The R-ratio from the lattice

Alessandro De Santis

Lattice Gauge Theory Contributions to New Physics Searches
(Muon g-2 session)

12-16 June 2023, Instituto de Fisica Teorica (IFT) UAM/CSIC Madrid

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Extended Twisted Mass Collaboration

INFN
ROMA TOR VERGATA

Based on:

Probing the energy-smeared R-ratio on the lattice

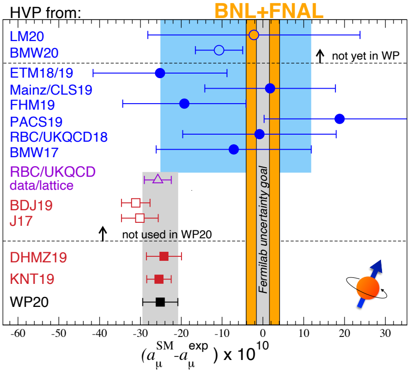
Constantia Alexandrou,[1,2] Simone Bacchio,[2] Alessandro De Santis,[3] Petros Dimopoulos,[4] Jacob Finkenrath,[2] Roberto Frezzotti,[3] Giuseppe Gagliardi,[5] Marco Garofalo,[6] Kyriakos Hadjiyiannakou,[1,2] Bartosz Kostrzewa,[7] Karl Jansen,[8] Vittorio Lubicz,[9] Marcus Petschlies,[6] Francesco Sanfilippo,[5] Silvano Simula,[5] Nazario Tantalo,[3] Carsten Urbach,[6] and Urs Wenger[10]
(Extended Twisted Mass Collaboration (ETMC))

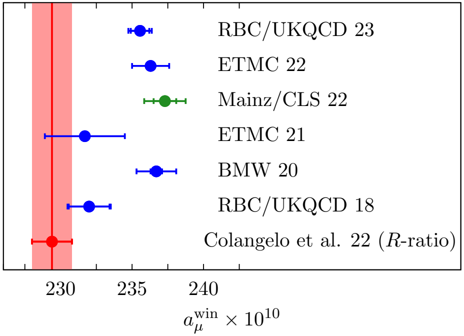
Soon on PRL ... now on arXiv:2212.08467

- ▷ Motivations
- ▷ **Spectral densities in Lattice QCD**
- ▷ **HLT method**
- ▷ Results for the R-ratio
- ▷ Outlook and conclusions

MOTIVATIONS

A picture is worth a thousand words (see talks on Monday morning's session):





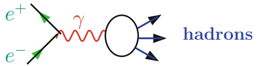
"Our accurate lattice results in the short and intermediate windows point to a possible ***deviation*** *of the* $\mathbf{e^+e^-}$ ***cross section data*** *with respect to Standard Model predictions in the low and intermediate energy regions."*

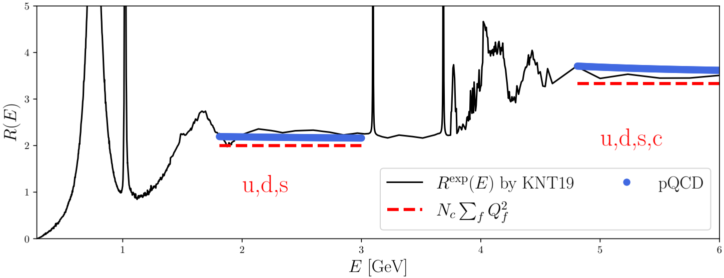
(ETMC PhysRevD.107.074506)

What is the R-ratio?

Experimentally:

$$R(E) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$





On the **Lattice** side:

$$\boldsymbol{C(t)} = \frac{1}{3}\sum_i \int \mathrm{d}^3x \left\langle \hat{J}_i(t,\boldsymbol{x})\hat{J}_i(0)\right\rangle = \int_{2m_\pi}^{\infty} \mathrm{d}\omega\, e^{-\omega t}\, \frac{\omega^2 \boldsymbol{R(\omega)}}{12\pi^2}$$

The R-ratio is the **spectral density** associated with $\boldsymbol{C(t)}$ so ...

SPECTRAL DENSITIES IN LATTICE QCD

Spectral densities from Euclidean correlation functions

$$C(t) = \int_0^\infty \mathrm{d}\omega \, e^{-\omega t} \rho_L(\omega)$$

t Euclidean time

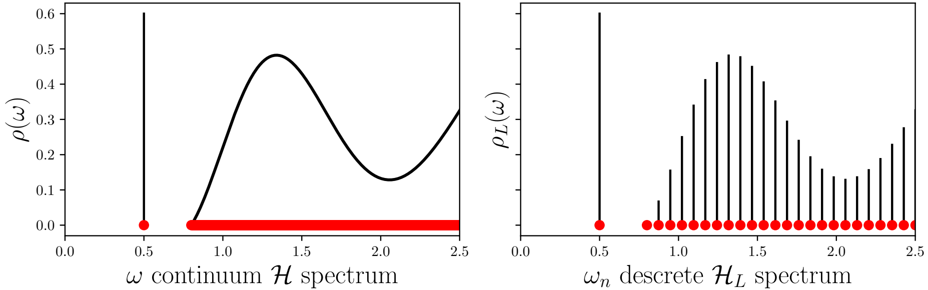
$L < \infty$ lattice volume

Numerically **ill-posed** inverse problem

- $t/a = 1, 2, 3, \cdots, T$ lack of information
- $C_i(t) = \bar{C}(t) + \delta C_i(t)$ imprecise data

Mathematically not well defined for $L < \infty$

- $\rho_L(\omega) = \sum_n f_n(L) \, \delta\big(\omega - \omega_n(L)\big)$ is a **distribution**



Even if the position and the coefficients of the peaks could be calculated exactly $\boldsymbol{\rho_L(\omega)}$ **cannot, in general, be associated with physical quantities** and

$$\lim_{L\to\infty} \rho_L(\omega) \qquad \text{is not defined}$$

Smeared spectral densities

$$\rho_{\sigma,L}(E) = \int_0^\infty \mathrm{d}\omega \, \Delta_\sigma(E,\omega) \, \rho_L(\omega)$$

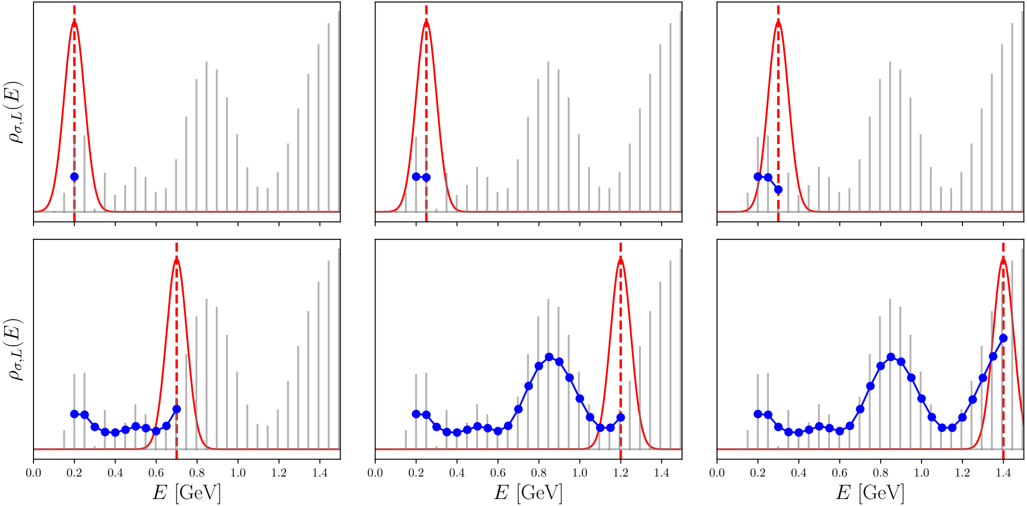
- The smearing kernel is such that $\Delta_\sigma(E,\omega) \mapsto \delta(E-\omega)$ when $\sigma \mapsto 0$
- The smeared spectral density is a **smooth** function of the energy

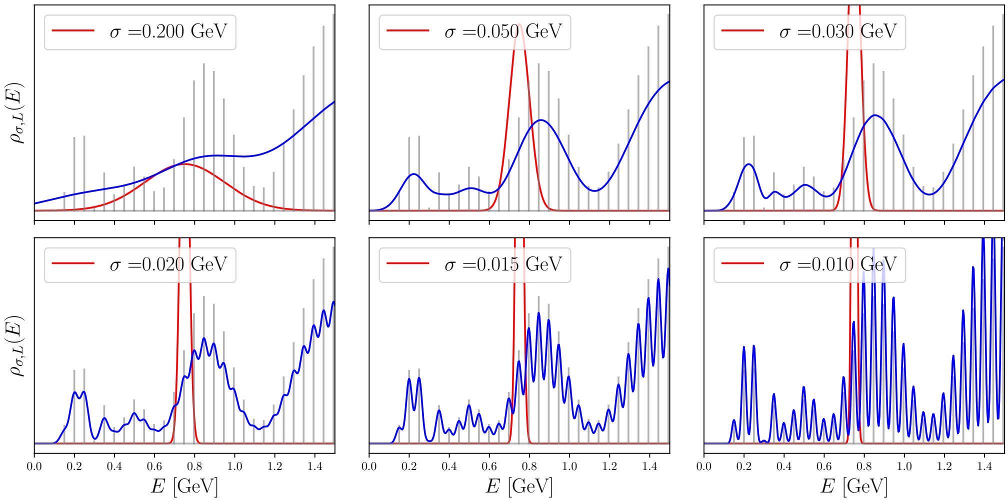
The infinite volume limit of $\rho_{\sigma,L}(E)$ is **well posed**

$$\rho(E) = \lim_{\sigma\to 0} \lim_{L\to\infty} \rho_{\sigma,L}(E)$$

Example

$$\rho_{\sigma,L}(E) = \int_0^\infty \mathrm{d}\omega \, \Delta_\sigma(\omega, E)\rho_L(\omega) \qquad \Delta_\sigma(\omega, E) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(E-\omega)^2}{2\sigma^2}\right)$$





$$\rho(E) = \lim_{\sigma\to 0} \lim_{L\to\infty} \rho_{\sigma,L}(E)$$

THE ORDER OF THE LIMITS IS IMPORTANT

In practice ...

- The $L \mapsto \infty$ limit is not a big deal (Bulava et al. JHEP 07 (2022) 034)

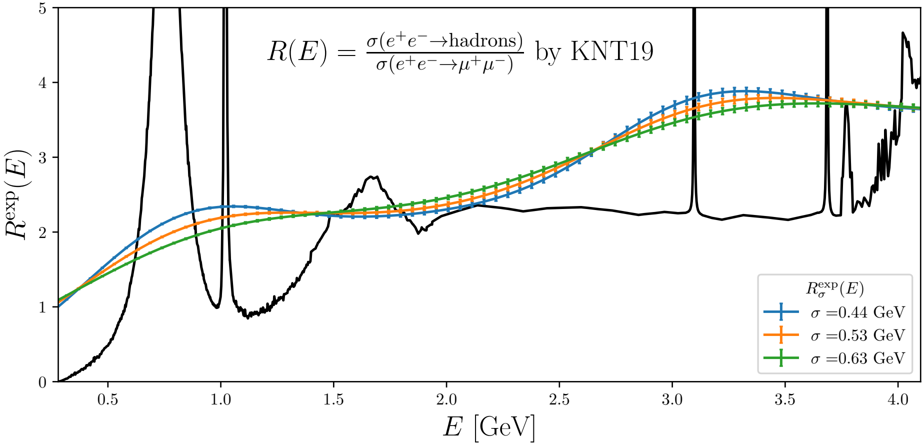
  $\rho_\sigma(E) - \rho_{\sigma,L}(E) = \mathcal{O}(L^{-\infty})$

- The $\boldsymbol{\sigma} \mapsto$ **0 limit is feasible for smooth unsmeared** spectral densities ...

- ... when possible we can instead **smear the experimental** result $\rightarrow$ $\rho_\sigma^{\mathrm{exp}}(E)$

What we actually calculated

We calculated $R_\sigma(E)$ for $\sigma = 0.44$ GeV, $\sigma = 0.53$ GeV and $\sigma = 0.63$ GeV.
Experimentally this corresponds to



APPROACHES TO SPECTRAL RECONSTRUCTION:

- Bayesian framework (MEM, BR, Gaussian Processes ecc.)
- Machine Learning (Neural networks ecc.)
- Chebyshev Polynomials
- Backus-Gilbert $\mapsto$ **HLT**:

On the extraction of spectral densities from lattice correlators

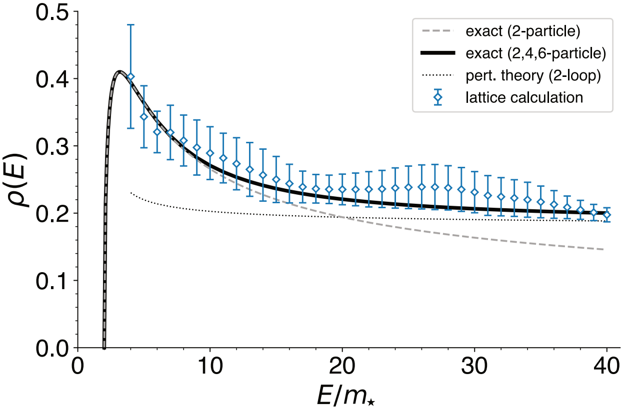
Martin Hansen,[1] Alessandro Lupo,[2] and Nazario Tantalo[3]

Phys. Rev. D 99, 094508

The mathematics of the *HLT method* had already been developed before (F. Pijpers, M. Thompson Astron.Astrophys. 262 (1992) L33). What we call the *HLT method* is the additional procedure used to estimate **reliably** the errors

This has been stringently and successfully tested ✓

- $O(3)$ non-linear σ-model
- Asymp.lly free, dynamically generated mass gap and **integrable** in 2D
- The spectral density is the analogous of the **R-ratio**
- Correlation functions calculated in Monte Carlo simulations



Inclusive rates from smeared spectral densities in the two-dimensional O(3) non-linear σ-model

John Bulava,[1,*] Maxwell T. Hansen,[2,†] Michael W. Hansen,[3,‡] Agostino Patella,[4,§] and Nazario Tantalo[5,¶]

(JHEP 07 (2022) 034)

$$\rho_\sigma(E) = \int_{E_{\rm th}}^{\infty} \mathrm{d}\omega \, \Delta_\sigma(E,\omega)\rho(\omega) \qquad \Delta_\sigma(E,\omega) \quad \textbf{smooth kernel} \quad \text{(e.g. the Gaussian)}$$

Approximate the target kernel with a finite number of basis functions

$$\Delta_\sigma(E,\omega) = \sum_{\tau=1}^{\infty} g_\tau e^{-a\omega\tau} \quad \mapsto \quad \Delta_\sigma^{\rm rec}(E,\omega) = \sum_{\tau=1}^{T<\infty} \boldsymbol{g_\tau} \, e^{-a\omega\tau}$$

and get the estimator for $\rho_\sigma(E)$

$$\rho_\sigma(E) \sim \sum_{\tau=1}^{T} g_\tau \underbrace{\int_{E_{\rm th}}^{\infty} \mathrm{d}\omega \, e^{-a\omega\tau} \rho(\omega)}_{C(a\tau)} = \sum_{\tau=1}^{T} \boldsymbol{g_\tau} \, C(a\tau)$$

LINEAR PROBLEM

We only need to estimate the **systematic error due to imperfect reconstruction** of the kernel since $T < \infty$

- The $\boldsymbol{g}$ coefficients are calculated by minimizing

$$W[\lambda, \boldsymbol{g}] = (1-\lambda)\frac{A_\alpha[\boldsymbol{g}]}{A_\alpha[\mathbf{0}]} + \lambda B[\boldsymbol{g}]$$

- Suppression of the **statistical error**

$$B[\boldsymbol{g}] \propto \boldsymbol{g}^T \cdot \mathrm{C\hat{O}V}\left[C(t)\right] \cdot \boldsymbol{g} \equiv \left(\Delta_\rho^{\mathtt{stat}}\right)^2$$

- **Accuracy of the approximated** kernel

$$A_\alpha[\boldsymbol{g}] = \int_{E_{\mathrm{th}}}^{\infty} \mathrm{d}\omega \left\{ \boldsymbol{\Delta_\sigma(\omega, E)} - \sum_{\tau=1}^{T} g_\tau e^{-a\omega\tau} \right\}^2 \cdot e^{\alpha\omega}$$

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$$B[\boldsymbol{g}] \propto \boldsymbol{g}^T \cdot \hat{\mathrm{COV}}\left[C(t)\right] \cdot \boldsymbol{g} \equiv \left(\Delta_\rho^{\mathrm{stat}}\right)^2$$

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- **Unbiased** procedure: exact result is recovered for $T \mapsto \infty$ and $\hat{\mathrm{COV}} \mapsto \hat{0}$

- λ is a **trade-off parameter** between statistical precision and relative error of the approximated kernel

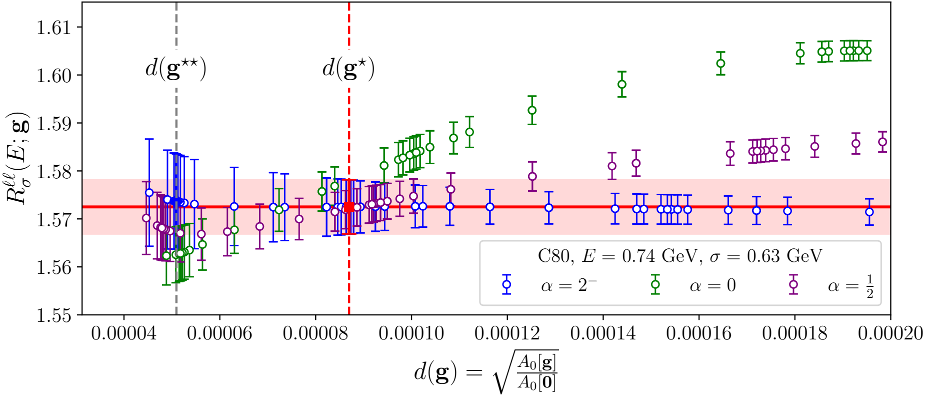
- α (< 2) defines different **norms** to calculate $\Delta_\sigma^{\mathrm{rec}}(\omega, E)$

- Reference norm

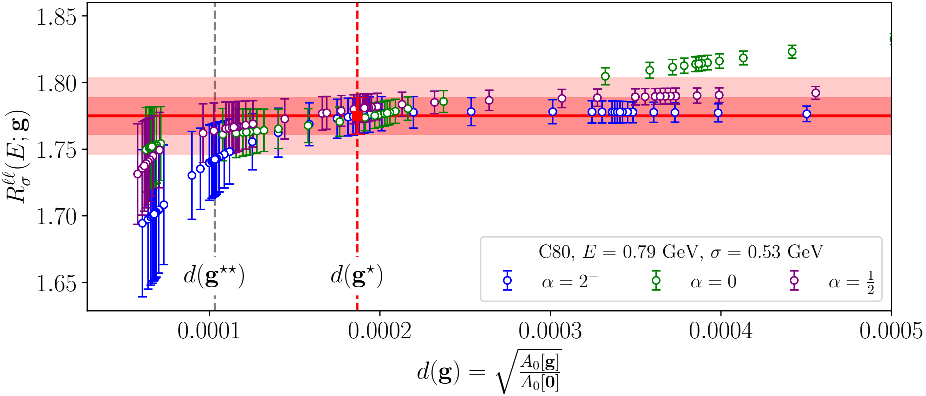
$$d(\boldsymbol{g}) = \sqrt{\frac{A_0[\boldsymbol{g}]}{A_0[\mathbf{0}]}} \quad \leftrightarrow \quad \lambda$$

Stability analysis to tune λ and estimate the errors (HLT procedure)

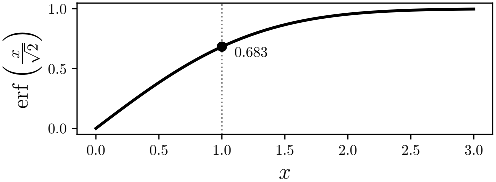
We look for a **stability region** in which systematic fluctuations are within statistical errors and **results for different norms are statistically compatible**



$$d(\mathbf{g}^\star): \quad \frac{A_{2^-}[\mathbf{g}]}{A_{2^-}[\mathbf{0}]} = 10B[\mathbf{g}] \qquad d(\mathbf{g}^{\star\star}): \quad \frac{A_{2^-}[\mathbf{g}]}{A_{2^-}[\mathbf{0}]} = B[\mathbf{g}] \qquad \alpha = 2^-$$



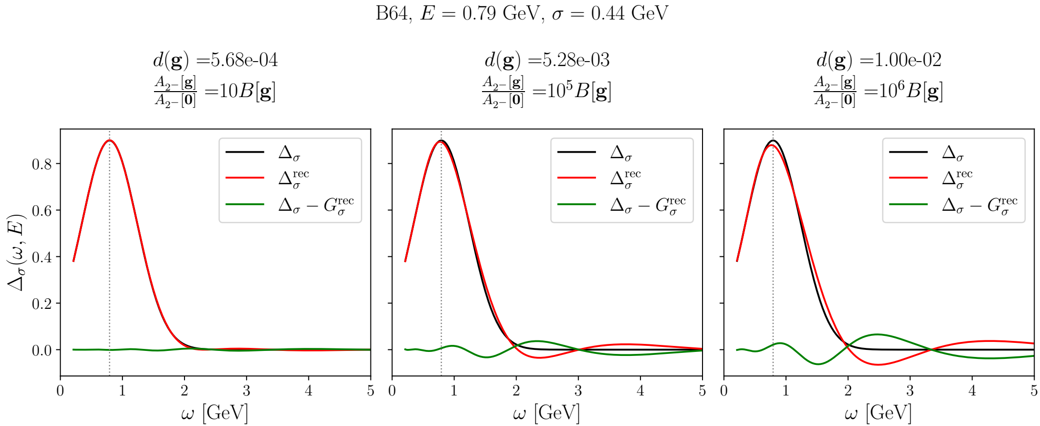
We include fluctuations larger than the statistical error as a **reconstruction systematic uncertainty**



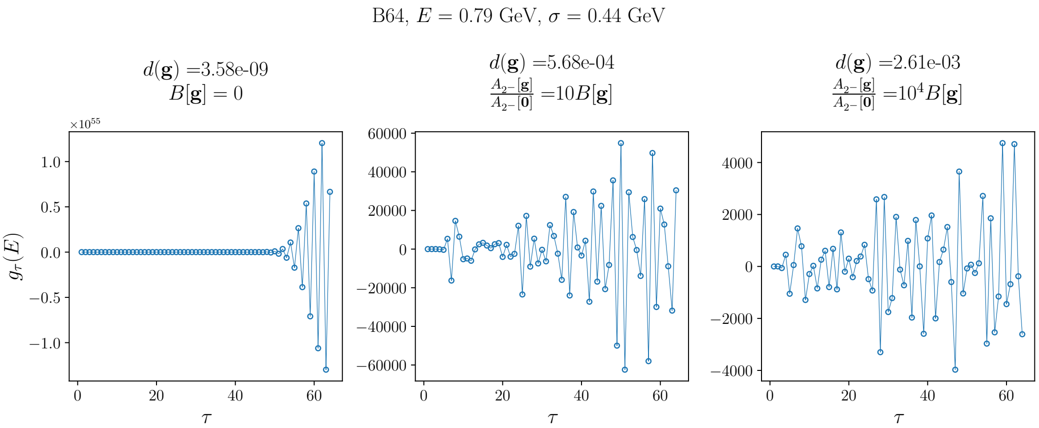
$$P_\sigma(E) = \frac{R_\sigma(E;\boldsymbol{g}^\star) - R_\sigma(E;\boldsymbol{g}^{\star\star})}{\Delta_\sigma^{\mathrm{stat}}(E;\boldsymbol{g}^{\star\star})}$$

$$\boldsymbol{\Delta_\sigma^{\mathrm{rec}}(E)} = |R_\sigma(E;\boldsymbol{g}^\star) - R_\sigma(E;\boldsymbol{g}^{\star\star})|\,\mathrm{erf}\left(\frac{P_\sigma(E)}{\sqrt{2}}\right)$$

Kernel reconstruction



The effect of the noise regulator (B[g] functional)



- In absence of a noise regulator the ill-posedness manifests itself through large and oscillating g_τ coefficients $\mapsto$ **Extended arithmetic precision is required**

Our analysis of the R-ratio (ETMC arXiv:2212.08467)

Ensemble details at PhysRevD.107.074506 (ETMC)

ensemble	$L^3 \cdot T$	a (fm)	L (fm)	M_π (MeV)	β
B96	$96^3 \cdot 192$	0.07961(13)	7.64	135.2(2)	1.778
B64	$64^3 \cdot 128$	0.07961(13)	5.09	135.2(2)	1.778
C80	$80^3 \cdot 160$	0.06821(12)	5.46	134.9(3)	1.836
D96	$96^3 \cdot 192$	0.05692(10)	5.46	135.1(3)	1.900

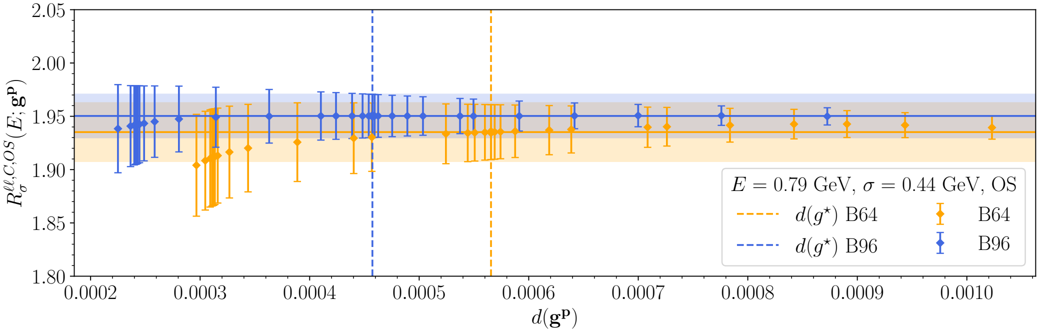
- Iso-symmetric QCD with $N_f = 2+1+1$ dynamical quarks
- Both **connected** and **disconnected** contributions to $C(t)$ included
- Two regularizations: Twisted Mass (**TM**) and Osterwalder-Seiler (**OS**)

✗ **Missing** QED and strong Isospin-Breaking corrections

$R_\sigma(E)$ calculated for $\sigma = 0.44$, 0.53 and 0.63 GeV from $E = 0.25$ GeV to 2.4 GeV.

(data-driven) Finite volume effects estimation

We check that different volumes are statistically compatible ($L_{\rm B64} \simeq 5$ fm, $L_{\rm B96} \simeq 7.6$ fm)



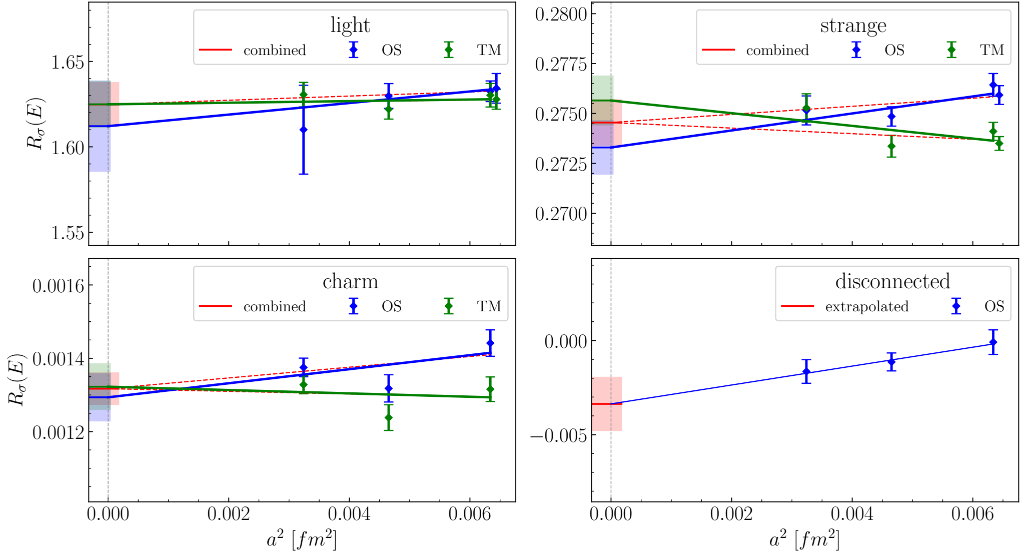
and consider as associated **systematic** uncertainty

$$P_\sigma^L(E) = \frac{R_\sigma(E;\mathrm{B96}) - R_\sigma(E;\mathrm{B64})}{\sqrt{\Delta_\sigma^{\rm stat}(E;\mathrm{B96})^2 + \Delta_\sigma^{\rm stat}(E;\mathrm{B64})^2}}$$

$$\boldsymbol{\Delta_\sigma^L(E)} = |R_\sigma(E;\mathrm{B96}) - R_\sigma(E;\mathrm{B64})|\,\mathrm{erf}\left(\frac{P_\sigma^L(E)}{\sqrt{2}}\right)$$

Continuum extrapolation

Linear constrained and unconstrained ansatz ($E = 0.79$ GeV, $\sigma = 0.63$ GeV),

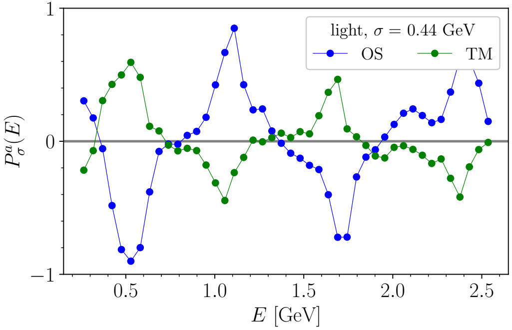


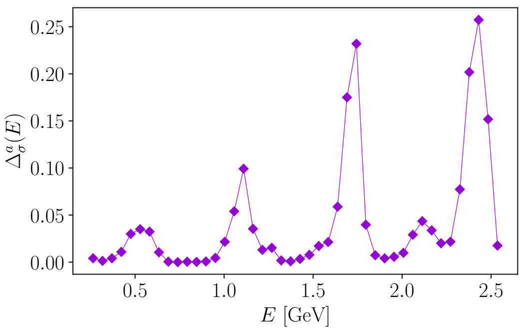
Continuum extrapolation and data-driven systematic error

We check that constrained and unconstrained continuum extrapolations are compatible and define as **systematic** uncertainty

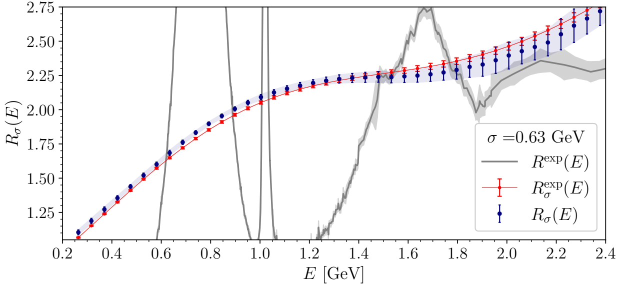
$$P_{\sigma,\mathrm{reg}}(E) = \frac{R_\sigma^{\mathrm{comb}}(E) - R_\sigma^{\mathrm{reg}}(E)}{\sqrt{\Delta^2_{\mathrm{comb}}(E) + \Delta^2_{\mathrm{reg}}(E)}} \qquad \boldsymbol{\Delta^a_\sigma(E)} = \max_{\mathrm{reg}=\{\mathrm{OS,TM}\}} |R_\sigma^{\mathrm{comb}}(E) - R_\sigma^{\mathrm{reg}}(E)| \mathrm{erf}\left(\frac{P^a_{\sigma,\mathrm{reg}}(E)}{\sqrt{2}}\right)$$

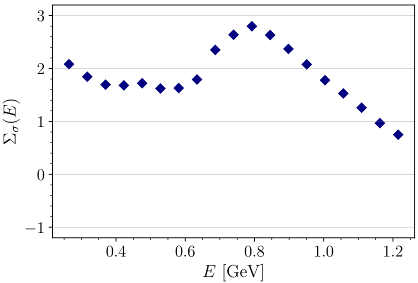
We do the pull

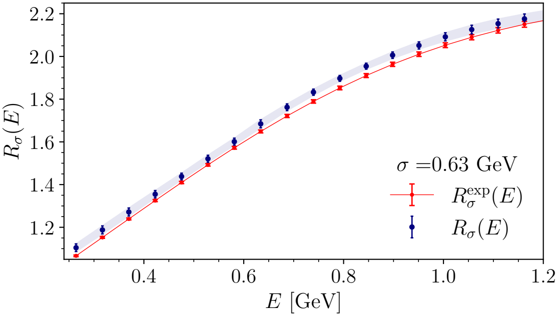




Final results for $\sigma = 0.63$ GeV:

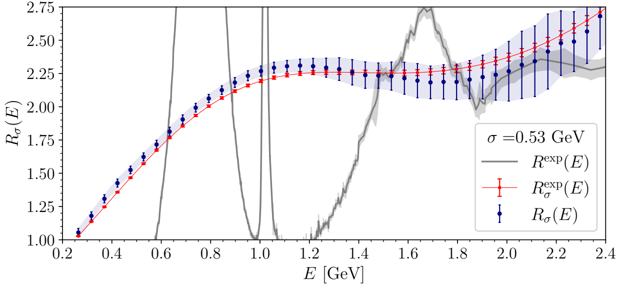


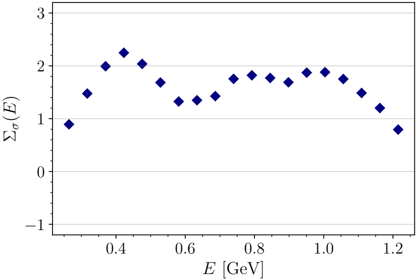


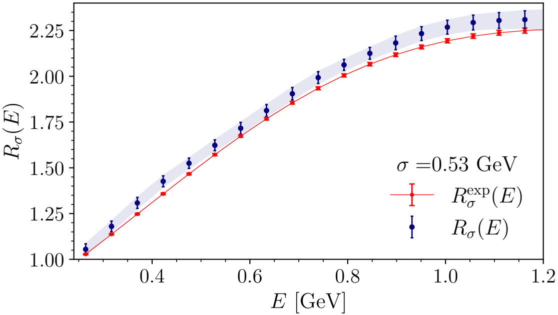


- ▷ 1% precision, enough to compare to experiments: general good agreement but
- ▷ ~ 3 standard deviation from $R^{\rm exp}_\sigma(E)$ around $E = 0.8$ GeV
- ▷ Our result is above $R^{\rm exp}_\sigma(E)$ consistently with a_μ^W and $a_\mu^{\rm HVP}$ determinations from lattice QCD

Final results for $\sigma = 0.53$ GeV:

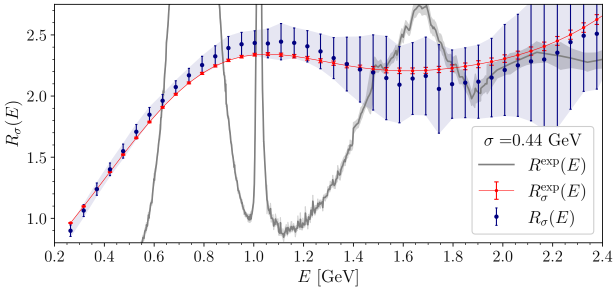


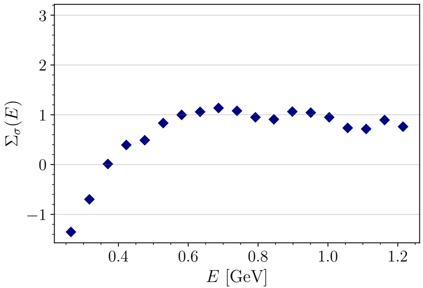


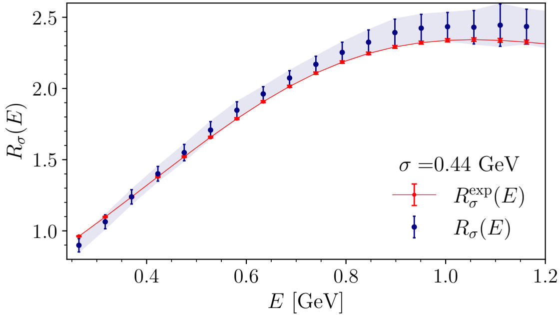


- Same trend when increasing the resolution (decrease σ)
- Our errors start increasing: the stability region moves towards larger statistical errors
- The increase of $R_\sigma(E)$ around 2.2 GeV shows sensitivity to charmonium states (J/Ψ, Ψ ecc.)

Final results for $\sigma = 0.44$ GeV:





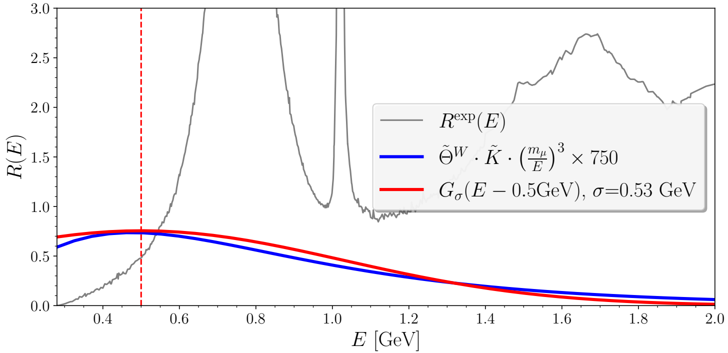


- The presence of low energy structures ($\rho - \omega$ mixing, ϕ resonance) is slightly appreciable but
- our total error is now that large that a comparison with experiments is no longer significant

FUTURE DIRECTIONS AND OUTLOOKS

Future directions 1: Isospin Breaking corrections

- Gaussian kernels are not much different from the kernel providing $a_\mu^{\mathrm{W}} \to a_\mu^{\mathrm{W}}(\mathrm{IB}) \sim 0.2\%$ (by BMW)
- However $\dfrac{R_\sigma(E)}{R_\sigma^{\mathrm{exp}}(E)} - 1 \sim \mathcal{O}(5\%)$ at $E = 0.5$ GeV

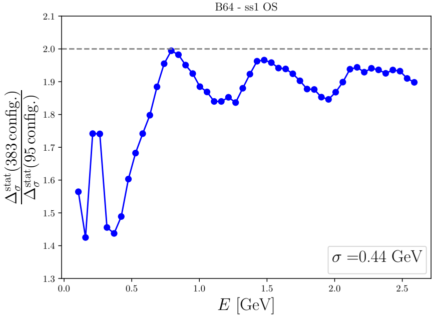


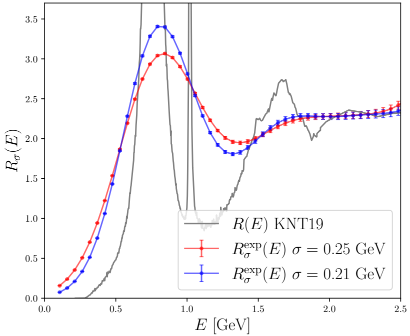
Anyway IB correction may become important around $E = 0.8$ GeV and for $\sigma << 0.44$ GeV

$\mapsto$ Isospin-breaking effects will be be computed from first principles (*à la* RM123)

Future directions 2: Reduce σ

- $\sigma \mapsto 0$ **hardly feasible** due to resonances
- Ease the conservative systematic error estimation
- The statistics has a major impact on the statistical error

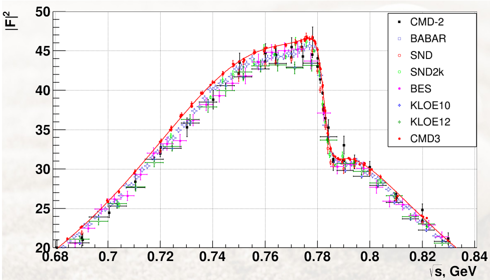


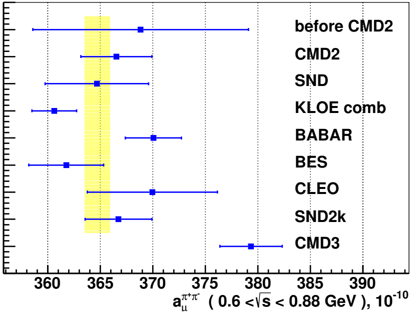


We aim at $\sigma \sim 250$ MeV: it may be enough to **better localize** the source of the discrepancy

Outlooks: recent CMD-3 results (arXiv:2302.08834) ...

Recent analysis on the dominant $e^+e^- \to \pi^+\pi^-$ channel ($\sim$ 70% of a_μ^{HVP}) significantly (and inexplicably) deviate from previous compilations exactly where we see deviation form $R_\sigma^{\text{exp}}(E)$





▷ Including a new data set compilation requires an accurate and delicate work

Conclusions

- Hadronic spectral densities, like the R-ratio, need to be **smeared in finite volume**. The **HLT** provides a method with **reliable error estimation** to the calculation of such quantities
- Our analysis of $R_\sigma(E)$ for $\sigma = 0.44$, 0.53 and 0.63 GeV, **although still missing the IB corrections**, **confirms the deviation from $\boldsymbol{R_\sigma^{\text{exp}}(E)}$** for energy below 1.2 GeV
- The study of the smeared R-ratio may shed light on the origin of one of the most puzzling tension in Standard Model VS Nature
- Increase of the statistics + improved optimal point selection criterion **should allow to go down to $\boldsymbol{\sigma = 250}$** MeV with controlled errors
- The origin of this puzzle is still open but **lattice results are very solid and in agreement**, so
  - either we are facing a clear signal of **new physics**
  - .. or the **experiments need to be revised** (unlikely they are all wrong but after CMD-3 results ...)

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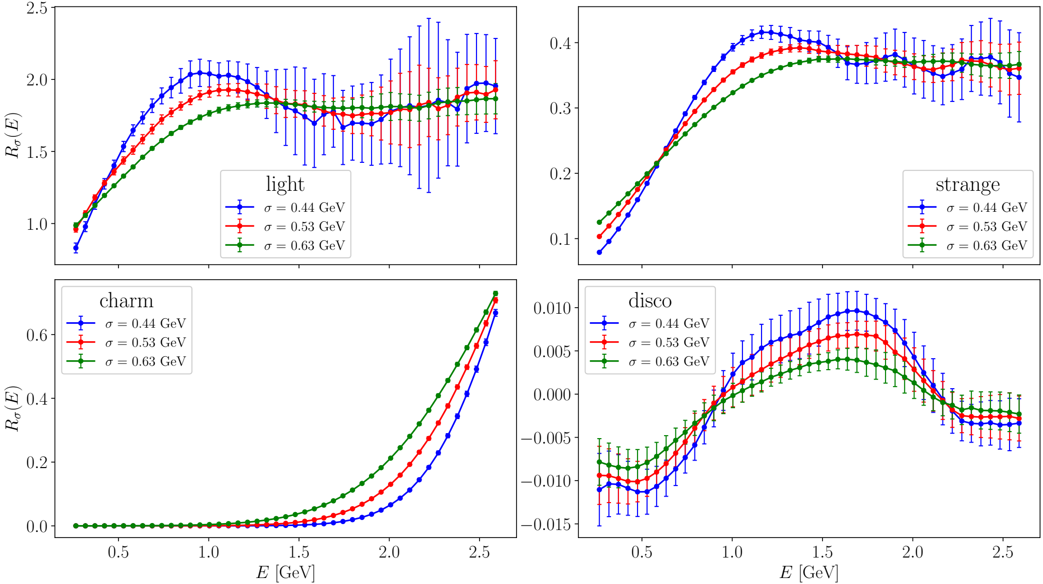
Thank you for the attention!

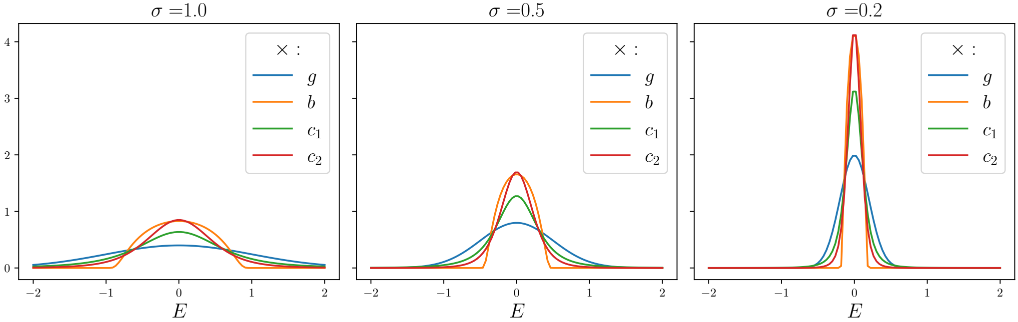
Backup slides

Final results separated by flavour



Smearing functions

$$\lim_{\sigma \to 0} \Delta_\sigma^{\times}(E,\omega) = \delta(E-\omega) \qquad \int_{-\infty}^{\infty} \mathrm{d}\omega \Delta_\sigma^{\times}(E,\omega) = 1$$

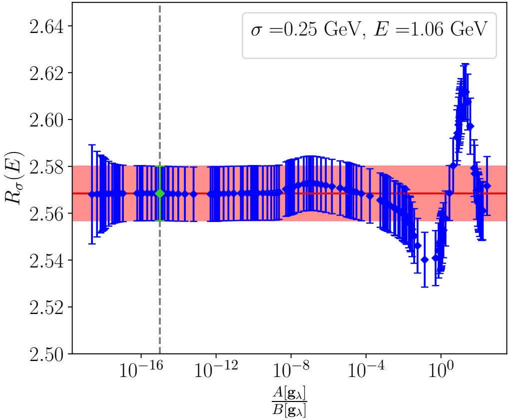


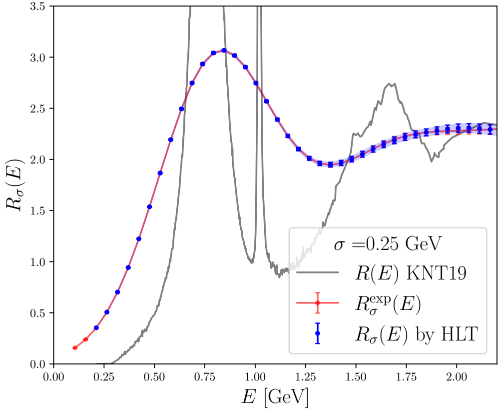
$x = E - \omega$

$$\Delta_\sigma^{g}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \qquad \Delta_\sigma^{b}(x) = \frac{1}{\sigma c} \exp\left[\frac{1}{\left(\frac{x}{\sigma}\right)^2 - 1}\right]$$

$$\Delta_\sigma^{c_1}(x) = \frac{2}{\pi} \frac{\sigma^3}{(x^2+\sigma^2)^2} \qquad \Delta_\sigma^{c_2}(x) = \frac{8}{3\pi} \frac{\sigma^5}{(x^2+\sigma^2)^3}$$

HLT applied to experimental $R(E)$ ✓





- ▷ Error consistent with the direct smearing of $R^{\text{exp}}(E)$
- ▷ Only diagonal part of covariance matrix

Motivation to the norm $e^{\alpha\omega}$

$$A[\boldsymbol{g}] = \int_{E_{\text{th}}}^{\infty} \mathrm{d}\omega \left\{ \Delta_{\sigma}^{\text{true}}(\omega, E) - \Delta_{\sigma}^{\text{approx}}(\omega, E) \right\}^2 \cdot e^{\alpha\omega}$$

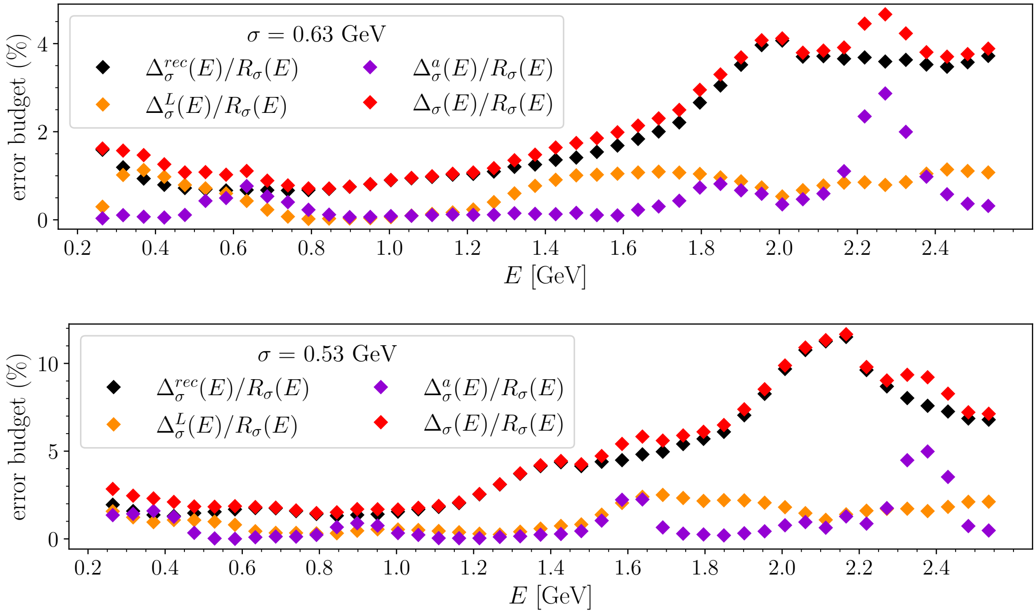
The exact systematic error is

$$\Delta^{\text{syst}}(E) \equiv \rho[\Delta_{\sigma}^{\text{true}}](E) - \rho[\Delta_{\sigma}^{\text{approx}}](E) = \int_{E_{\text{th}}}^{\infty} \mathrm{d}\omega \rho(\omega) \left| \Delta_{\sigma}^{\text{true}}(\omega, E) - \Delta_{\sigma}^{\text{approx}}(\omega, E) \right|$$

$\rho(\omega)$ in general increases as a power of the energy (Axiomatic QFT).

If $\left|\Delta_{\sigma}^{\text{true}}(\omega, E) - \Delta_{\sigma}^{\text{approx}}(\omega, E)\right|$ is forced to decrease exponentially fast thanks to $e^{\alpha\omega}$ with $\alpha > 0$, then $\Delta^{\text{syst}}(E)$ does not exhibit wildly oscillation for large E giving more stability to $\rho_{\sigma}(E)$

Relative error budget



Relative error budget