Holographic Complexity Beyond Proposals and AdS Holography

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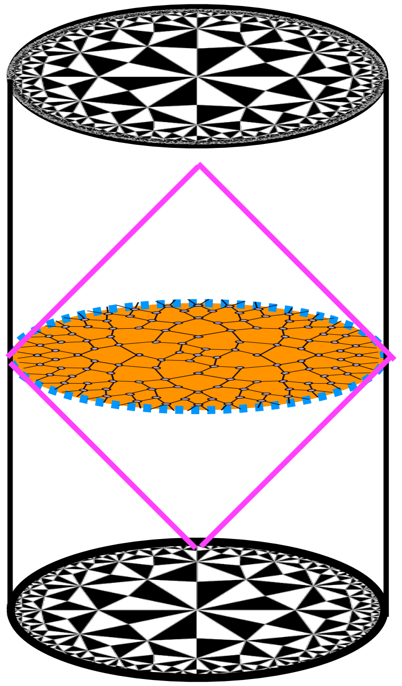
based on **2212.00043** with Erdmenger, Gerbershagen, Weigel, **2112.12158** + Flory
2305.11280 with Aguilar-Gutierrez and Van der Schueren

Introduction

Holographic complexity till 2016

1402.5674 by Susskind, 1509.07876 by Brown et al., 1610.02038 by Couch et al., …



$\mathcal{C}_V \sim$ volume of $\begin{matrix}\text{max (Lorentzian)} \\ \text{min (Euclidean)}\end{matrix}$ volume time slice

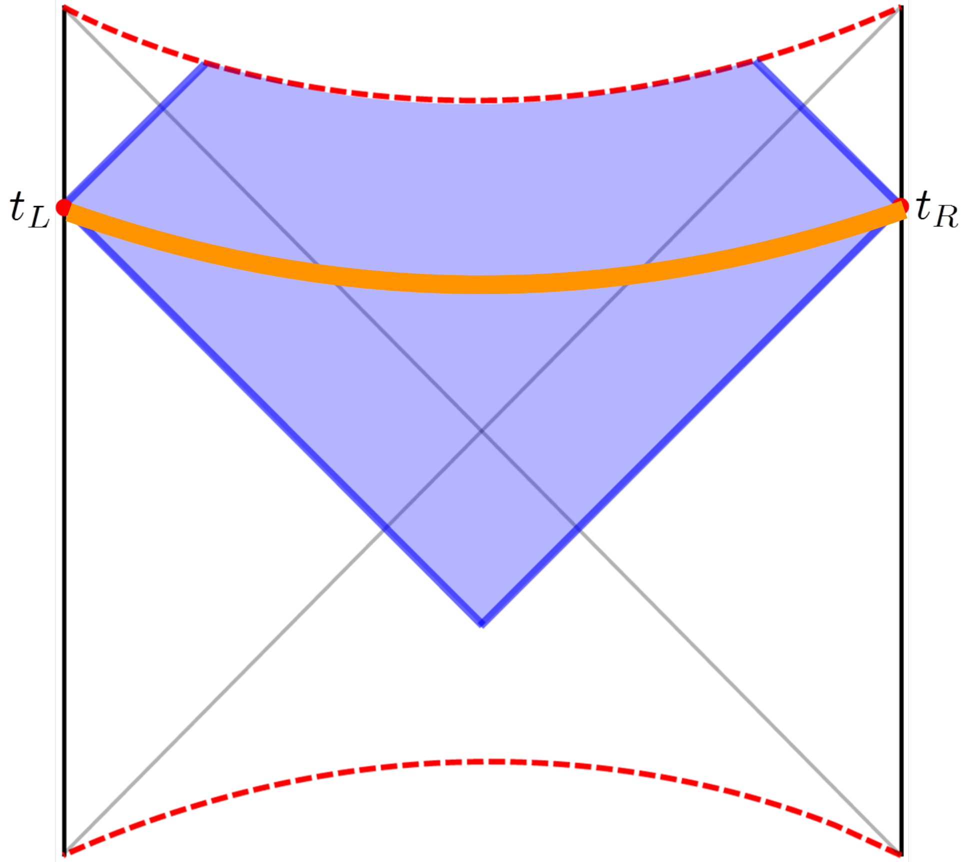
$\mathcal{C}_A \sim$ bulk action in the Wheeler - de Witt patch

$\mathcal{C}_{V\,2.0} \sim$ bulk volume of the Wheeler - de Witt patch

As of mid 2010s, novel ways of characterizing states in holographic QFTs

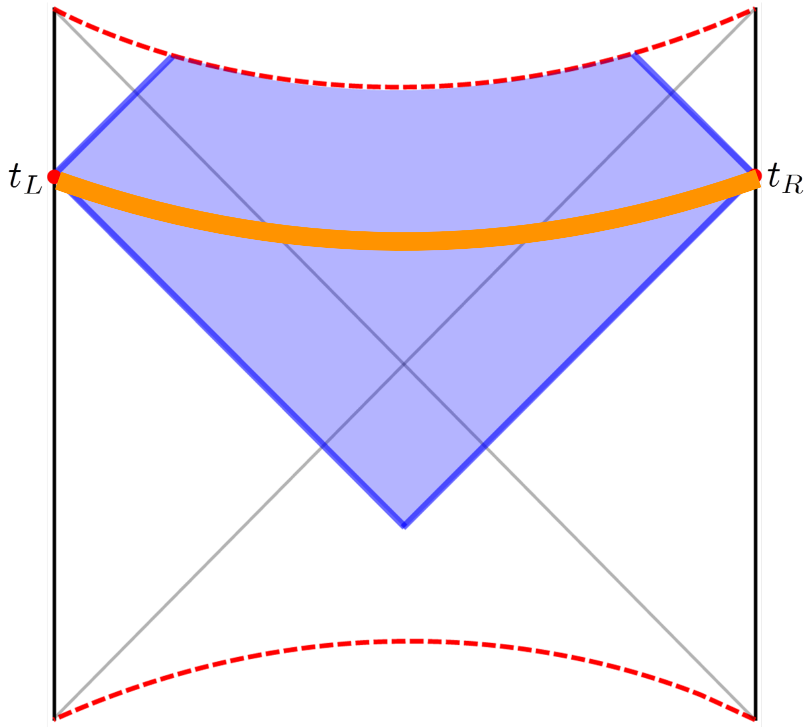
Why interesting for string theory?

Mainly because holographic complexity are natural probes of black hole interior



In particular, they capture its persistent growth in GR: $C_{V,A,V2.0}|_{t_L+t_R\gg\beta} \sim t_L + t_R$

Why is it called holographic complexity?



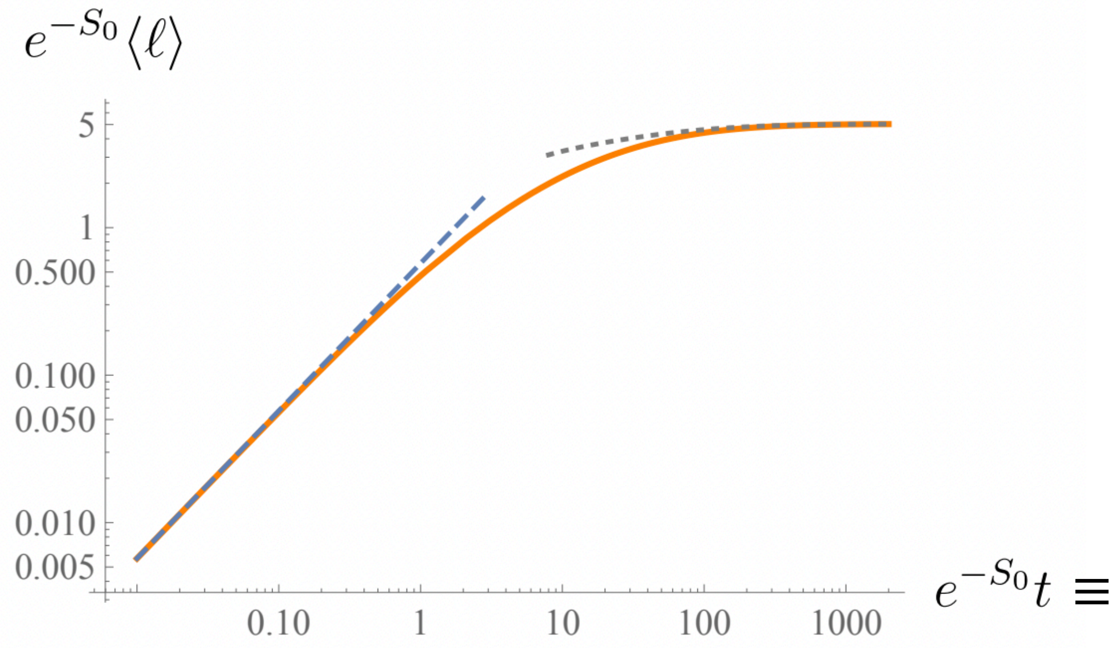
$$\longleftrightarrow \qquad |TFD(t_L, t_R)\rangle \sim e^{-iH_L t_L} e^{-iH_R t_R} \sum_E e^{-\beta E/2} |E\rangle_L |E\rangle_R$$

If we represent Hamiltonian time evolution as a tensor network and count the number of tensors, then one gets the wanted linear growth

However, if one additionally requires that that this tensor network is optimized, then at some point shortcut circuits not requiring further growth will appear

Post 2016: Saturation of holographic complexity

2107.06286 by Iliesiu, Mezei and Sárosi considered a quantum generalization of CV in JT gravity path integral and obtained that it saturates after exp time in BH entropy



$$= \langle \ell(t) - \ell(0) \rangle \approx \begin{cases} \hat{C}_1 \hat{t} + \dots & \hat{t} \ll 1 \\ C_0 - e^{S_0} A \, \frac{\hat{t}^2 \exp\left[-\frac{\beta \log^2(2\pi\hat{t})}{8\pi^2}\right]}{\log^2(2\pi\hat{t})} + \dots & \hat{t} \gg 1 \end{cases}$$

Post 2016: QFT complexity

The approach that naturally applies to QFTs comes from quant-ph/0502070 by Nielsen :

$$|T\rangle \sim U|R\rangle$$

with

$$U = \mathcal{P}e^{-i\int_0^1 d\tau\, Q(\tau)}$$

$$Q(\tau) = \sum_I O_I\, \epsilon^I(\tau)$$

different costs

$$\mathcal{C}_{L_1} \sim \min\left[\int_0^1 \mathrm{d}\tau \sum_I \Pi_I\, |\epsilon^I(\tau)|\right]$$

$$\mathcal{C}_{L_2} \sim \min\left[\int_0^1 \mathrm{d}\tau \sqrt{\sum_{I,J} \Pi_{IJ}\, \epsilon^I(\tau)\, \epsilon^J(\tau)}\right]$$

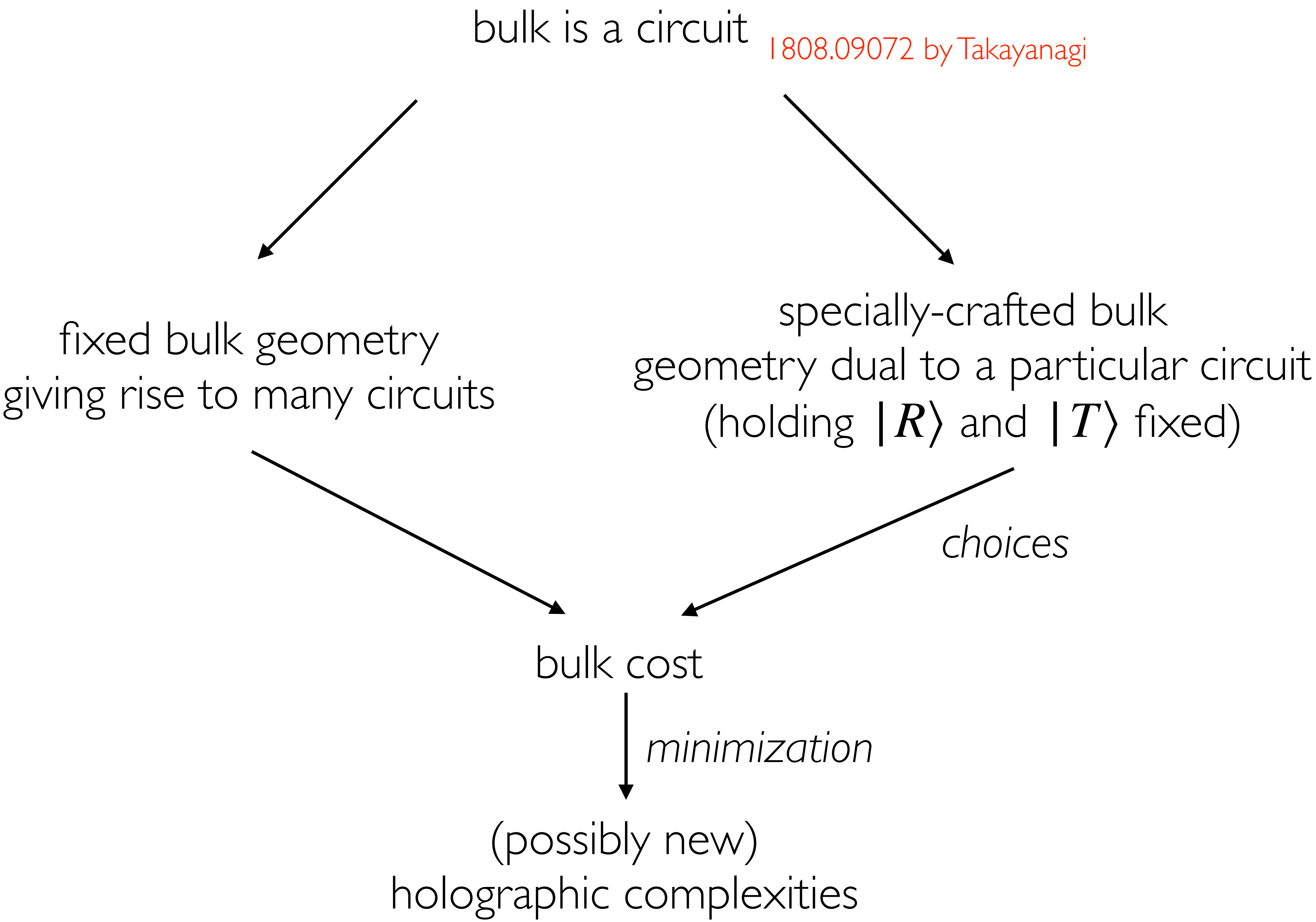
$$\mathcal{C}_{FS} \sim \min\left[\int_0^1 \mathrm{d}\tau \sqrt{\langle Q^2\rangle - |\langle Q\rangle|^2}\right]$$

Implementations of these ideas in free QFTs reproduce some of the static properties of holographic complexity, but crucially rely on Gaussianity

1707.08570 by Jefferson & Myers, 1707.08582 with Chapman, Marrochio, Pastawski, …

Holographic Complexity Beyond Proposals

A Perspective



bulk is a circuit 1808.09072 by Takayanagi

fixed bulk geometry
giving rise to many circuits

specially-crafted bulk
geometry dual to a particular circuit
(holding $|R\rangle$ and $|T\rangle$ fixed)

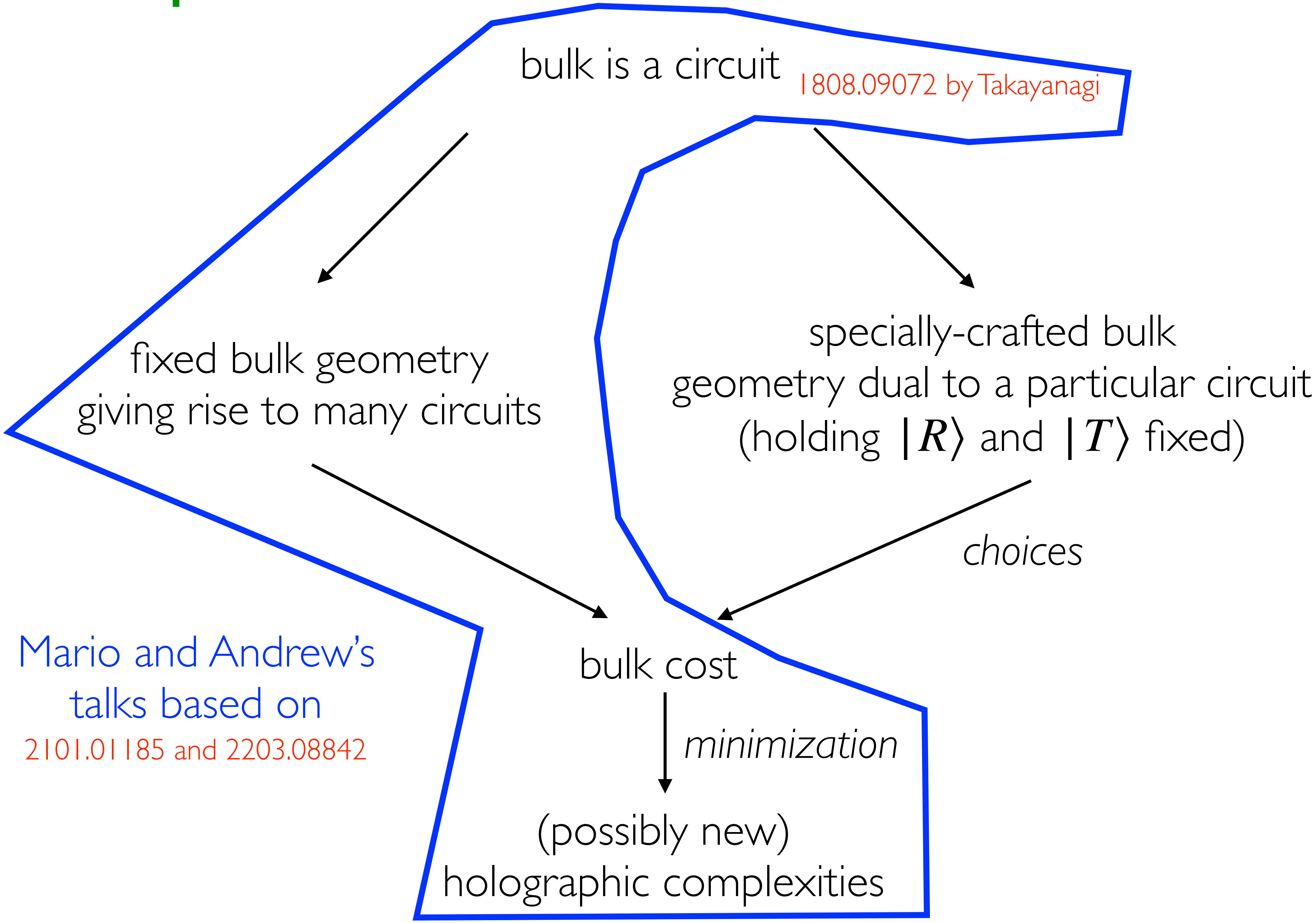
choices

bulk cost

minimization

(possibly new)
holographic complexities

A Perspective



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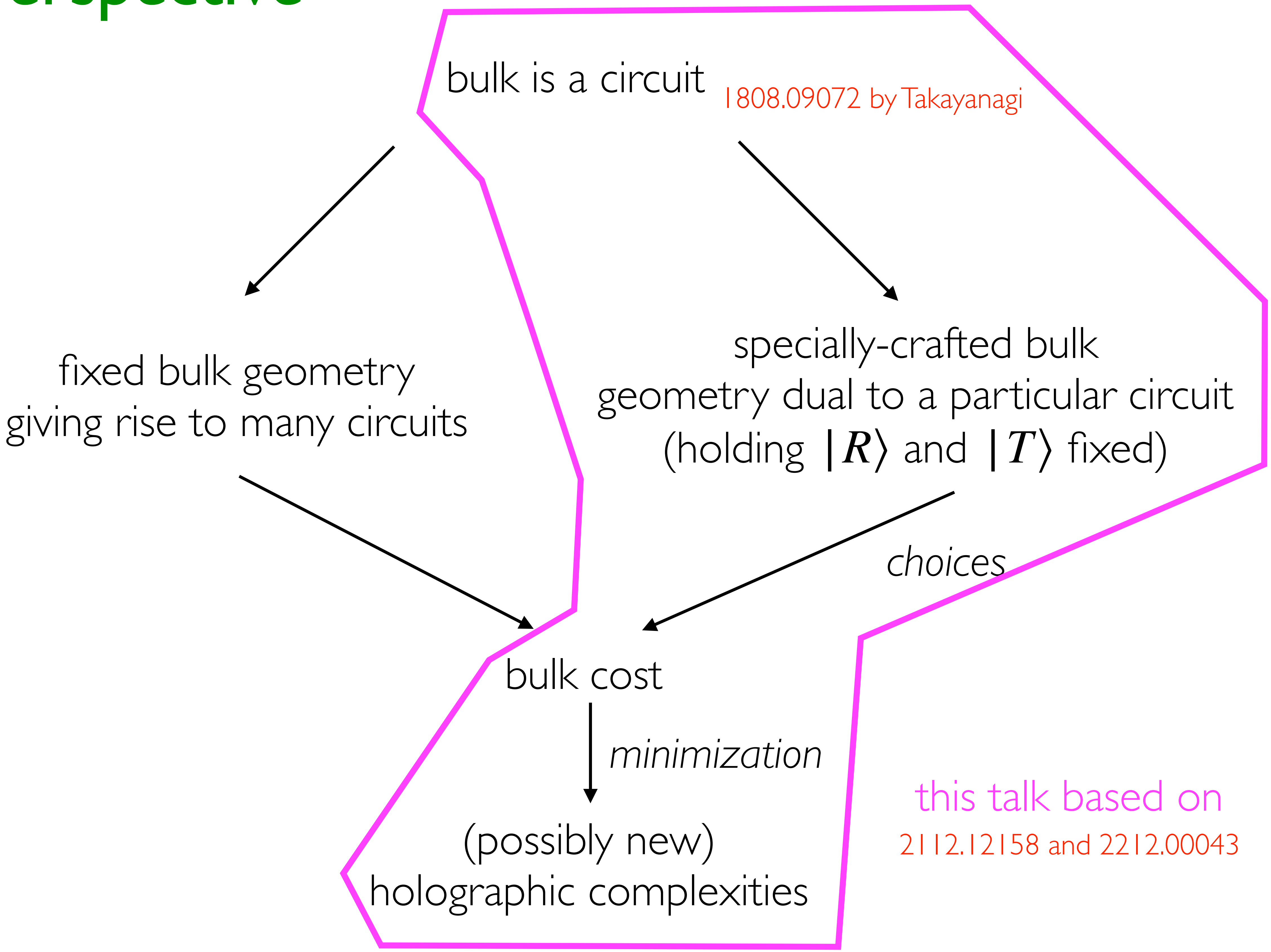
Mario and Andrew's talks based on
2101.01185 and 2203.08842

A Perspective



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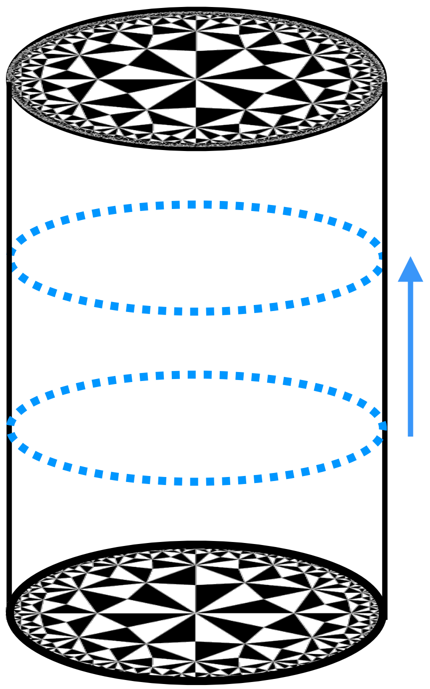
this talk based on

2112.12158 and 2212.00043

Fubini-Study Cost Is Holographic…

2112.12158 with Erdmenger, Flory, Gerbershagen, Weigel; 2212.00043 with Erdmenger, Gerbershagen, Weigel

Let's zoom out quite a bit and adopt much broader lenses:



$|\, some\, T \rangle$ at $\tau \equiv t = 1$

$|\, some\, R \rangle$ at $\tau \equiv t = 0$

local holographic QFT operators

$$Q(\tau) \equiv H(t) = \int_{t=const} \sum_I J_I O^I$$

holographic QFT sources acting as boundary conditions for dual bulk fields

The Fubini-Study cost for one time step ~ $\langle \psi(t) \,|\, Q(t)^2 \,|\, \psi(t) \rangle - \langle \psi(t) \,|\, Q(t) \,|\, \psi(t) \rangle^2$

↓

This reduces to a sum of non-equilibrium boundary 2-point functions!

…, However, In General Is Subtle to Extract

2212.00043 with Erdmenger, Gerbershagen, Weigel

One point functions easy to get from the asymptotic fall-offs of bulk fields

Fine grained (von Neumann) entropy is also directly encoded in the geometry via Ryu-Takayanagi / Hubeni-Rangamani-Takayanagi formulas

However, Renyi entropy already need backreaction on a given geometry

It is similar with 2-point functions in general, but no for the pure AdS_3 gravity!

Cost of conformal transformations

1807.04422 by Caputa, Magan;

2004.03619 by Erdmenger et al.; 2005.02415 and 2007.11555 with Flory

The stress tensor sector of 1+1D CFTs offers a soluble example of cost and complexity problem, which is universal and, therefore, should map to gravity

Such circuits are realized by unitaries of the form $U = \mathcal{P} e^{-i \int_0^1 \mathrm{d}\tau \, Q(\tau)}$

with $Q(\tau) = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} T(\sigma) \, \dot{f}(\tau, F(\tau,\sigma))$ and $f(\tau, F(\tau,\sigma)) = \sigma$

right- or left-moving component of $T_{\mu\nu}$

$$\epsilon(\tau,\sigma) = -\frac{\dot{F}(\tau,\sigma)}{F'(\tau,\sigma)}$$

and can be thought of as a gradual diffeomorphism of a circle

$\sigma \to f(\sigma)$ via $f(\tau,\sigma)$ with $f(\tau = 0, \sigma) = \sigma$ and $f(\tau = 1, \sigma) = f(\sigma)$

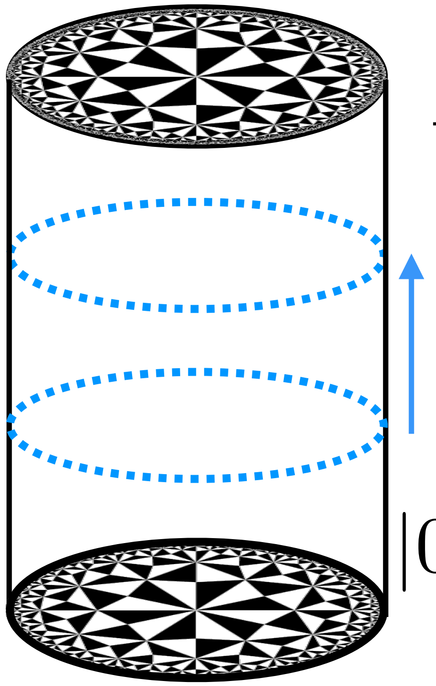
Towards a precise bulk dual to a circuit

2112.12158 with Erdmenger, Flory, Gerbershagen, Weigel

Local conformal transformations lead to a transformation of the stress tensor

$$\langle T \rangle \to \frac{1}{(\partial_\sigma f)^2}\left(\langle T \rangle - \frac{c}{12}\{f,\sigma\}\right) \text{ with } \langle \bar{T} \rangle \text{ and } \langle T^\mu{}_\mu \rangle \text{ staying the same}$$

The key idea: embed this kind of circuit on the boundary of AdS3



trivial foliation

$\tau = 1$

non-trivial foliation

$\tau = 0$

trivial foliation

trivial time evolution of $|excited\rangle$

circuit in $\boldsymbol{\tau \equiv t}$

with a nontrivial $\boldsymbol{Q(\tau) \equiv H(t)}$

$|0\rangle$

The gravity dual is obtained by using the exact Fefferman-Graham expansion

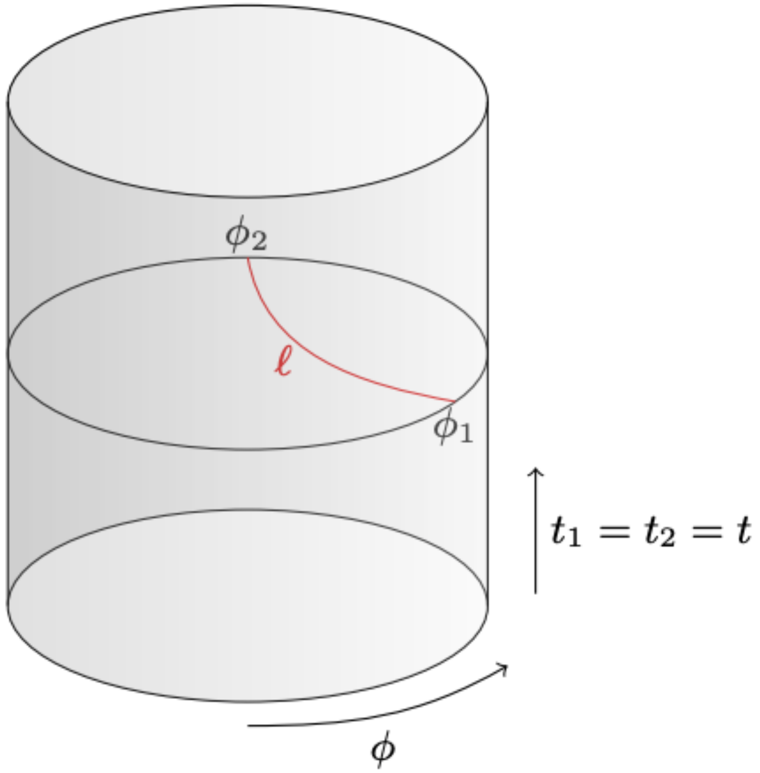
Excellent testbed for holographic complexity ideas, since both the bulk is known and the circuit is 100% under control

The FS Cost in AdS3 Gravity Is Geometric

2212.00043 with Erdmenger, Gerbershagen, Weigel

Indeed, in this case the operator is $T_{\mu\nu}$ and its correlator is fixed and given geometrically in terms of kinematic space formulas, for example:

$$\langle T(z_1)T(z_2)\rangle = \frac{c}{32}\frac{1}{\sin((z1-z2)/2)^4} = \frac{c}{2}(\partial_{z_1}\partial_{z_2}\ell)^2$$



This generalizes to our AdS3 circuit geometry gives explicit gravity dual to the Fubini-Study costs:

$$\langle\psi(t)\,|\,Q(t)^2\,|\,\psi(t)\rangle - \langle\psi(t)\,|\,Q(t)\,|\,\psi(t)\rangle^2 =$$

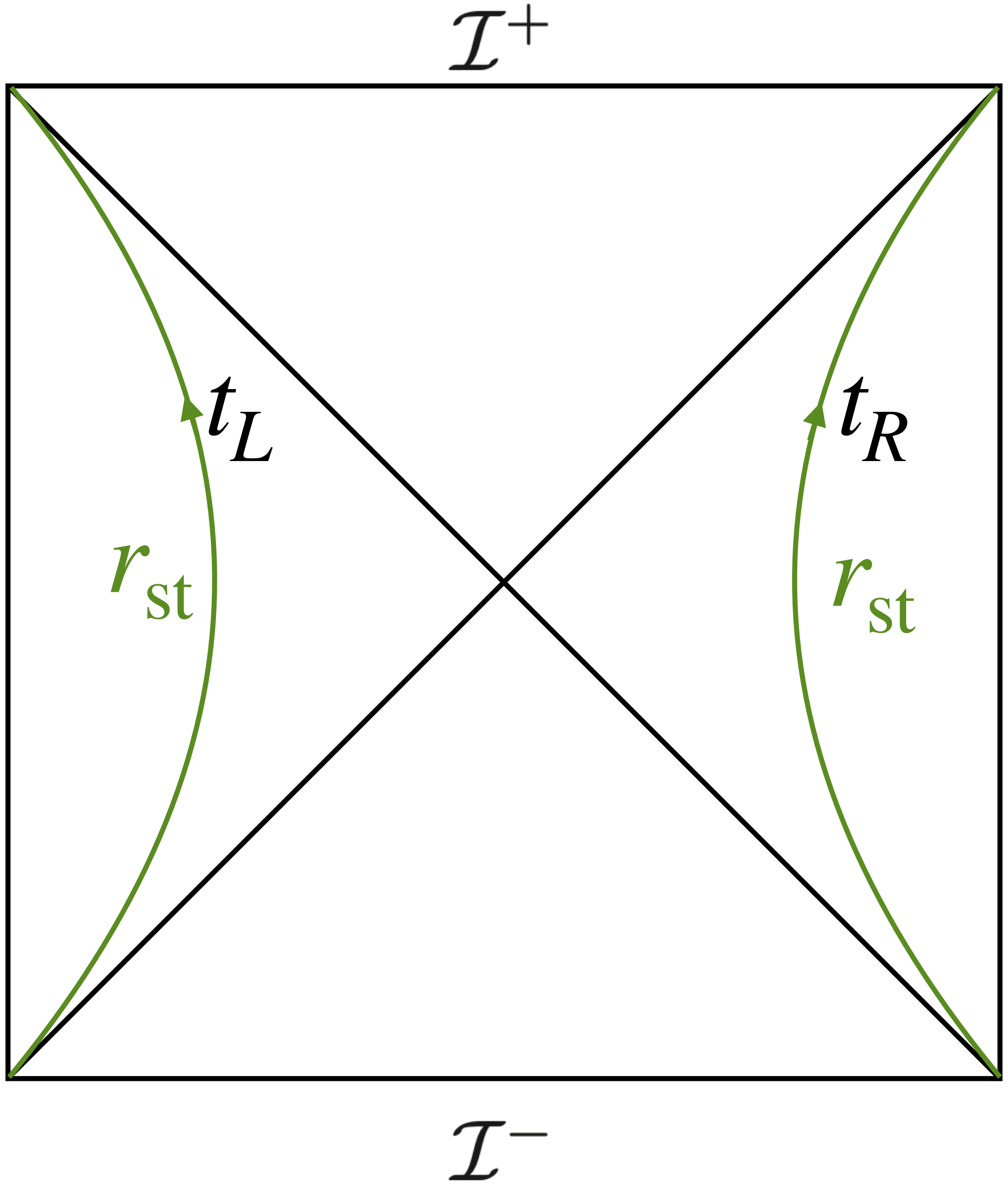
$$\int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2\ \frac{c}{4}\left[(\partial_{\phi_1}\partial_{\phi_2}\ell)(\partial_{t_1}\partial_{t_2}\ell) + (\partial_{\phi_1}\partial_{t_2}\ell)(\partial_{t_1}\partial_{\phi_2}\ell) - \frac{1}{2}g^{(0)}_{t_1\phi_1}g^{(0)}_{t_2\phi_2}g^{ij}_{(0)}(t_1,\phi_1)g^{kl}_{(0)}(t_2,\phi_2)(\partial_i\partial_k\ell)(\partial_j\partial_l\ell)\right]$$

Holographic Complexity
Beyond AdS Holography

A perspective: holographic complexity matured and became robuster.
Therefore, it is not unreasonable to start using it beyond AdS black holes.

de Sitter Static Patch Holography

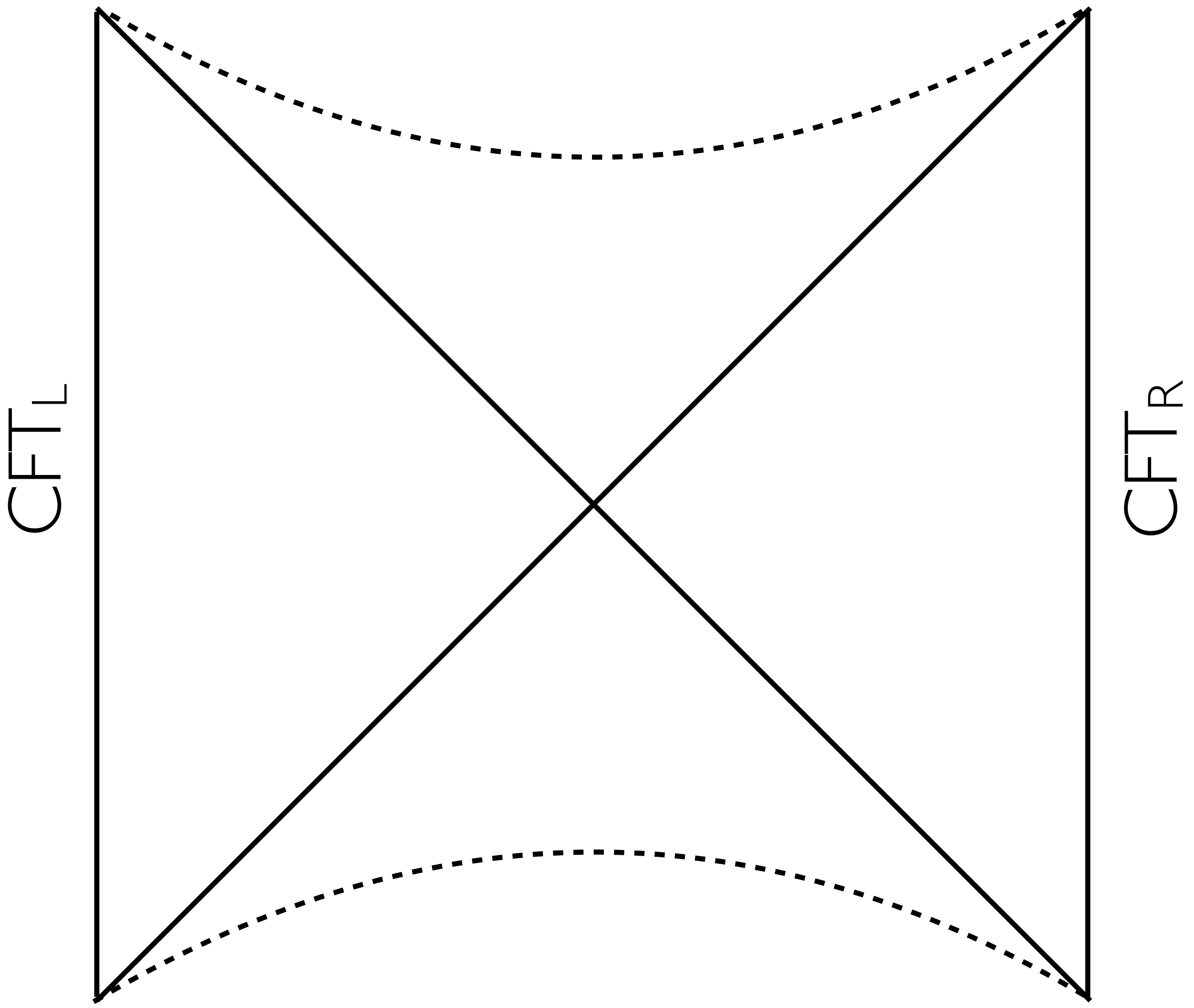
2109.14104 by Susskind



$$\mathrm{d}s^2 = -f(r)\mathrm{d}t_{L/R}^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 d\Omega_{d-1}^2 \text{ with } f = 1 - r^2$$

Anti-de Sitter Eternal Black Brane

hep-th/0106112 by Maldacena



$$|TFD(t_L, t_R)\rangle \sim e^{-iH_L t_L} e^{-iH_R t_R} \sum_E e^{-\beta E/2} |E\rangle_L |E\rangle_R$$

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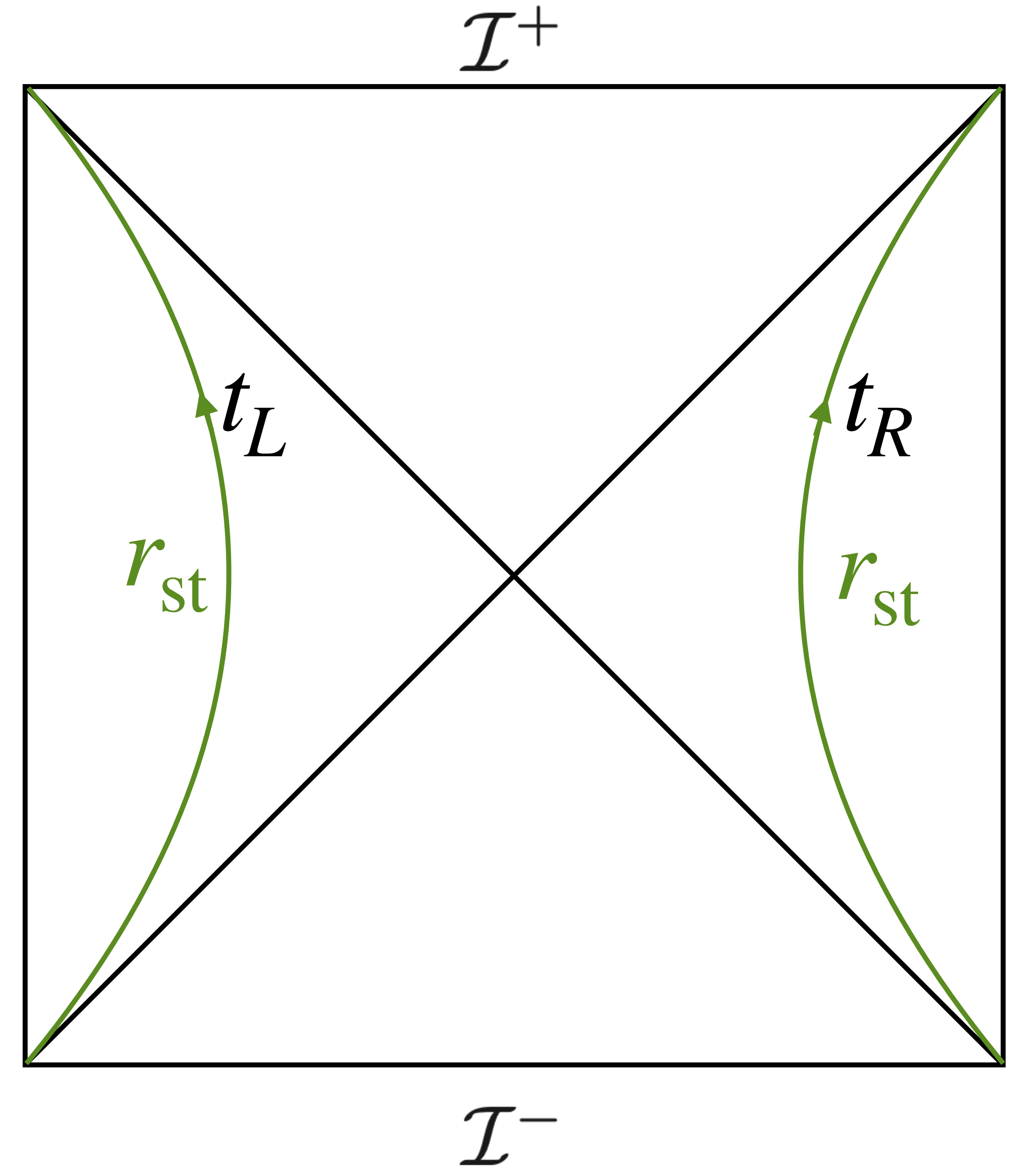
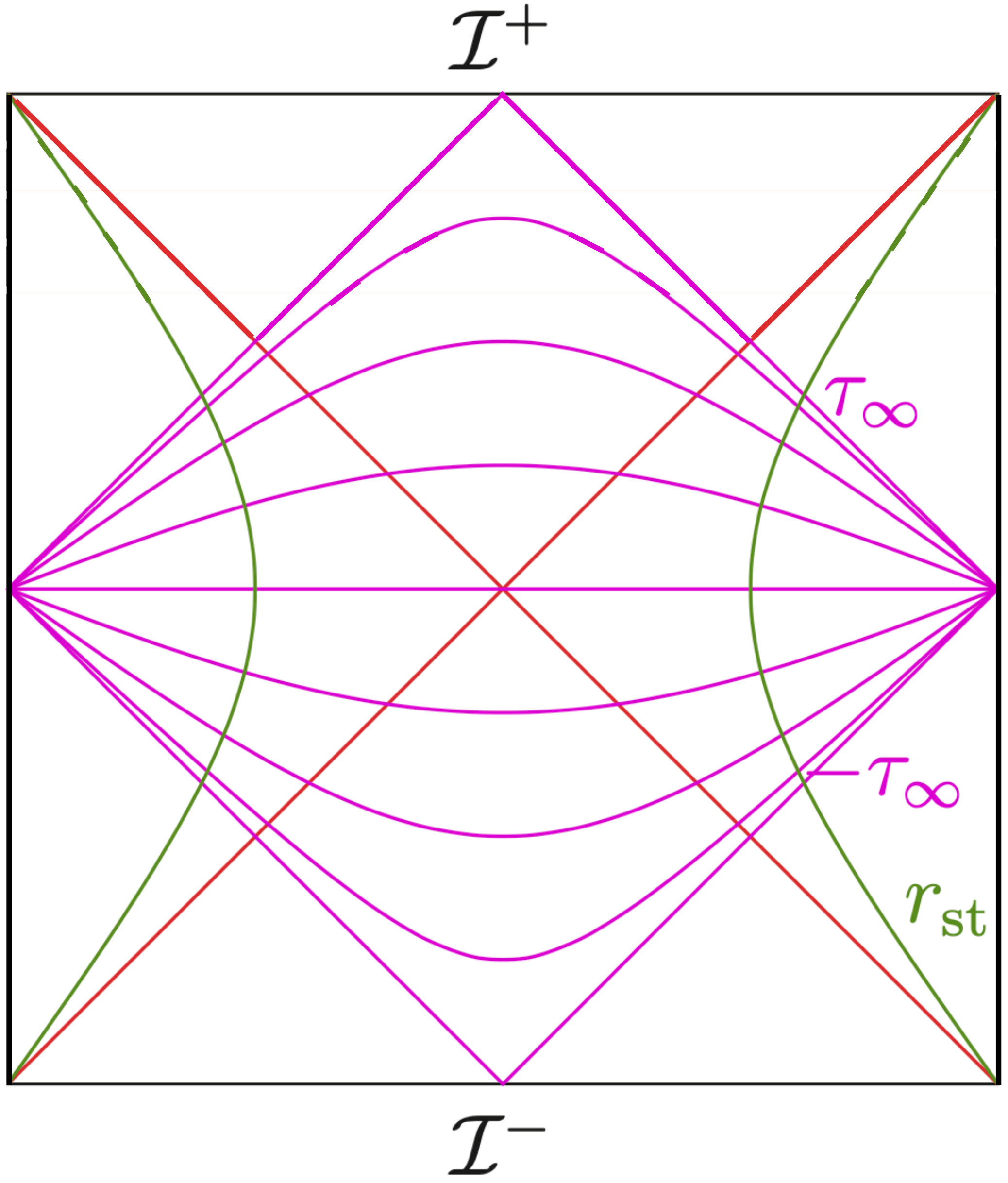


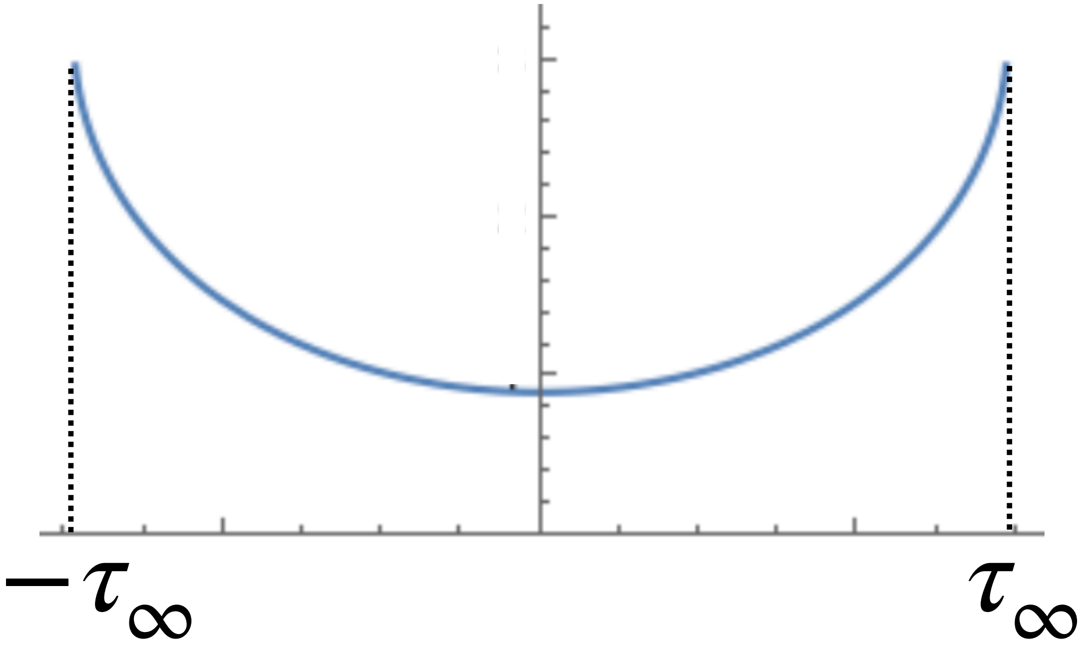
Two putative quantum systems in an entangled state with two time evolutions

Hyperfast Growth

2109.14104 by Susskind
2110.05222 by Chapman, Galante, Kramer
2202.10684 by Jørstad, Myers, Ruan



$\mathcal{C}_V$ in dS$_{1+1}$
(spatial geodesic length)



In higher number of dimensions, $\mathcal{C}_V$ diverges at $\pm\tau_\infty$; similar story for $\mathcal{C}_{A,\,V2.0}$

New Game in Town 2111.02429 and 2210.09647 by Belin, Myers, Ruan, Sarosi, Speranza

In the meantime it was realized that there is a holographic complexity landscape

These so call complexity = anything objects are obtained in a two step procedure

1) optimization of a covariant functional to get a geometric carrier

2) evaluating (possibly other) covariant functional to get a non-negative number

subject to linear growth in AdS black hole background + one more condition

There is a continuum of options, e.g. spatial volume 2) of constant K slices 1)

(the original $\mathcal{C}_V$ has $K = 0$)

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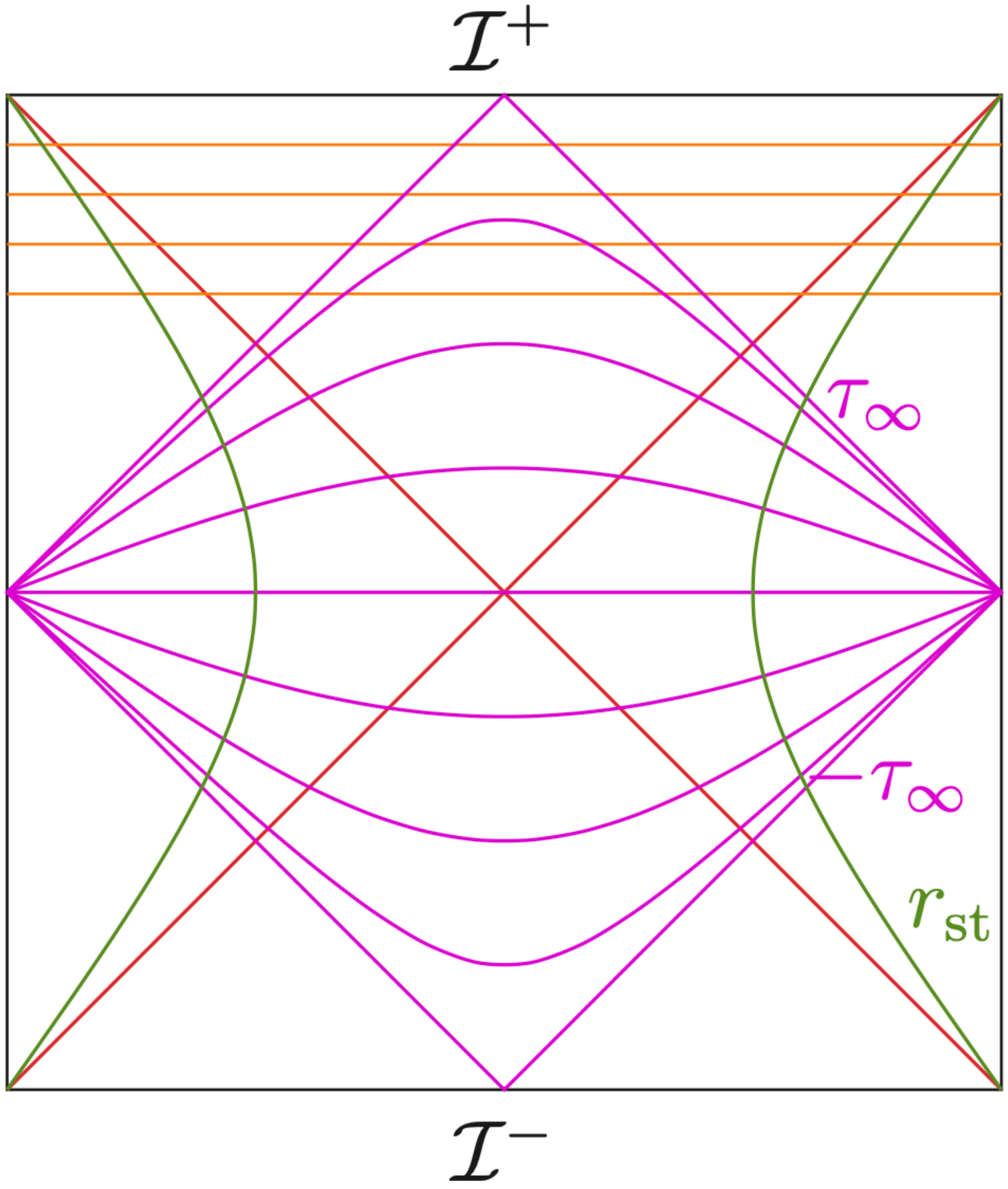
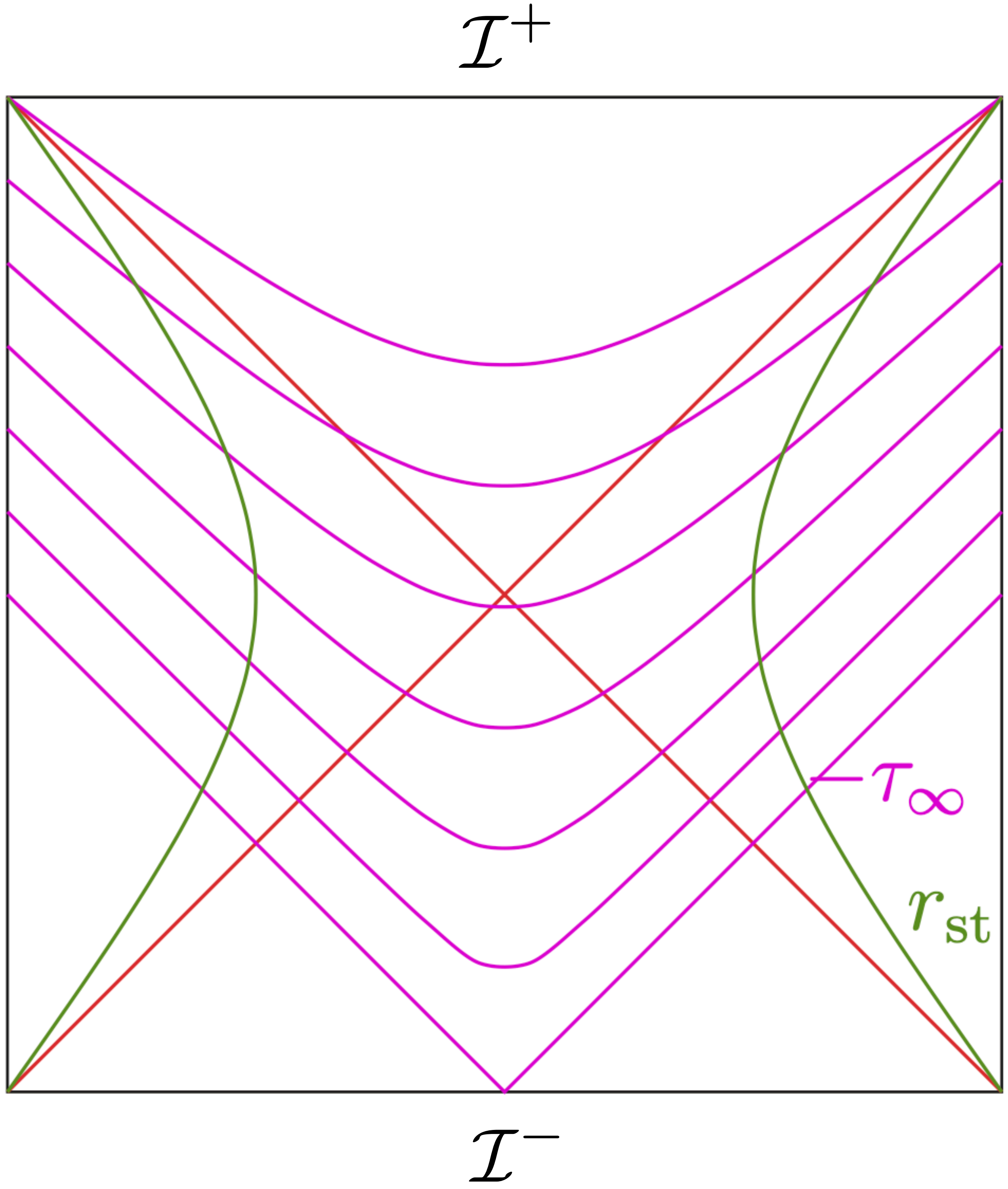


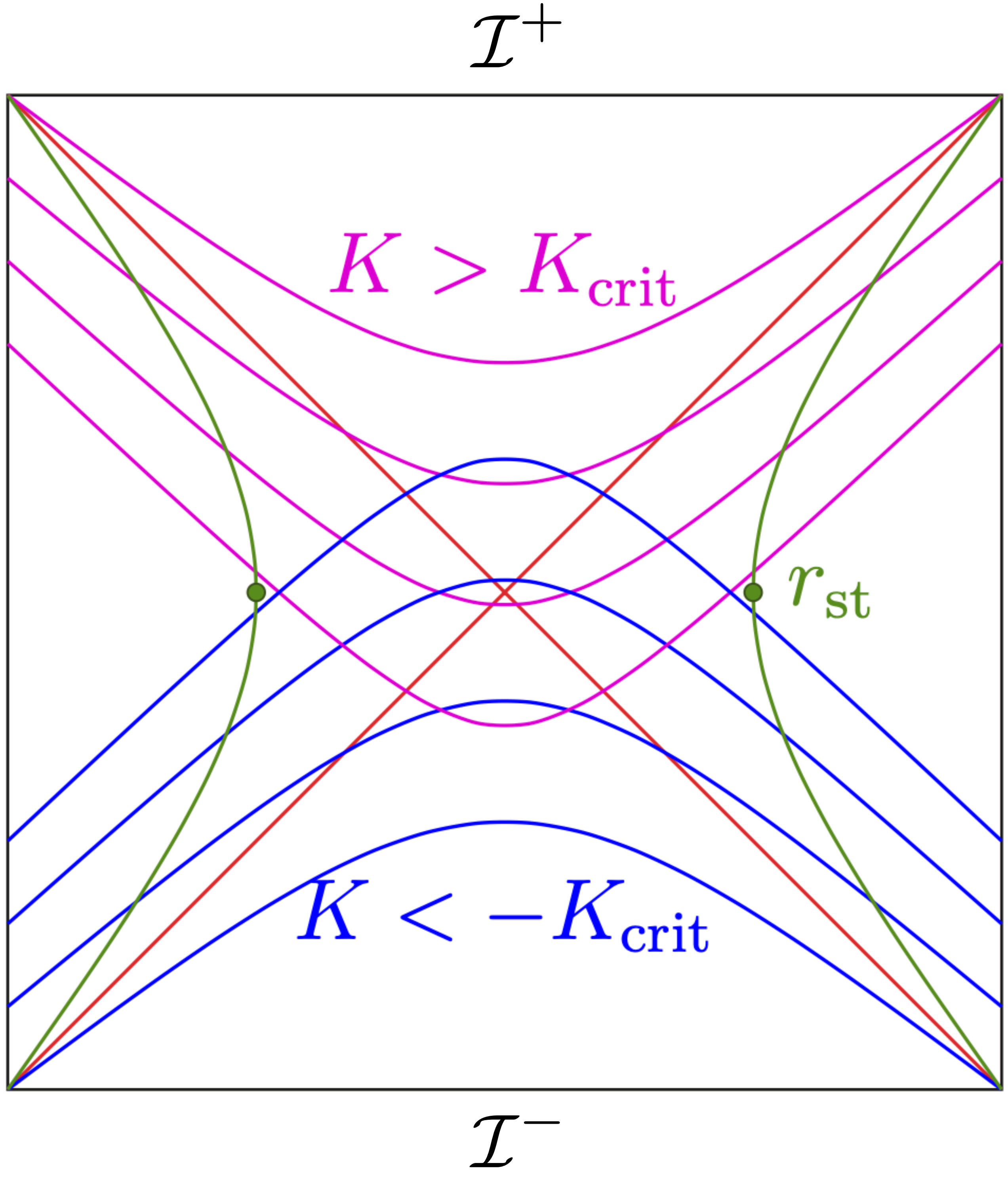
Our Idea

For large enough K
our surfaces do not touch $\mathcal{I}^+$



Implementation

$$\mathcal{C}_{\rm us} = \min_{\pm K} \mathrm{vol}$$



Properties of Our Proposal

For $K < K_{\text{crit}}$ we do get the hyperfast growth

For $K > K_{\text{crit}}$ we get a linear late time growth (and early time decay)

In dS_{1+1} and dS_{1+2} at K_{crit} linear ⟶ exponential

Outlook

Outlook

Holographic complexity got matured

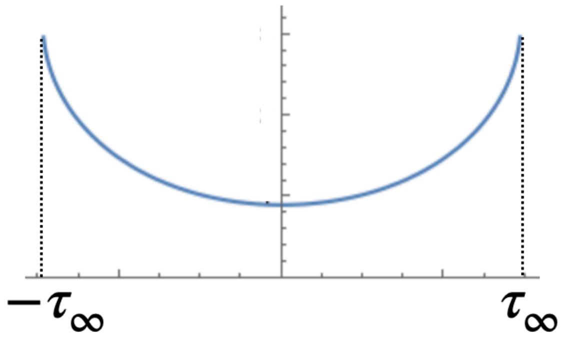
In particular, 2212.00043 with Erdmenger, Gerbershagen, Weigel, 2112.12158 + Flory we established that the Fubini-Study cost is holographic for any geometric quantum circuit

Time to start thinking about it outside AdS/CFT

New in dS: the hyperfast growth

$\mathcal{C}_V$ in dS_{1+1}

(spatial geodesic length)



In 2305.11280 with Aguilar-Gutierrez and Van der Schueren we show complexity = anything does not make the hyperfast growth a necessity