCD perturbation theory is used in the inclusive region



Some tension between pt. QCD and recent BES-III results (in magenta)

We add duality violation contributions and enlarge the pt. QCD error

isospin breaking

Isospin breaking (IB) contributions must be subtracted to compare with isospin symmetric lattice results.

To $O(\alpha, m_u - m_d)$, pure I=0,1 EM only, mixed isospin is a combination of EM + strong IB

Pure *I* = 0,1 Electromagnetic (EM) IB contributions

Inclusive. Extracted from (a combination of) BMW results.

The only (small) lattice input to our final results

Mixed-isospin contribution (strong IB + EM)

Expected to be dominated by $\rho - \omega$ interference

Results for the dominant 2π and 3π channels obtained from fits to data (VMD or dispersive).

Colangelo, Hoferichter, Kubis and Stoffer, 2208.08993

O(1%) estimate for other, subdominant, channels used as an additional IB uncertainty.

(We safely ignore IB corrections to the already small contributions in the inclusive region.)

lqc and s+lqd contributions from data

Example: breakdown of contributions to $a_\mu^{\rm lqc;IL}$

(results based on KNT19)

$$a_\mu^{\rm lqc;IL} = 543.2(2.1) + 2.9(1.0) + 28.27(2) + 0.26(12) + 0.93(59) - 4.09(47)$$

(88% of this result comes from pi+pi-)

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ambiguous modes DVs

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unambiguous modes with I=1 — pt. QCD above 1.937 GeV — EM IB — MI IB

95% 0.46% 5.0% 0.15% -0.64%

(88% of this result comes from pi+pi-)

The DV central value is used as an uncertainty in the perturbative contribution in the final result.

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95% — 0.46% — 5.0% — 0.15% — -0.64%

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The DV central value is used as an uncertainty in the perturbative contribution in the final result.

Final results of lqc and s+lqd contributions to $a_\mu^{\rm HVP}$ from data

light-quark connected (lqc)

$$a_\mu^{\rm lqc;IL} = 635.0(2.7) \qquad \text{(KNT)}$$
$$a_\mu^{\rm lqc;IL} = 638.1(4.1) \qquad \text{(DHMZ)}$$

DB, Golterman, Maltman, and Peris, 2211.11055

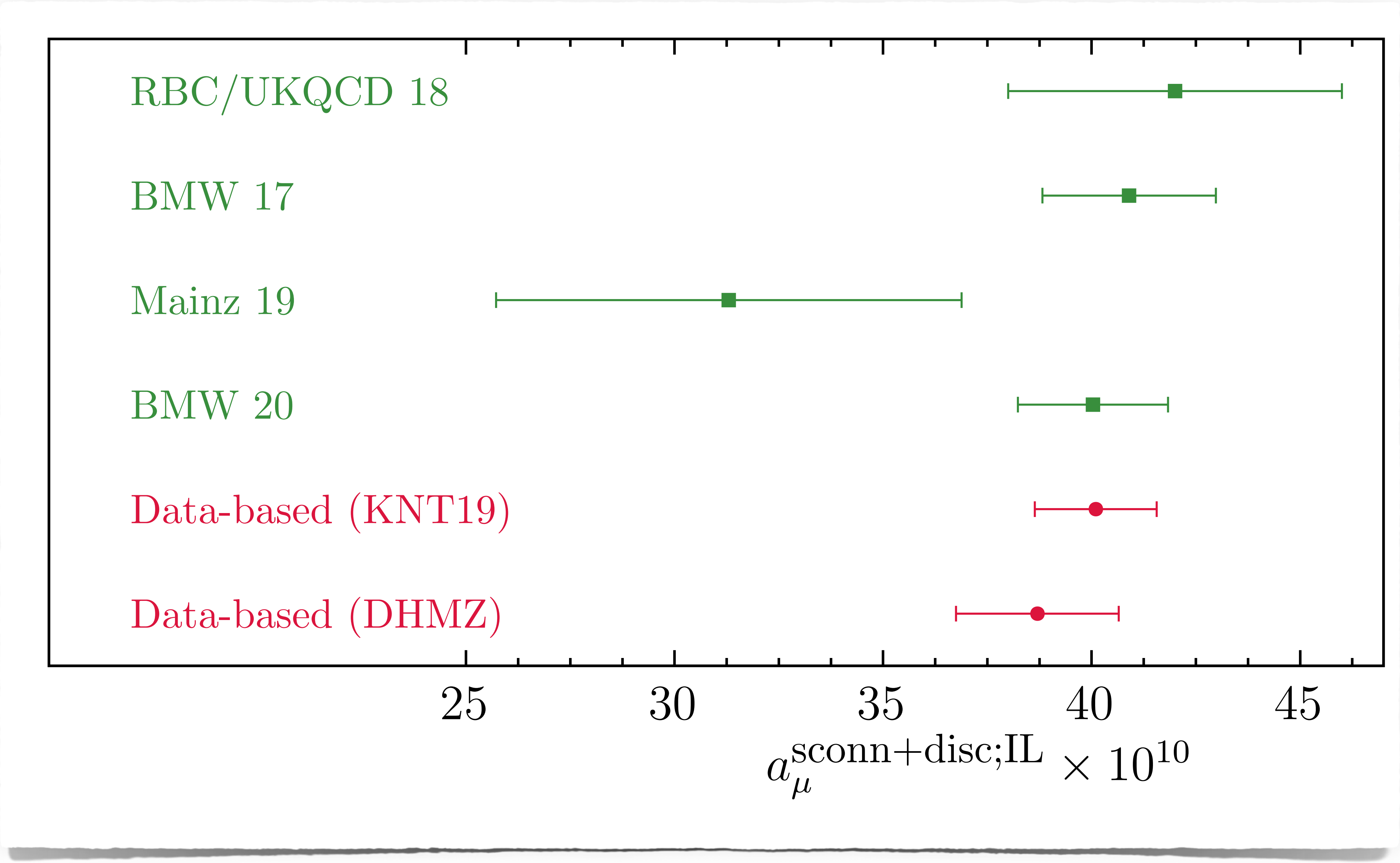
strange + light-quark disconnected (s+lqd)

$$a_\mu^{\rm s+lqd;IL} = 40.1(1.5) \qquad \text{(KNT)}$$
$$a_\mu^{\rm s+lqd;IL} = 38.7(2.0) \qquad \text{(DHMZ)}$$

DB, Golterman, Maltman, and Peris, 2203.05070

s+lqd cont. from data

s-quark + light-quark disconnected contributions (strongly dominated by $K\bar{K}$)

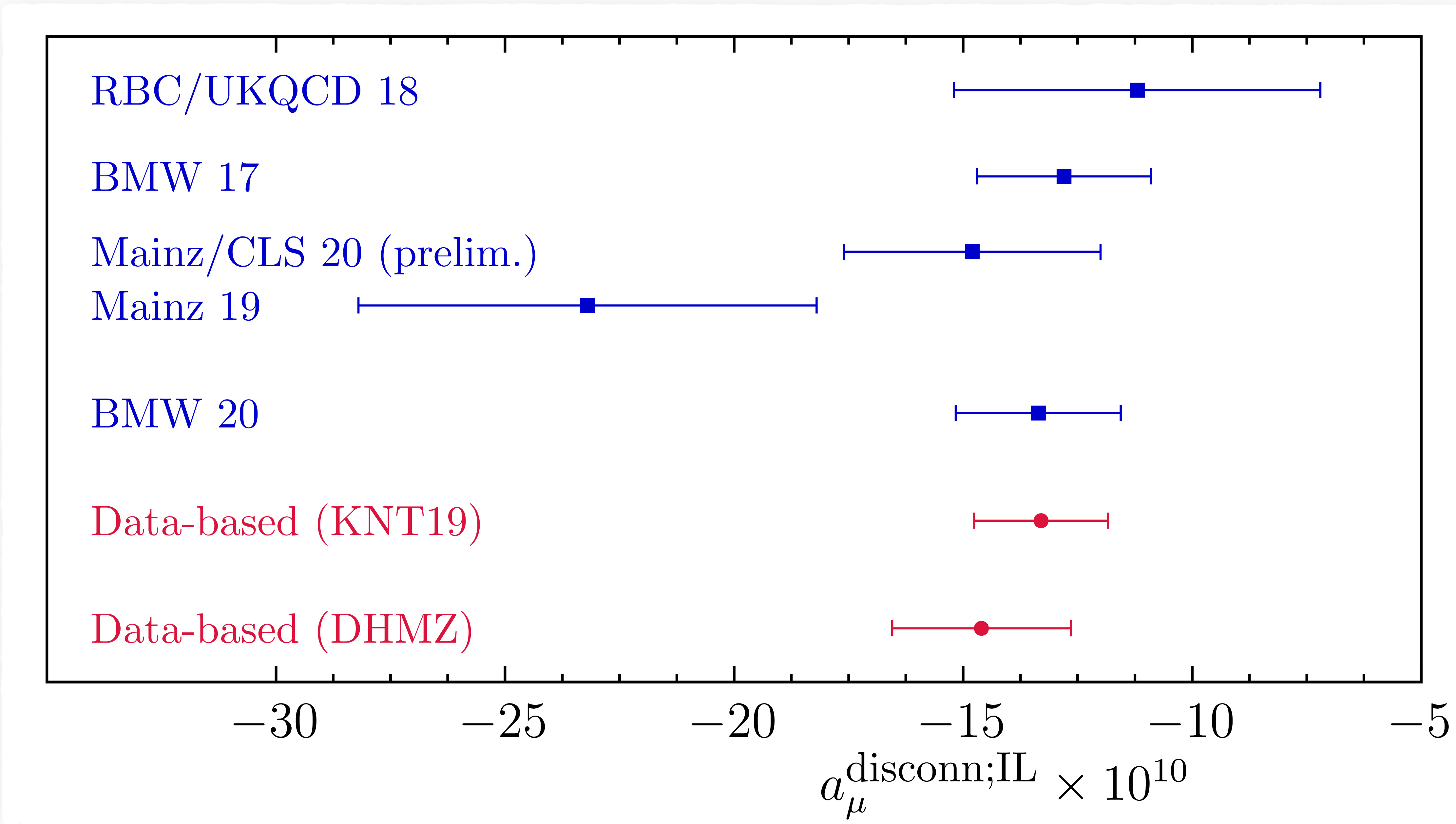


DB, Golterman, Maltman, and Peris, 2203.05070

No sign of tension in s + disconnected contribution

s+lqd cont. from data

disconnected contributions



DB, Golterman, Maltman, and Peris, 2203.05070

(here we use the average of lattice results for the s-quark connected contributions)

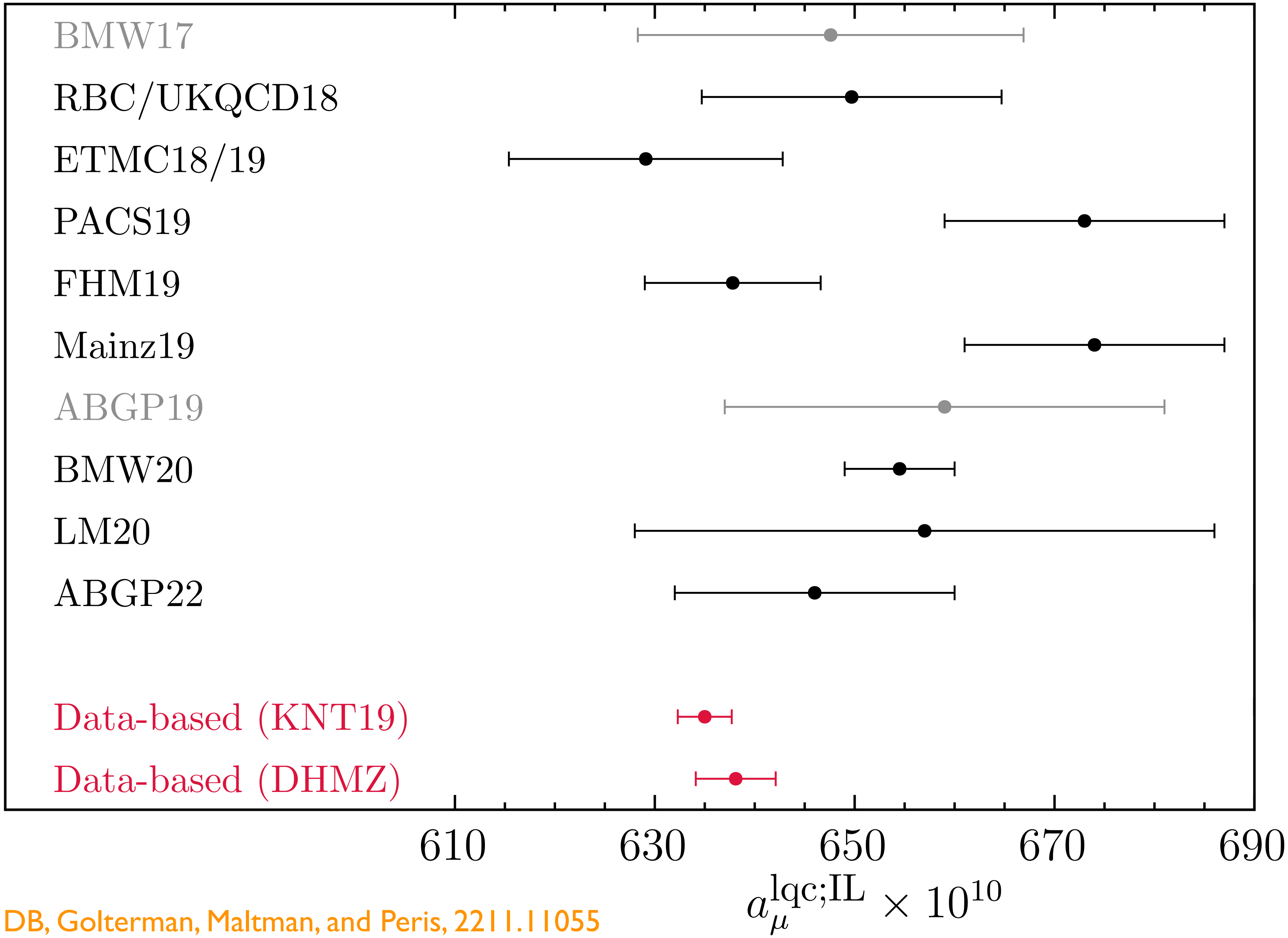
g-2 Theory Initiative white paper 2006.04822

light-quark connected contribution from data

Results for $a_\mu^{\text{lqc;IL}}$

light-quark connected

(88% of this result comes from pi+pi-)



DB, Golterman, Maltman, and Peris, 2211.11055

Tension between data and some of the lattice results (mainly BMW20)

results for windows

e+e- data

$$a_\mu^W(t_0,t_1;\Delta) = 2\int_0^\infty dt\, W(t;t_0,t_1;\Delta)\, w(t)C(t) = \frac{4\alpha^2 m_\mu^2}{3}\int_{m_\pi^2}^\infty ds\, \frac{\hat{K}(s)}{s^2}\, \hat{W}(s;t_0,t_1;\Delta)\, \rho_{\rm EM}(s)$$

lattice results

RBC/UKQCD windows

$$a_\mu^{\rm HVP,LO} = [a_\mu^{\rm HVP,LO}]^{\rm SD} + [a_\mu^{\rm HVP,LO}]^{W1} + [a_\mu^{\rm HVP,LO}]^{\rm LD}$$

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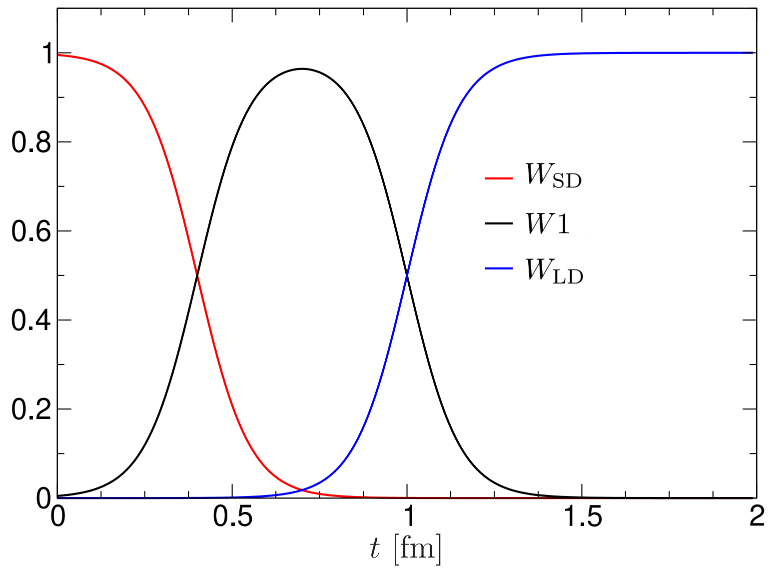
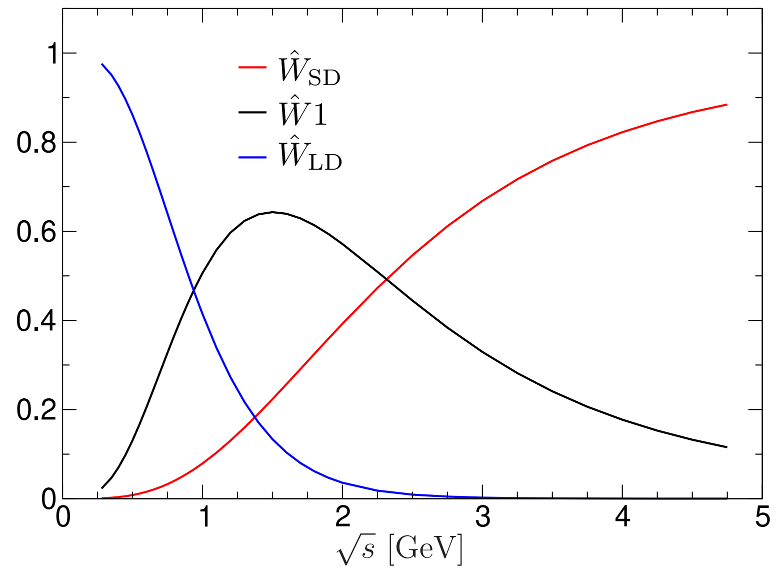
adapted from Colangelo et al, 2205.12963



$W(t)$



$\hat{W}(s)$

Intermediate window (*W*1, black) has many advantages:

- cuts out lattice artifacts at short and long distance (lattice spacing, large volume)
- can be computed very precisely on the lattice
- all lattice collaborations are computing this quantity
- several recent results for the light-quark connected component (90% of the total)

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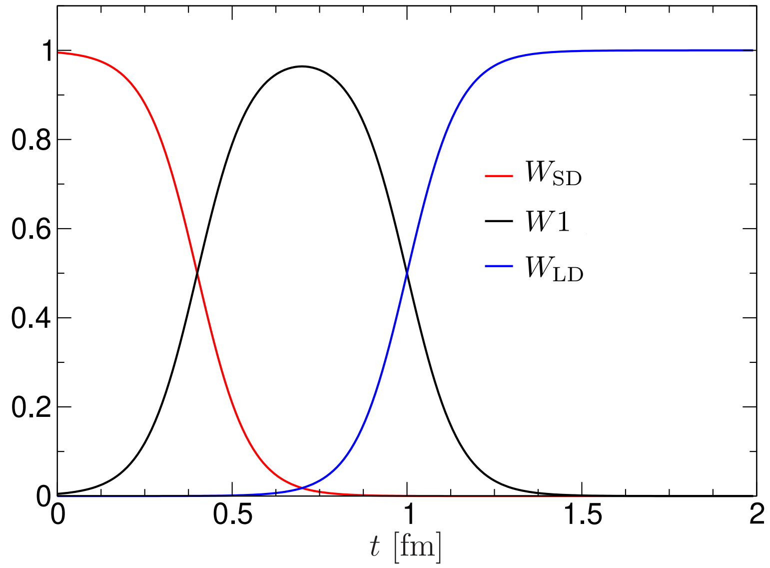
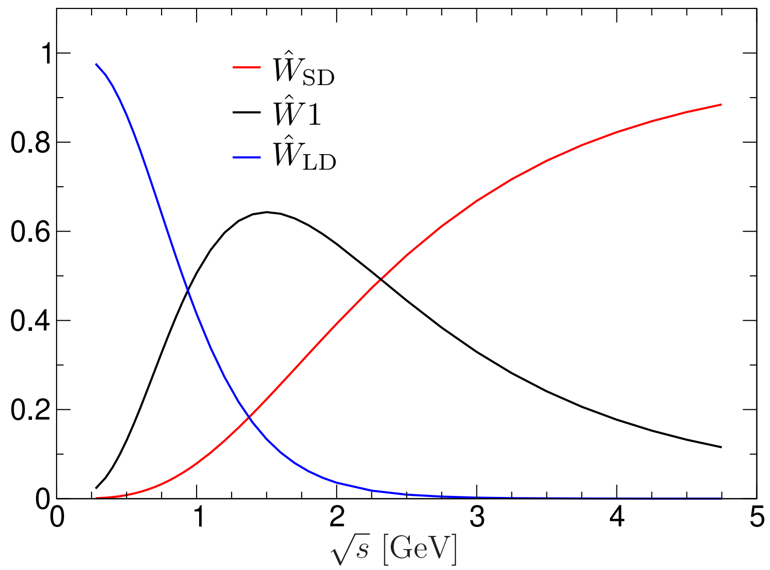
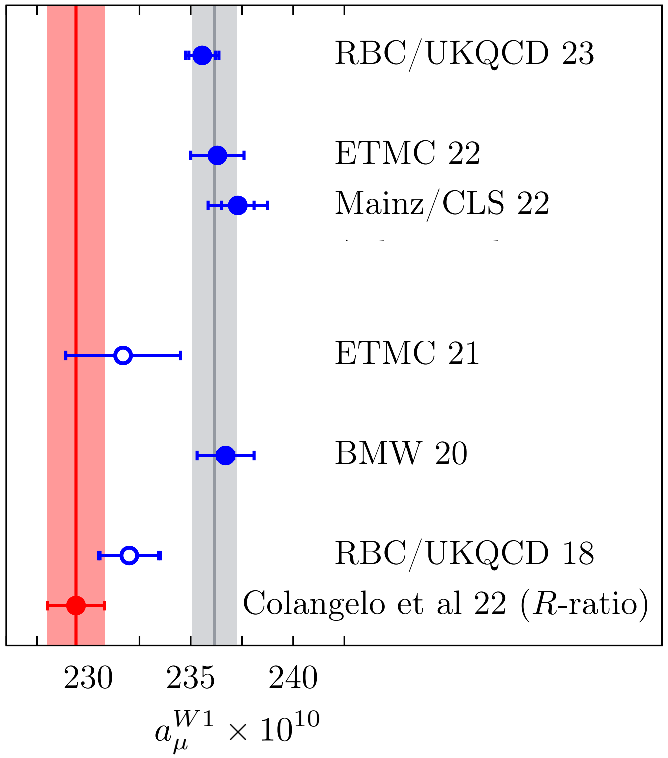


$W(t)$



$\hat{W}(s)$

H. Wittig, 2306.04165



very good recent results exist for the intermediate window

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- several recent results for the light-quark connected component (90% of the total)

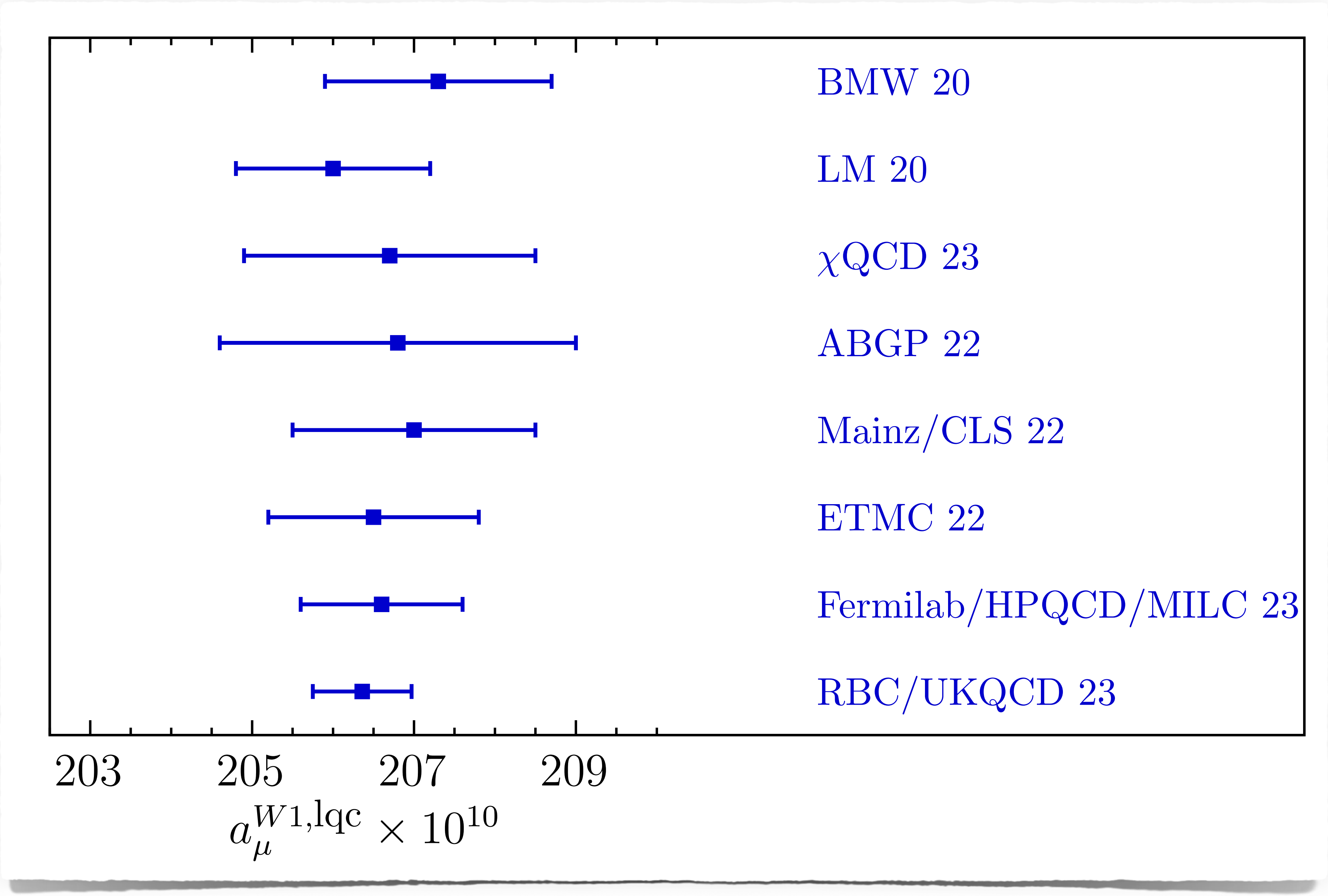
light-quark connected intermediate window

$$a_\mu^{W1}(t_0,t_1;\Delta) = 2\int_0^\infty dt\, W1(t;t_0,t_1;\Delta)\, w(t)C(t)$$

$$W(t;t_0,t_1;\Delta) = \frac{1}{2}\left(\tanh\frac{t-t_0}{\Delta} - \tanh\frac{t-t_1}{\Delta}\right)$$

$$t_0 = 0.4\,\text{fm},\;\; t_1 = 1.0\,\text{fm},\;\; \Delta = 0.15\,\text{fm}$$

light-quark connected lattice results



several, recent, lattice results that are fully compatible and have small errors

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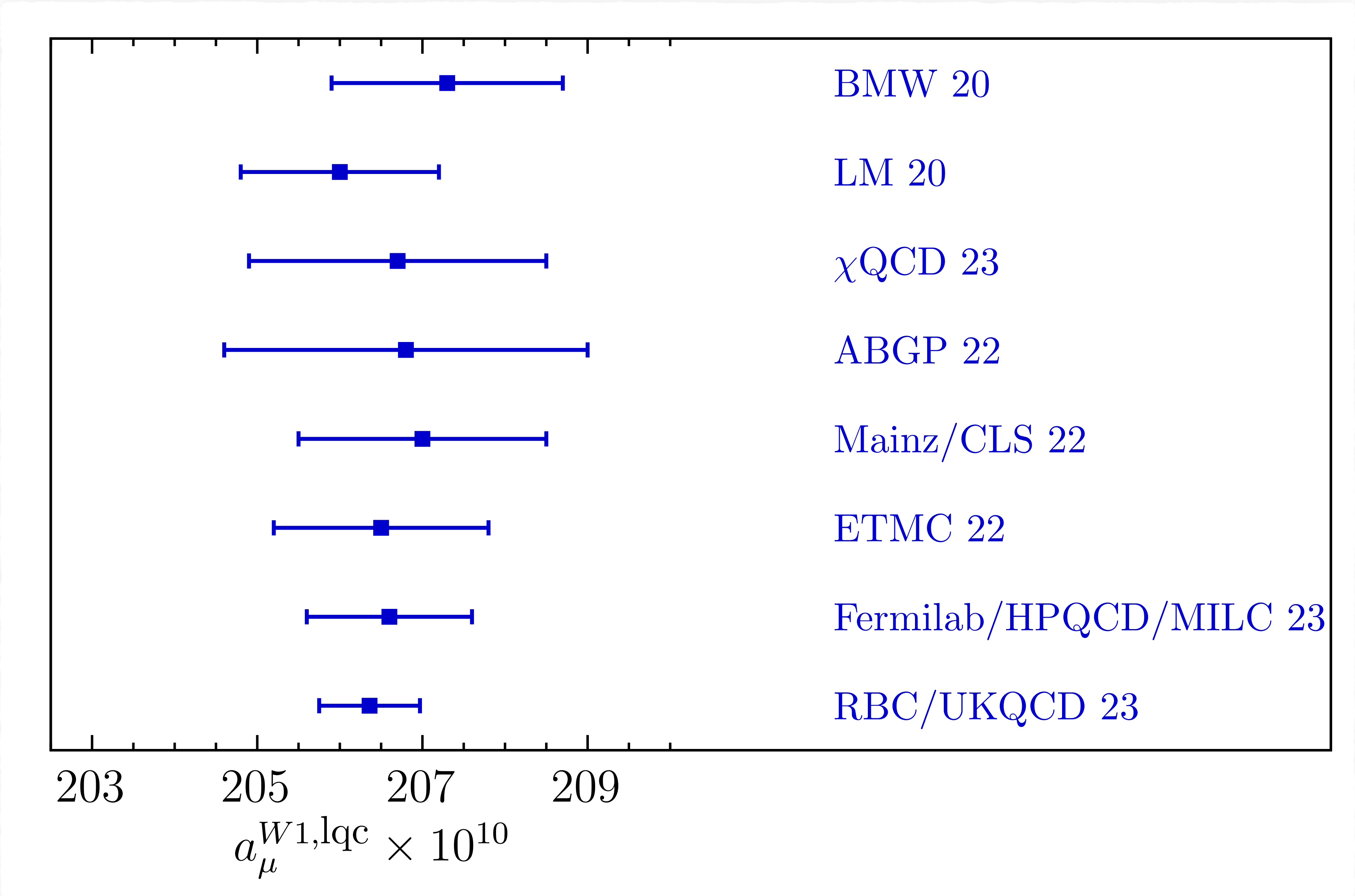

several, recent, lattice results that are fully compatible and have small errors

What about data??

light-quark connected intermediate window

Getting the lqc and s+lqd contributions to window quantities is straightforward *provided* we have the exclusive channel spectra (which we do from KNT19)

We can still get the EM IB corrections for the intermediate window using BMW results. For other windows we cannot and will neglect this sub-percent correction

Mixed-isospin IB contribution depend on fits to data (here dispersive) but can be estimated for the dominant 2π and 3π channels

thanks to M. Hoferichter and P. Stoffer for running the MI IB contributions to window quantities

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Following the same steps as before we find

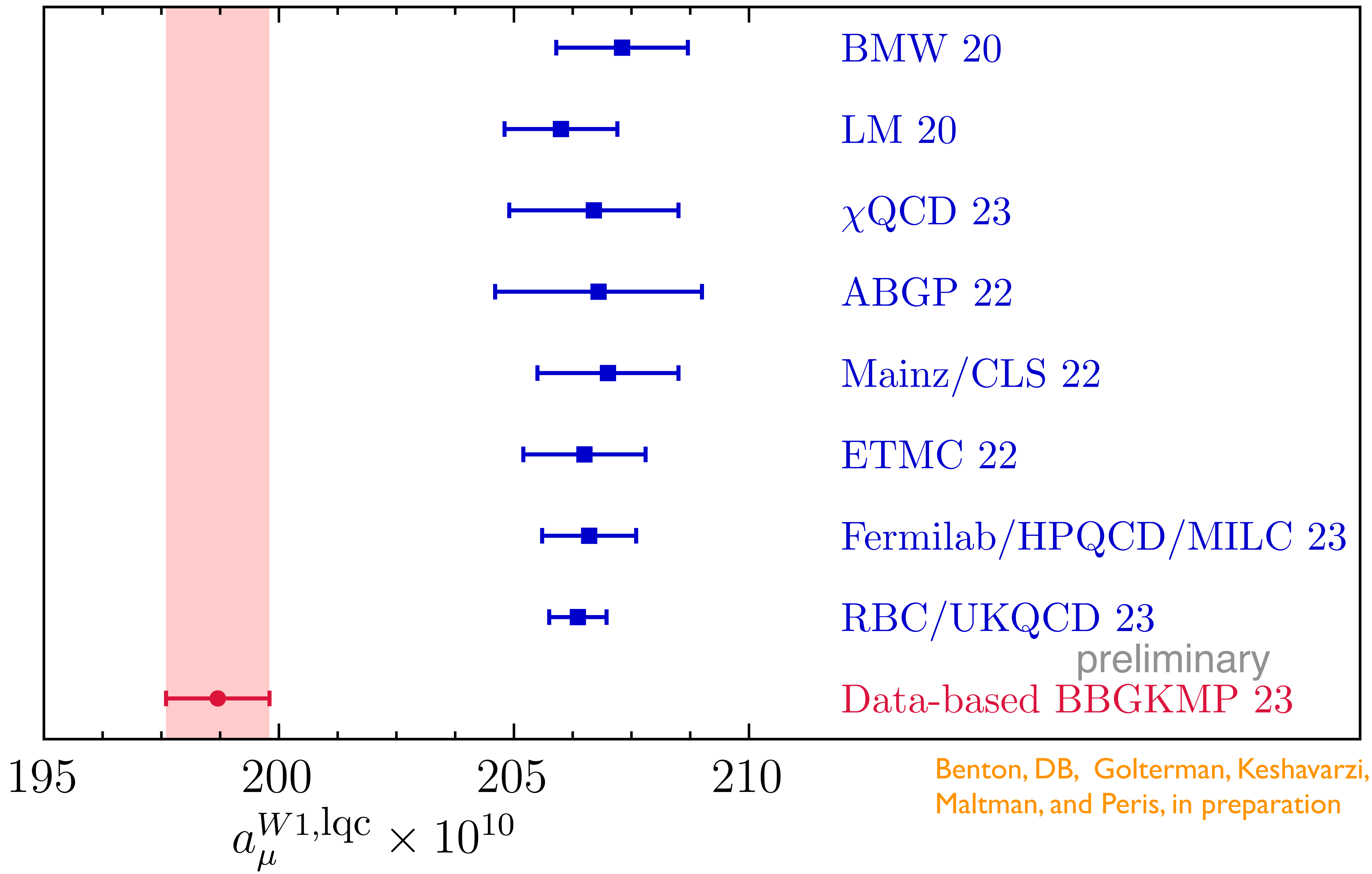
$$a_\mu^{W1,\mathrm{lqc}} = 198.7(1.1) \times 10^{-10}$$

Benton, DB, Golterman, Keshavarzi, Maltman, and Peris, in preparation

(81% of this result comes from pi+pi-…)

light-quark connected intermediate window

Light-quark connected: very significant tension between lattice QCD and the dispersive approach



Benton, DB, Golterman, Keshavarzi, Maltman, and Peris, in preparation

accounts for nearly all the discrepancy in the total result

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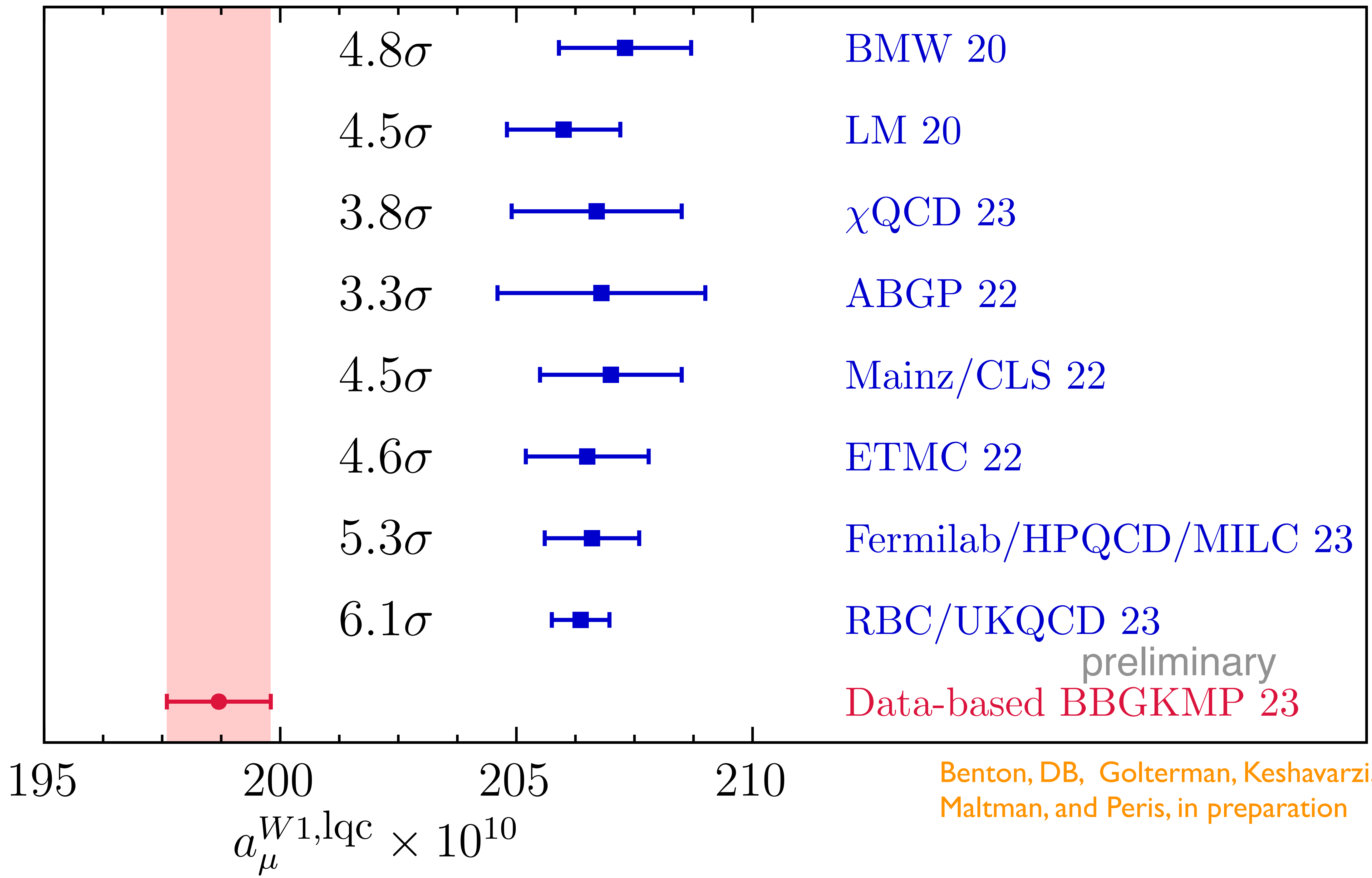

Benton, DB, Golterman, Keshavarzi, Maltman, and Peris, in preparation

accounts for nearly all the discrepancy in the total result

light-quark connected: other windows

$$a_\mu^{W2}(t_0,t_1;\Delta) = 2\int_0^\infty dt\, W2(t;t_0,t_1;\Delta)\, w(t)C(t)$$

$$W(t;t_0,t_1;\Delta) = \frac{1}{2}\left(\tanh\frac{t-t_0}{\Delta} - \tanh\frac{t-t_1}{\Delta}\right)$$

$$t_0 = 1.5\,\mathrm{fm},\;\; t_1 = 1.9\,\mathrm{fm},\;\; \Delta = 0.15\,\mathrm{fm}$$

Aubin, Blum, Golterman, Peris (ABGP) ‘22

Similar advantages but more “long distance”; within the reach of chiral perturbation theory

light-quark connected from KNT19 R(s) data

$$a_\mu^{W2,\mathrm{lqc}} = 93.70(36)\times 10^{-10}$$

Benton, DB, Golterman, Keshavarzi, Maltman, and Peris, in preparation

lattice results

Aubin, Blum, Golterman, Peris ‘22

$$a_\mu^{\mathrm{W2,lqc}} = 102.1(2.4)\times 10^{-10}$$

Fermilab/HPQCD/MILC ‘23

$$a_\mu^{W2,\mathrm{lqc}} = 100.7(3.2)\times 10^{-10}$$

1. Notation and definitions

2. Light-quark connected and strange plus light-quark disconnected results from data

3. Sum rules

lattice vs exp. data: new sum rules

One cannot get the spectral function locally from lattice data.

$$C(t) = \int_{E_{\rm th}}^{\infty} dE E^2 e^{-Et} \rho(E)$$

This Laplace transform cannot be inverted if all we have from the lattice is a discrete data set affected by errors (ill-posed problem).

see e.g., Hansen, Meyer and Robaina '17 (based on Backus and Gilbert '68); Hansen, Lupo and Tantalo '19; Bailas, Hashimoto and Ishikawa '20

- New set of **sum rules** for the comparison of spectral functions from experimental data and lattice simulations

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- New set of **sum rules** for the comparison of spectral functions from experimental data and lattice simulations

- Starting point is a narrow window on the spectral function (not in Euclidean time)

- Allows for the choice of weight functions that are well localized in energy
    - Comparison between R-ratio data and lattice data
    - Potentially useful in reconsidering results from tau decay, combining a more precise tau decay vector spectral function with future isospin-breaking results from the lattice

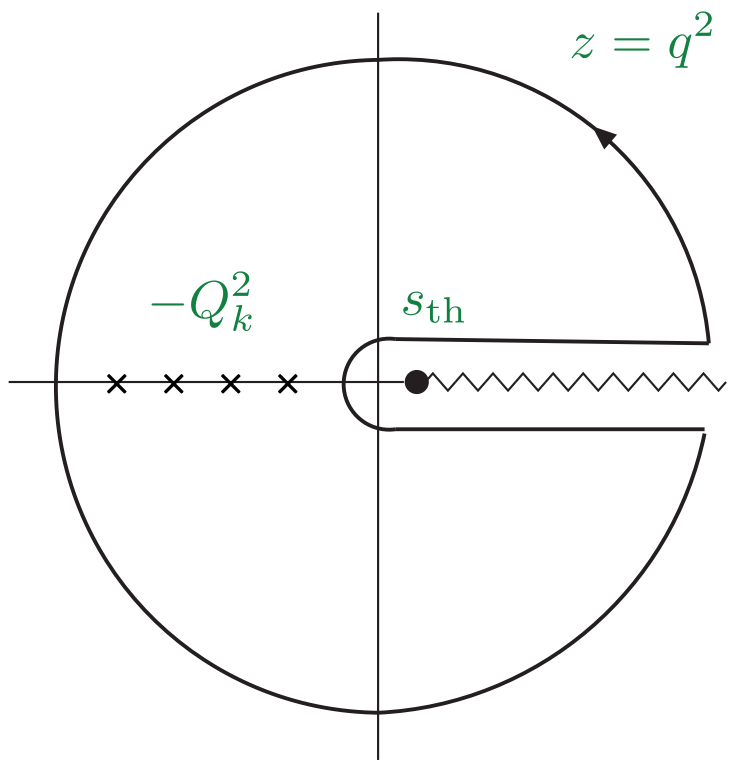
DB, Golterman, Maltman, Peris, Rodrigues, Schaaf ’20
mainly ALEPH and OPAL data but no MC input needed

M Bruno et al

rational-weight sum rules

Consider a class of functions $W_{m,n}(s) = \mu^{2(n-m-1)} \frac{(s-s_{\rm th})^m}{\prod_{\ell=1}^n (s+Q_\ell^2)}$, Q_k^2 Euclidean and fixed.

Boyle et al (RBC/UKQCD) '18



$$\frac{1}{2\pi i}\oint_C dz\, W_{m,n}(z)\Pi(-z) = (-1)^m \mu_\tau^{2(n-m-1)} \sum_{k=1}^n \frac{(Q_k^2+s_{\rm th})^m}{\prod_{\ell\neq k}(Q_\ell^2 - Q_k^2)}\,\Pi(Q_k^2)$$

$$= \int_{s_{\rm th}}^\infty ds\, W_{m,n}(s)\rho(s)$$

Taking radius to infinity we get a relation between a spectral function integral and an Euclidean quantity

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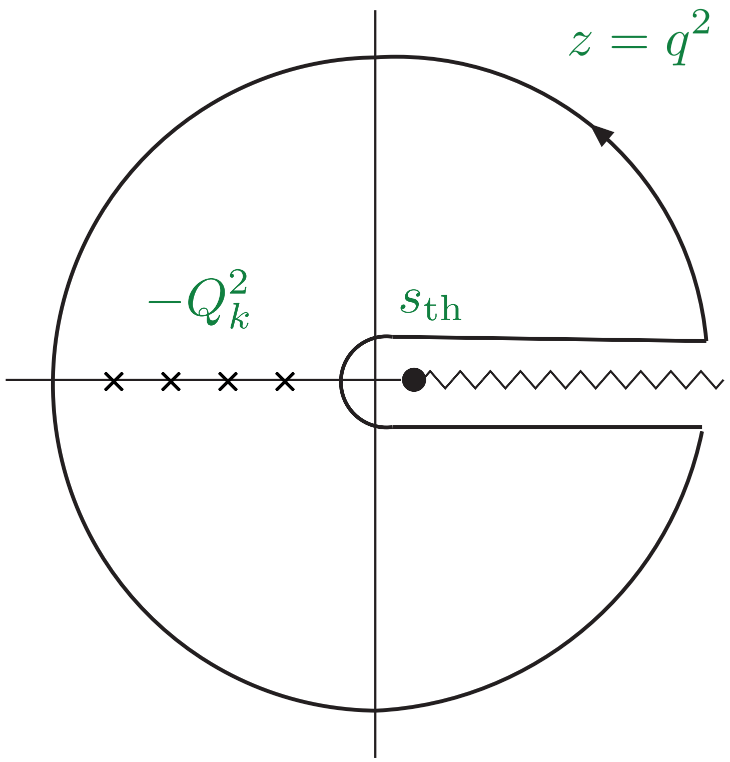

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$$= \int_{s_{\rm th}}^\infty ds\, W_{m,n}(s)\rho(s)$$

Taking radius to infinity we get a relation between a spectral function integral and an Euclidean quantity

In terms of lattice data for $C(t)$ we get

exp. data — lattice results

$$\int_{s_{\rm th}}^\infty ds\, W_{m,n}(s)\rho(s) = \int_0^\infty dt\, c^{(m,n)}(t)\, C(t)$$

$$c^{(m,n)}(t) = (-1)^m \mu^{2(n-m-1)} \sum_{k=1}^n \frac{(Q_k^2+s_{\rm th})^m}{\prod_{\ell\neq k}(Q_\ell^2-Q_k^2)} \left(\frac{4\sin^2(Q_k t/2)}{Q_k^2} - t^2\right)$$

We can tune the profile of W(s) by adjusting the position of the poles.

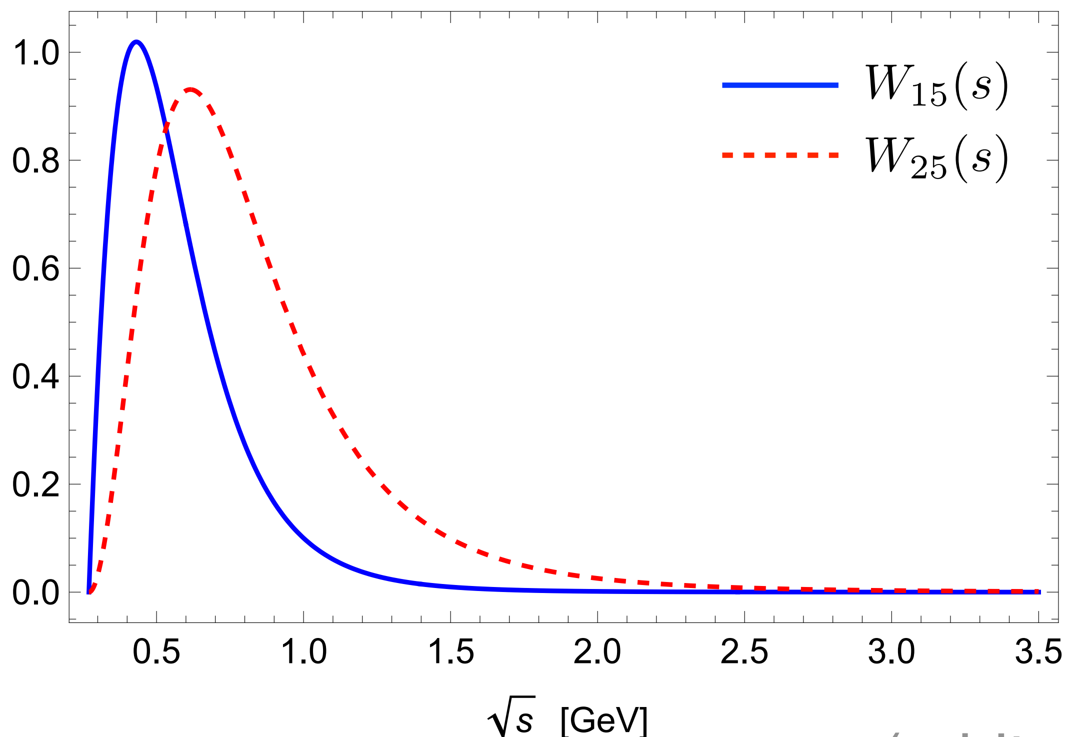
rational-weight sum rules

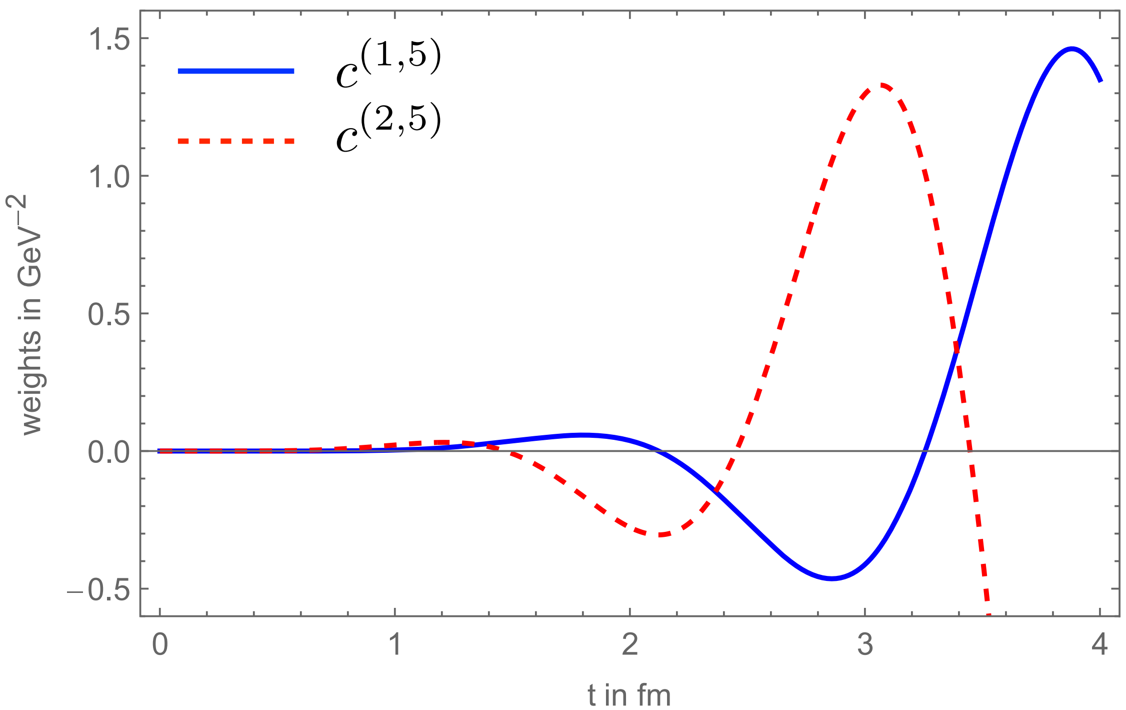
$$W_{m,n}(s) = \mu^{2(n-m-1)} \frac{(s - s_{\rm th})^m}{\prod_{\ell=1}^{n}(s + Q_\ell^2)}$$

$W_{15}(s)$

$W_{25}(s)$

Examples: choose $Q_k^2 = 0.25,\ 0.325,\ 0.4,\ 0.475,\ 0.55\ \text{GeV}^2$ and $n = 1,\ 2$:





(arbitrary vertical scales)

rational-weight sum rules

$$W_{m,n}(s) = \mu^{2(n-m-1)} \frac{(s - s_{\rm th})^m}{\prod_{\ell=1}^{n}(s + Q_\ell^2)}$$

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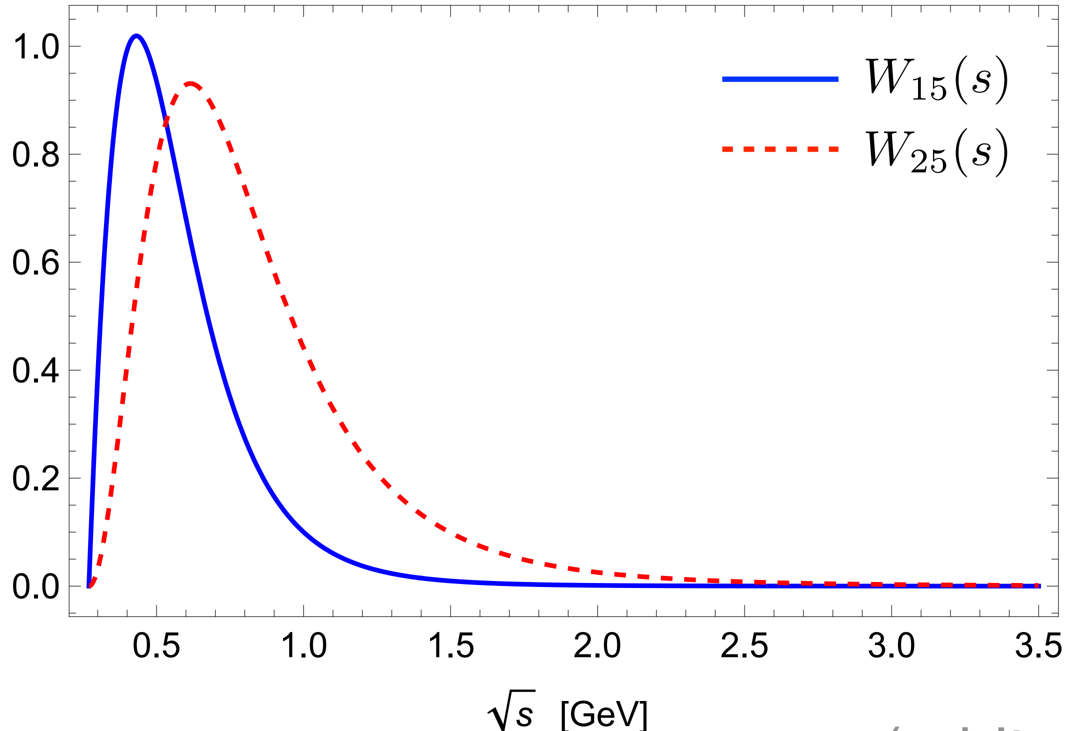
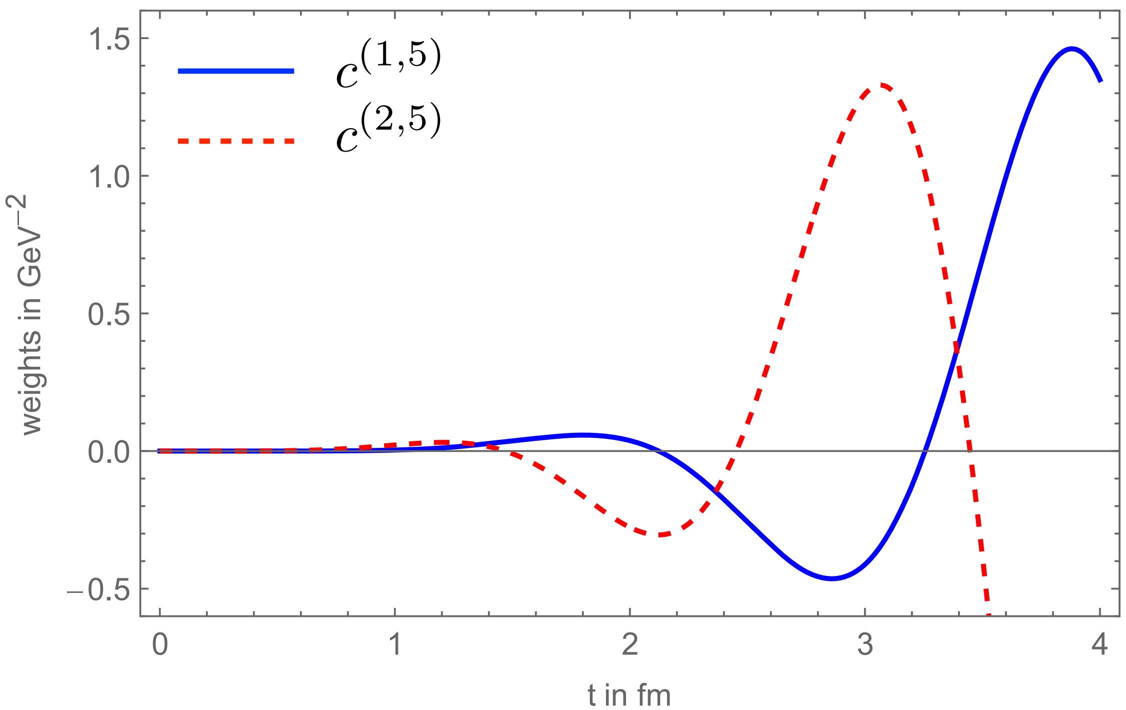




(arbitrary vertical scales)

- KNT19 data
- Lattice: ABGP 22 lqc only! **Central values should not be directly compared.** (4 lattice spacings, a=0.06, 0.09, 0.12, 0.15 fm, extrapolated to continuum limit)

	R-ratio	rel. error	lattice	rel. error
W_{15}	0.4756(16)	0.3%	0.468(26)	5.6%
W_{25}	0.08912(34)	0.4%	0.0838(33)	3.9%

DB, Golterman, Maltman, and Peris, '22

Large errors on the lattice side

exponential-weight sum rules

Because $C(t) = \int_{E_{\rm th}}^{\infty} dE E^2 e^{-Et} \rho(E) \qquad t > 0$

weight functions in terms of exponentials provide a new type of sum rule

$$w_n(E) = \sum_{j=1}^{n} x_j E^2 e^{-Et_j} \quad \Rightarrow$$

Exponential-weight sum rules

$$\int_{E_{\rm th}}^{\infty} dE w_n(E) \rho(E) = \sum_{j=1}^{n} x_j C(t_j)$$

exponential-weight sum rules

Because $C(t) = \int_{E_{\rm th}}^{\infty} dE E^2 e^{-Et} \rho(E) \qquad t > 0$

weight functions in terms of exponentials provide a new type of sum rule

$$w_n(E) = \sum_{j=1}^{n} x_j E^2 e^{-Et_j} \quad \longrightarrow \quad \int_{E_{\rm th}}^{\infty} dE w_n(E) \rho(E) = \sum_{j=1}^{n} x_j C(t_j)$$

Exponential-weight sum rules

Strategy:

- Chose a physically interesting weight function (that we call the mold) $2E\,W(s = E^2)$
- Build an approximation to it using a small number of t_j values (the cast) $w(E)$
- Throw away the mold and work with the cast! **Exact sum rule for the cast function.**

DB, Golterman, Maltman, and Peris, '22

One can chose t_j such that $C(t_j)$ has small errors.

exponential-weight sum rules

Given the mold function, minimize $\int_{E_{\rm th}}^{\infty} dE \Big| w_n(E;\{t_j\},\{x_j\})/E^2 - 2W(E^2)/E \Big|^2$

which has the solution

$$x_i = \sum_{j=1}^{n} A^{-1}_{ij} f_j$$

with

$$A_{ij} = \int_{E_{\rm th}}^{\infty} dE e^{-(t_i+t_j)E},$$

$$f_i = 2\int_{E_{\rm th}}^{\infty} dE e^{-t_i E} W(E^2)/E$$

Hansen, Lupo, Tantalo '19

For a chosen set of time values this gives the coefficients x_j

exponential-weight sum rules

Given the mold function, minimize $\displaystyle\int_{E_{\rm th}}^{\infty} dE \left| w_n(E;\{t_j\},\{x_j\})/E^2 - 2W(E^2)/E \right|^2$

which has the solution

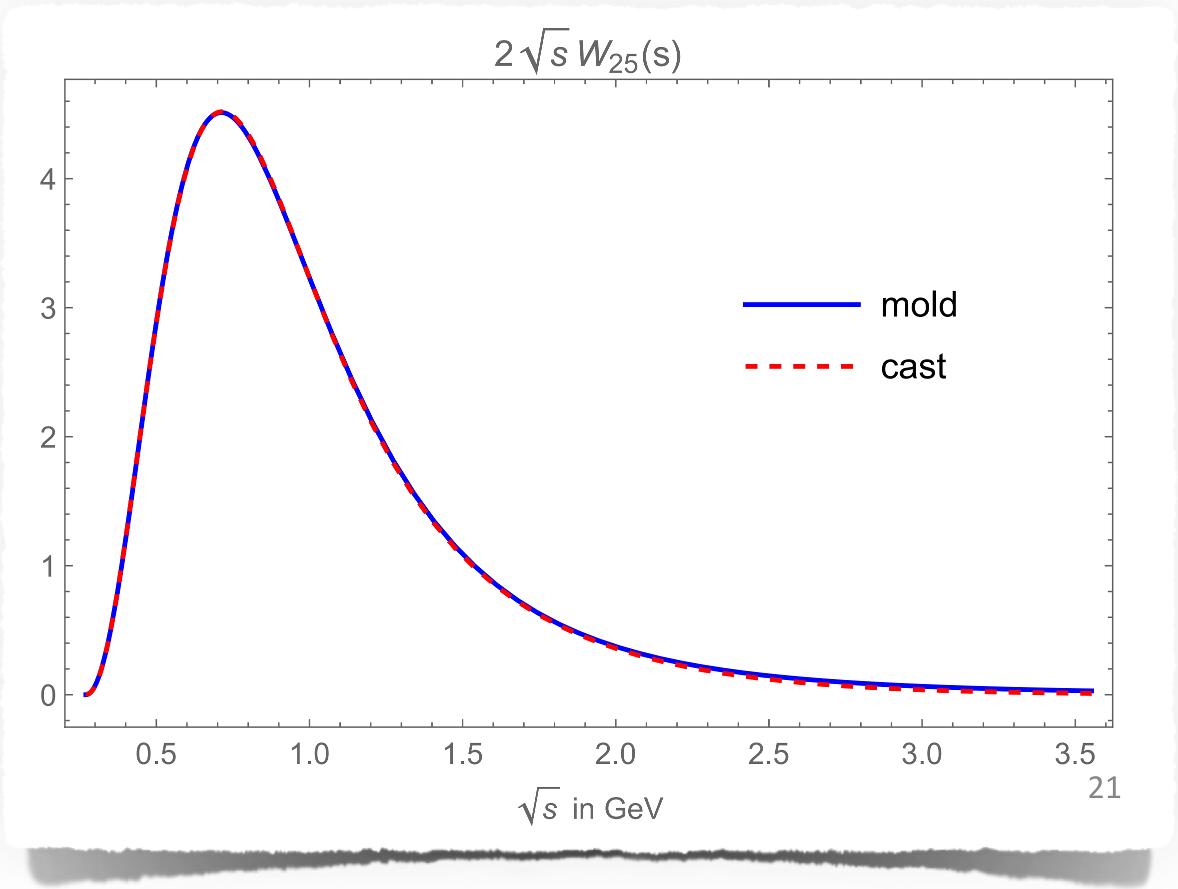
$$x_i = \sum_{j=1}^{n} A^{-1}_{ij} f_j \quad \text{with} \quad A_{ij} = \int_{E_{\rm th}}^{\infty} dE e^{-(t_i+t_j)E}, \quad f_i = 2\int_{E_{\rm th}}^{\infty} dE e^{-t_i E} W(E^2)/E$$

Hansen, Lupo, Tantalo '19

For a chosen set of time values this gives the coefficients x_j

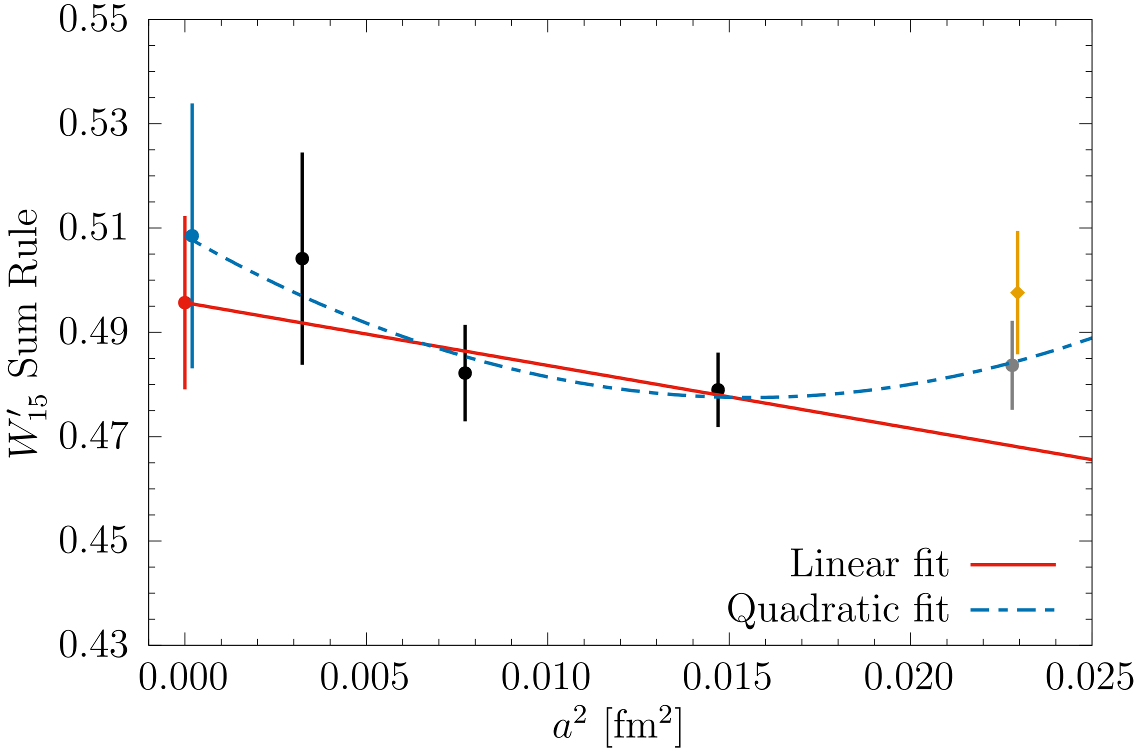
$$t_j = 3,\ 6,\ 9,\ 12,\ 15\ \mathrm{GeV}^{-1} \approx 0.6,\ 1.2,\ 1.8,\ 2.4,\ 3\ \mathrm{fm}$$

$$W'_{2,5}:\ x_1 = 34.0249, \quad x_2 = 870.640, \quad x_3 = -5501.14,$$
$$x_4 = 9933.01, \quad x_5 = -5284.24.$$



exponential-weight sum rules

lqc data from ABGP 22



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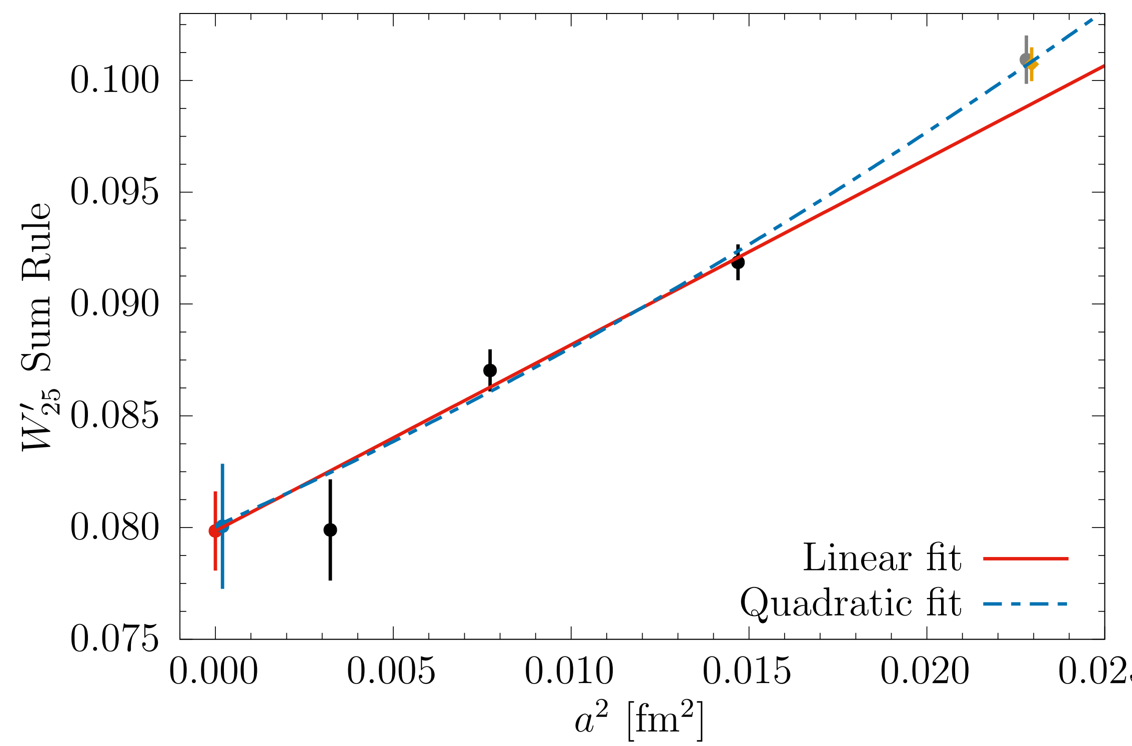
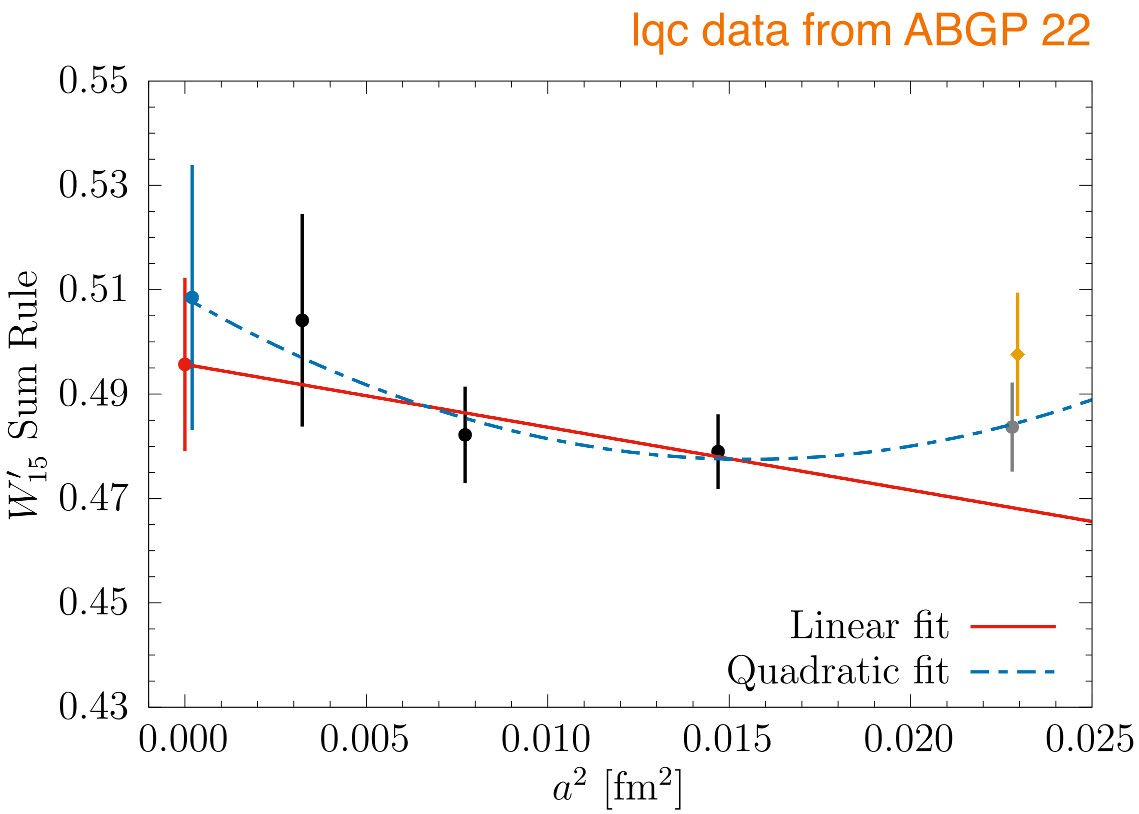
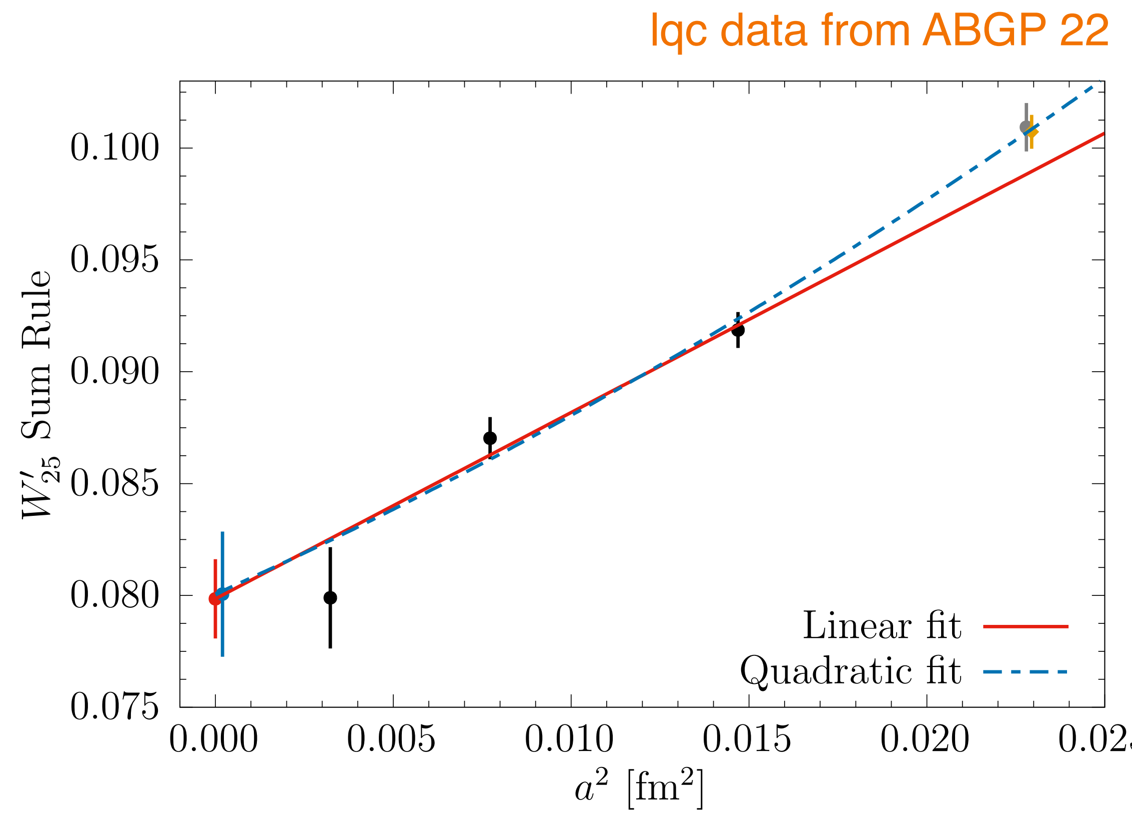


systematic errors on the lattice still to be assessed
(pion mass mistuning, finite volume and taste breaking)

exponential-weight sum rules





systematic errors on the lattice still to be assessed
(pion mass mistuning, finite volume and taste breaking)

- KNT19 data
- Lattice: ABGP 22 lqc only! **Central values should not be directly compared.**
  (4 lattice spacings, a=0.06, 0.09, 0.12, 0.15 fm, extrapolated to continuum limit)

R(s) data — rational weight — exponential weight

	R-ratio	rel. error	lattice	rel. error	lattice	rel. error
W_{15}	0.4756(16)	0.3%	0.468(26)	5.6%	0.496(17)	3.4%
W_{25}	0.08912(34)	0.4%	0.0838(33)	3.9%	0.0798(18)	2.3%

DB, Golterman, Maltman, and Peris, '22

improved exponential weights

Improving on the previous lattice errors

The minimization of $\displaystyle\int_{E_{\rm th}}^{\infty} dE \Big| w_n(E;\{t_j\},\{x_j\})/E^2 - 2W(E^2)/E\Big|^2$

has the solution

$$x_i = \sum_{j=1}^{n} A_{ij}^{-1} f_j$$

with

$$A_{ij} = \int_{E_{\rm th}}^{\infty} dE e^{-(t_i+t_j)E},$$

$$f_i = 2\int_{E_{\rm th}}^{\infty} dE e^{-t_i E} W(E^2)/E$$

Small eigenvalues of the matrix *A* can be removed using

$$\hat{A}(\lambda) = (1-\lambda)A + \lambda \mathbf{1}_n$$

This removes eigenvalues $< \lambda$ and reduce the range of the values of $\{x_i\}$.

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This removes eigenvalues $< \lambda$ and reduce the range of the values of $\{x_i\}$.

$\lambda = 10^{-9}$

$$\hat{W}_{2,5}: x_1 = 44.8916, \quad x_2 = 590.933, \quad x_3 = -3373.53,$$
$$x_4 = 3716.86, \quad x_5 = 879.149.$$

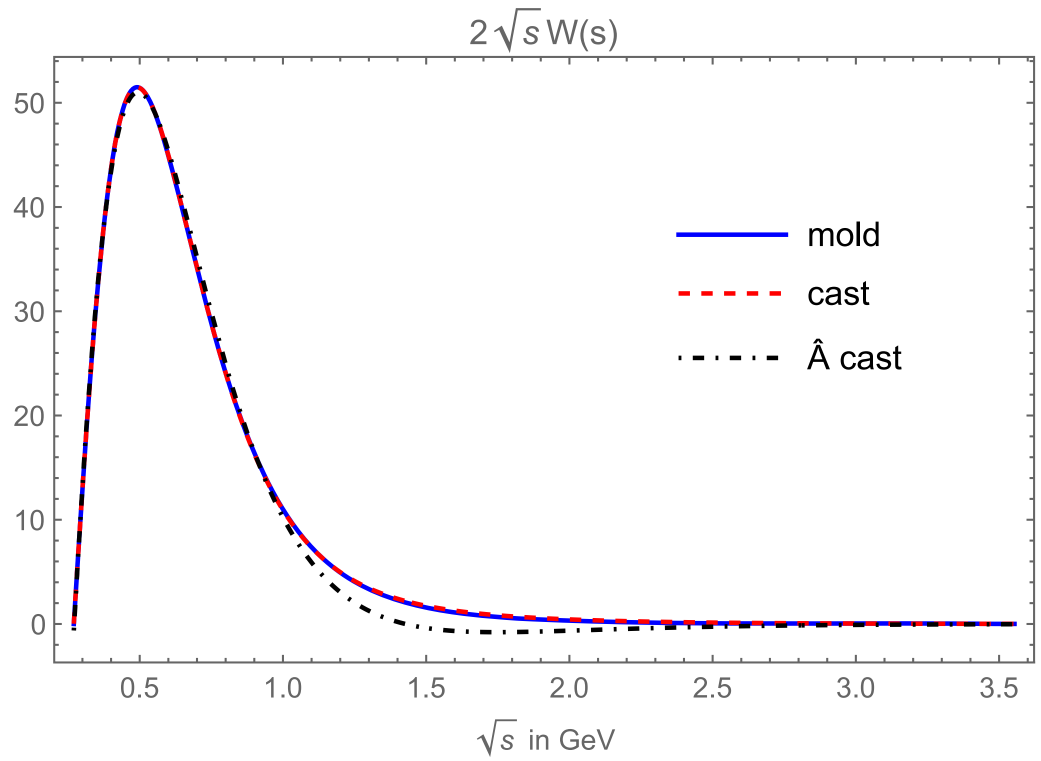
significantly reduced values of x_4 and x_5

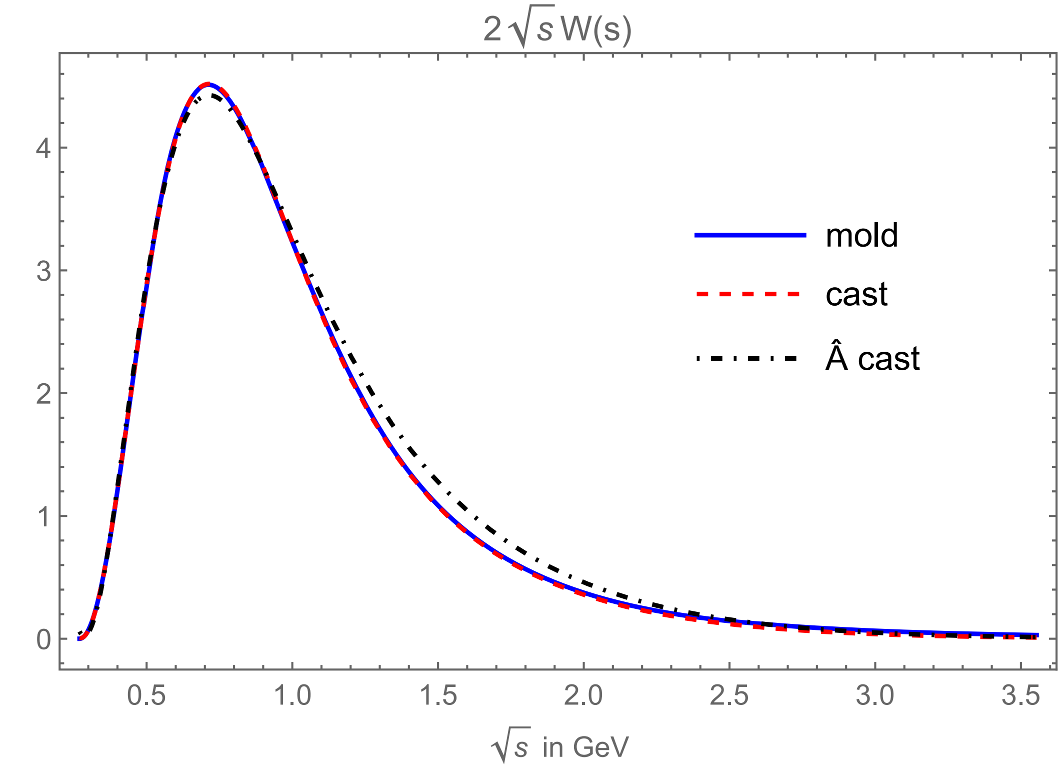
$$W'_{2,5}: x_1 = 34.0249, \quad x_2 = 870.640, \quad x_3 = -5501.14,$$
$$x_4 = 9933.01, \quad x_5 = -5284.24.$$

improved exponential weights

Price to pay: final cast less similar to the original mold

This is acceptable if we are interested in the general profile (zooming in a region in energy)





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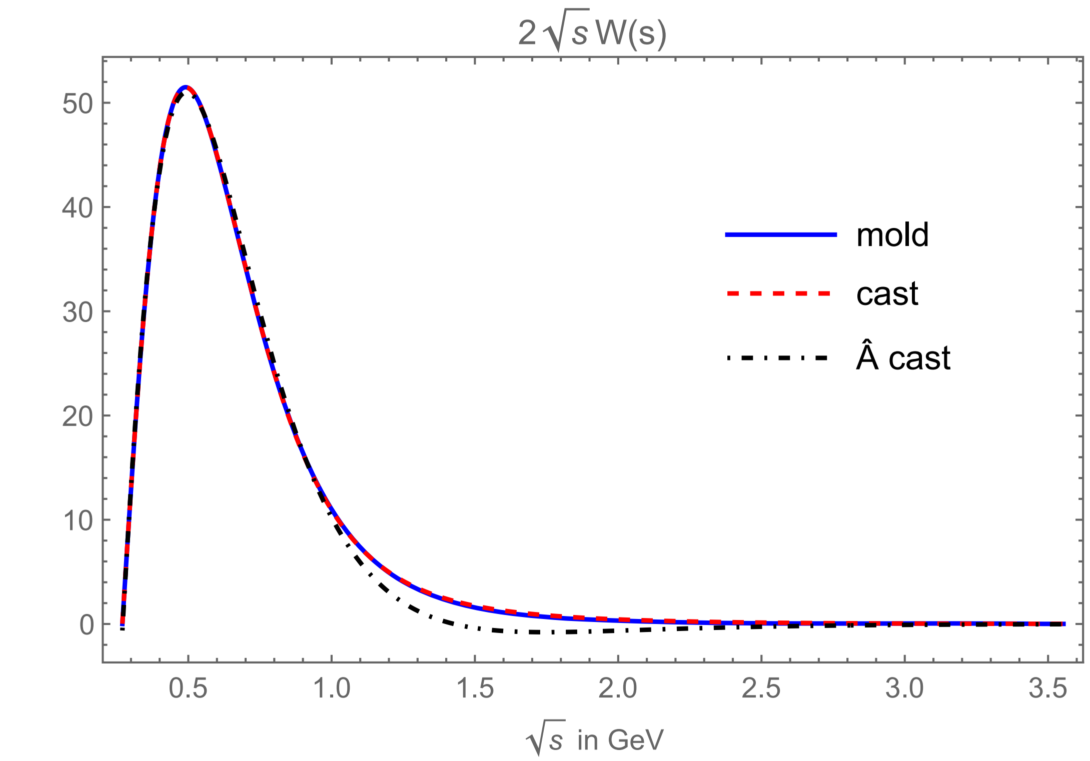
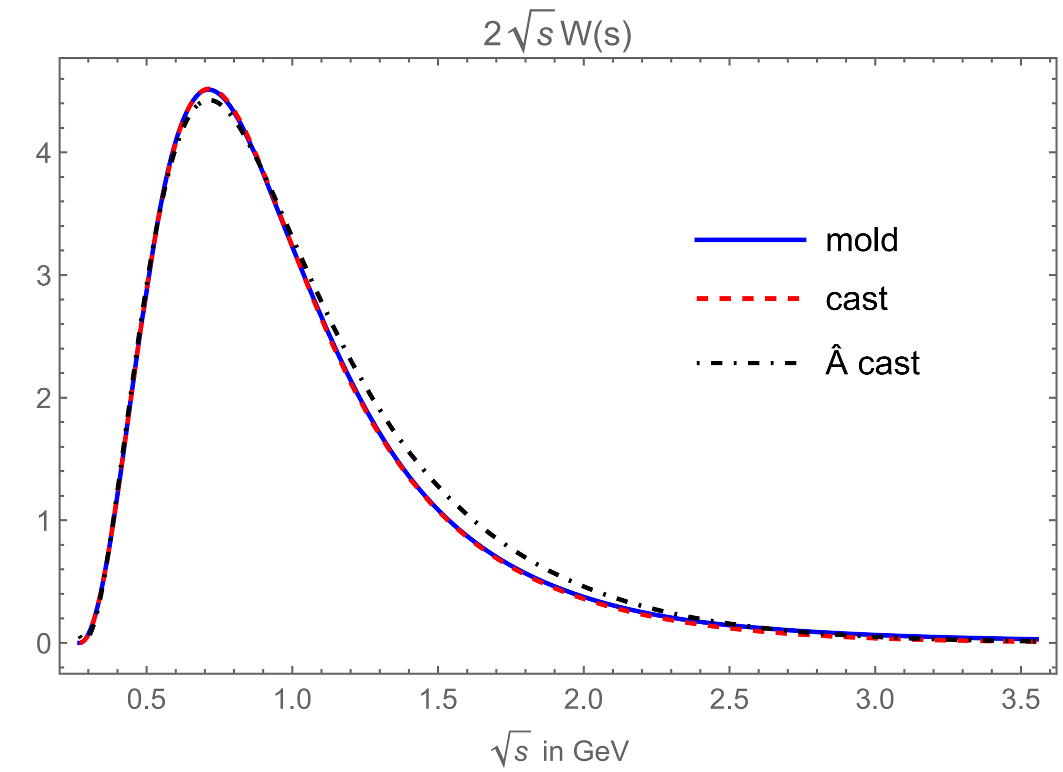
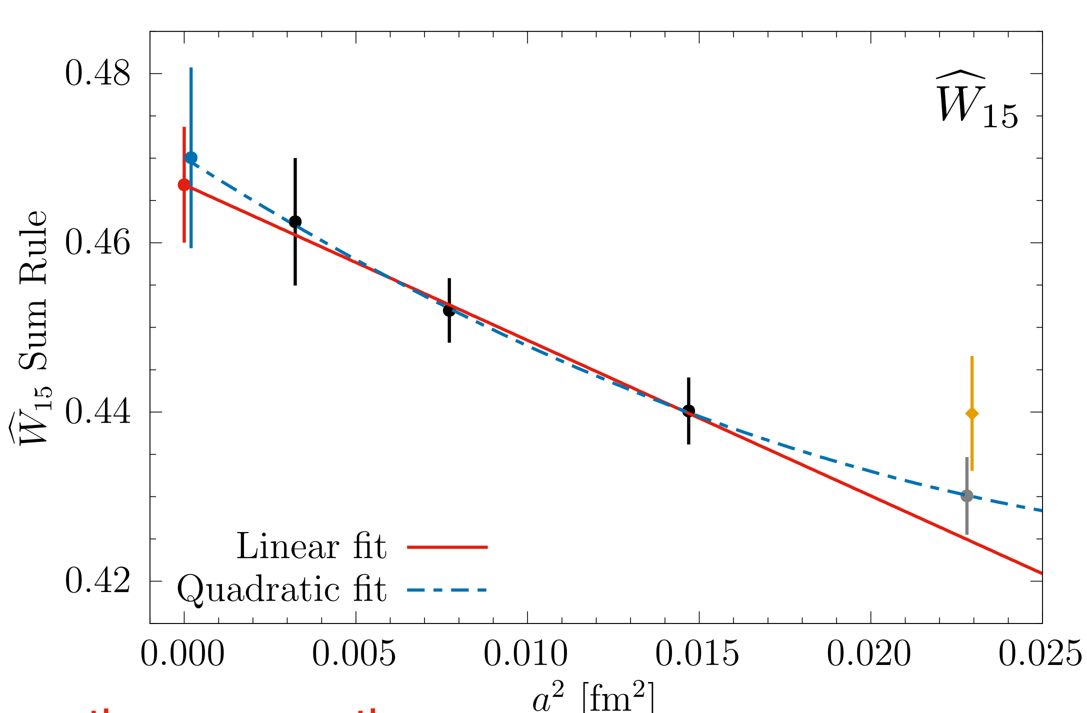
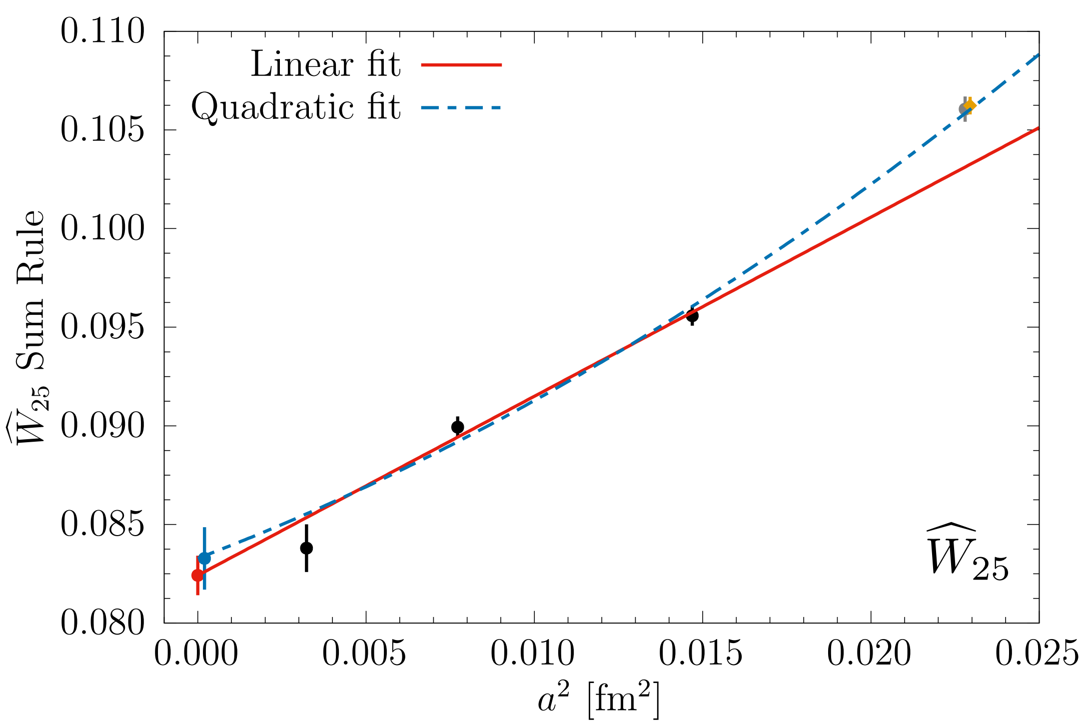








systematic errors on the lattice still to be assessed

lqc data from ABGP 22

improved exponential weights

Lattice: ABGP 22 lqc only! **Central values should not be directly compared.**

	R-ratio (R(s) data)	rel. error	lattice (rational weight)	rel. error	lattice (W', exponential weight)	rel. error
W_{15}	0.4756(16)	0.3%	0.468(26)	5.6%	0.496(17)	3.4%
W_{25}	0.08912(34)	0.4%	0.0838(33)	3.9%	0.0798(18)	2.3%

improved exponential weights

	lattice	rel. error
$\widehat{W}_{15}$	0.4669(68)	1.5%
$\widehat{W}_{25}$	0.0824(10)	1.2%

DB, Golterman, Maltman, and Peris, '22

Significant reduction in errors but still a factor ~3 to 5 larger than dispersive.

The procedure can still be fine tuned for a given data set.

Error reduction on the lattice data also expected.

improved exponential weights: lqc contribution

Merging the strategies: results for lqc contribution with improved exponential-weight sum rules

lqc from KNT19 R(s) data

$$I^{\rm lqc}_{\widehat{W}_{15}} = 42.78(16) \times 10^{-2}$$

$$I^{\rm lqc}_{\widehat{W}_{25}} = 78.85(46) \times 10^{-3}$$

Benton, DB, Golterman, Keshavarzi, Maltman, and Peris, in preparation

5.6σ

3.2σ

ABGP lqc lattice data

$$I^{\rm lqc}_{\hat{W}_{15}} = 46.69(68) \times 10^{-2}$$

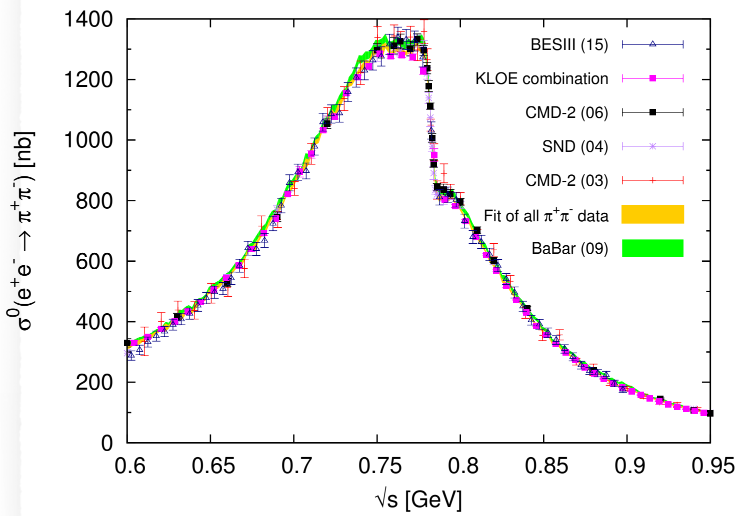
$$I^{\rm lqc}_{\hat{W}_{25}} = 82.4(1.0) \times 10^{-3}$$

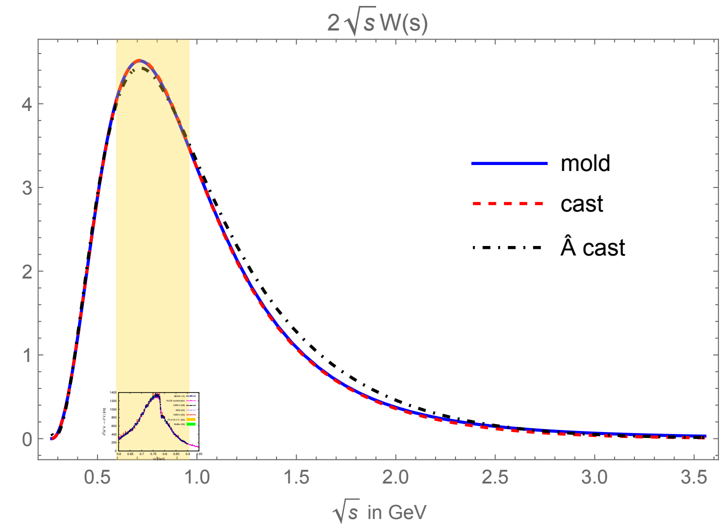
systematic errors on the lattice still to be assessed

Another indication of a tension between lattice and dispersive lqc results

lattice vs exp. data

- A potential application: BaBar/KLOE/(CMD-3) discrepancies





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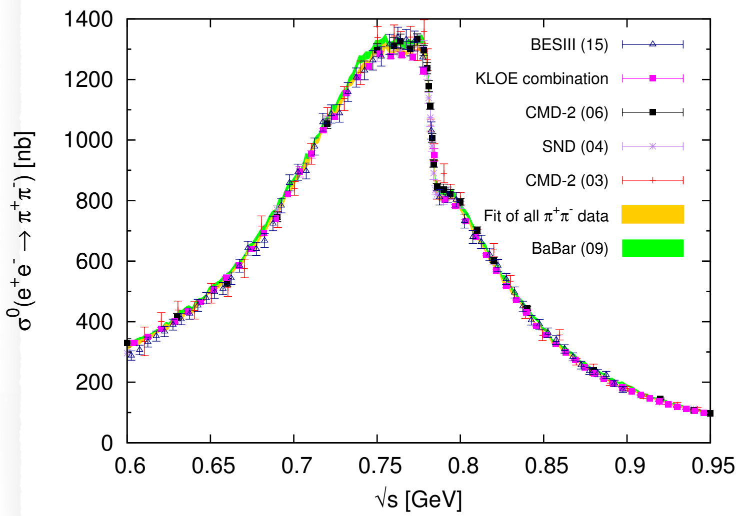

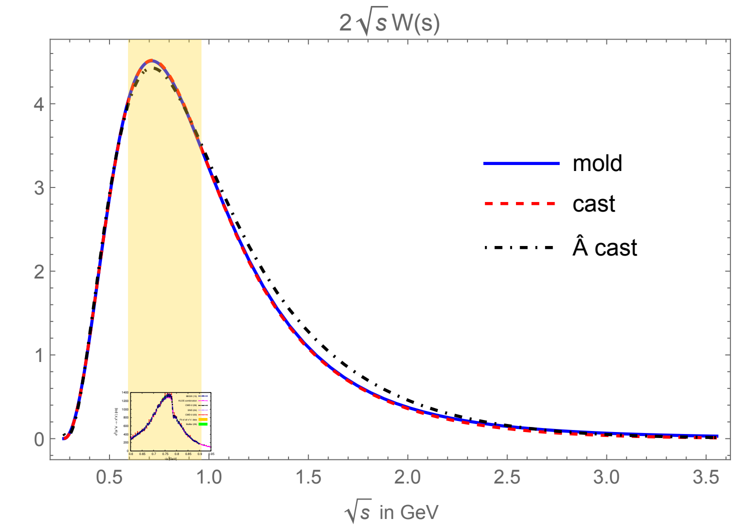

Babar and KLOE exp. data

$$I^{\pi\pi}_{\hat{W}_{1,5}}(BABAR) - I^{\pi\pi}_{\hat{W}_{1,5}}(\mathrm{KLOE}) = 0.0094(40),$$
$$I^{\pi\pi}_{\hat{W}_{2,5}}(BABAR) - I^{\pi\pi}_{\hat{W}_{2,5}}(\mathrm{KLOE}) = 0.00150(51)$$

ABGP lqc data

$$I_{\hat{W}_{1,5}}(\mathrm{rhs}) = 0.4669(68)$$
$$I_{\hat{W}_{2,5}}(\mathrm{rhs}) = 0.0824(10)$$

With smaller errors on the lattice side (factor of less than 2), exponential-weight sum rules can already be used to investigate the KLOE/Babar/(CMD-3) discrepancies.

conclusions

- We have **more than one** discrepancy in g - 2:

  experiment vs (dispersive based) Standard Model

  lattice HVP vs dispersive (R-ratio) HVP

  KLOE/Babar/CMD-3

  discrepancy in pt. QCD below charm (much smaller impact)

- The method of windows is a very important tool in the investigation of these discrepancies.
- Work needed on the lattice but many results in excellent agreement (e.g., lqc int. window)
- Lattice/*R*-ratio discrepancy resides mostly in the light-quark connected contribution which is dominated by pi+pi- on the data side (81%)
- New sum rules can help comparing lattice and *R*(s) data
- Exponential-weight sum rules can be tuned in order to reduce the error on the lattice side
- New sum rules may also be useful in a combination of tau data and lattice IB corrections

conclusions