State Dependence of Krylov Complexity in 2d CFT

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Complexity: Between Field Theory & Gravity

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Outline

Motivation & Introductory Remarks

Krylov Complexity & Chaos

What do we learn from them

Lanczos coefficients & Universal Operator Growth

A hypothesis

What is generic and what is special

The choice of inner product in Krylov basis

Motivation for this

Technical details

Physics of the Krylov complexity

Large-c CFT, Ising CFT, free CFT, etc

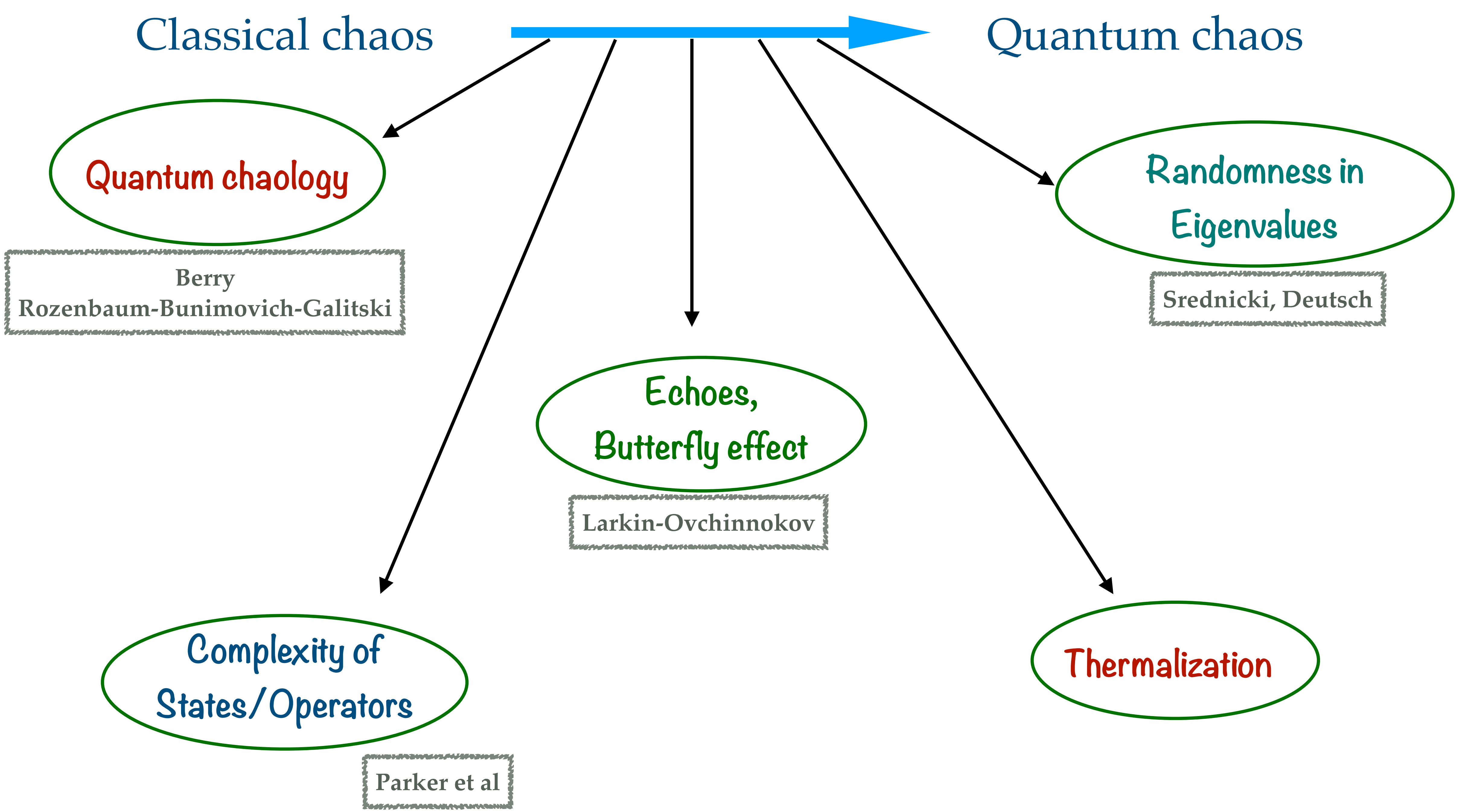
Conclusions and Outlook

Summary and Future directions

Motivation & Introduction

Dynamical aspects of a physical system are extremely important

Particularly interesting: Quantum Chaos



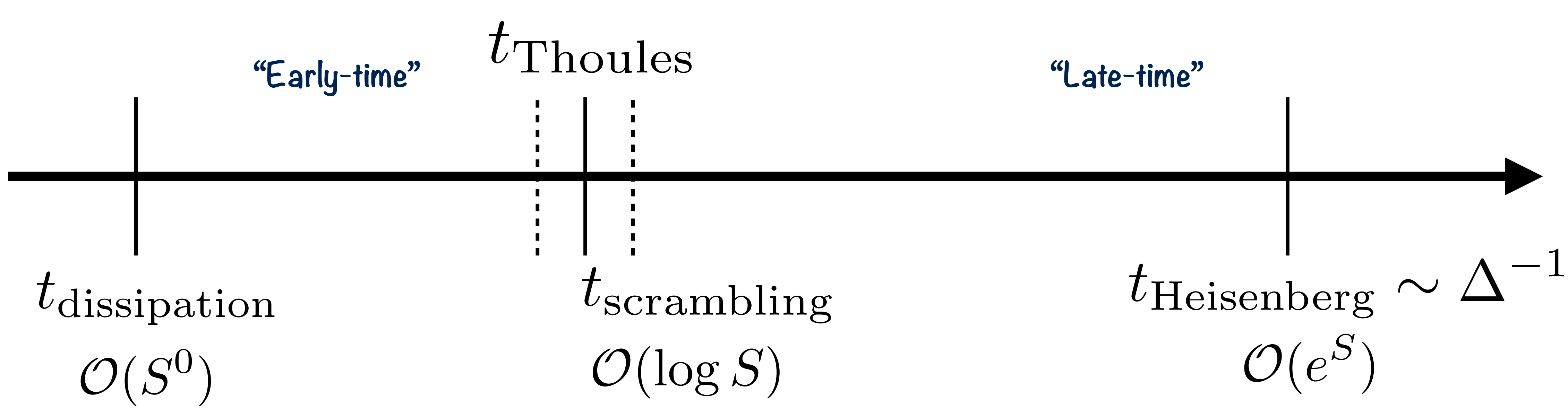
Motivation & Introduction

Why should we think about Quantum Chaos?

Experimental Strides: increasingly possible to access long-time dynamics of quantum systems

Theoretical Connections: ideas of quantum chaos seem to interconnect various fields, quantum information, quantum gravity and strongly coupled systems

Dynamics: Various Emerging time-scales



Probes at different time-scales

Quantum chaos splits into two parts: Early time & Late time

The notion of evolution & growth is natural in the Operator Hilbert Space

Heisenberg Evolution:

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}$$

$$\mathcal{O}(t) = \mathcal{O}(0) + it\,[H, \mathcal{O}(0)] + \frac{(it)^2}{2}\,[H, [H, \mathcal{O}(0)]] + \dots$$

The Operator spreads through non-trivial commutators with the Hamiltonian/Quantization

Spreading nature depends on the Operator & the Hamiltonian

Technically: Local or k-local operators

Krylov (sub)-Space

Time Evolution, redefined:

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} \equiv e^{i\mathcal{L}t}\mathcal{O}(0)$$

Liouvillian Operator

The Krylov (sub)-Space:

$$\mathcal{H}_{\mathcal{O}} = \text{span}\left\{\mathcal{L}^n\mathcal{O}\right\} = \text{span}\left\{\mathcal{O}, [H,\mathcal{O}], [H,[H,\mathcal{O}]], \ldots\right\}$$

The linear span of each element of the above space forms the invariant Krylov sub-space

This comes with a definition of the inner product, defined in the Krylov space

Parker et al, 2019; Viswanath & Muller, 2008

Constructing Krylov (sub)-Space

Use the inner product & construct a Gram-Schmidt orthogonalisation

The Louivillian matrix: $L_{mn} := (\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m)$

$$(\mathcal{O}_m|\mathcal{L}|\mathcal{O}_n) = a_m$$

$$(\mathcal{O}_{n-1}|\mathcal{L}|\mathcal{O}_n) = b_n = (\mathcal{O}_n|\mathcal{L}|\mathcal{O}_{n-1})$$

Non-vanishing elements: tri-diagonal matrix

Lanczos Coefficients: $b_n = \left(\tilde{\mathcal{O}}_n|\tilde{\mathcal{O}}_n\right)^{1/2}$

$$\left|\tilde{\mathcal{O}}_n\right) = \mathcal{L}\,|\mathcal{O}_{n-1}) - b_{n-1}\,|\mathcal{O}_{n-1}) - a_{n-1}\,|\mathcal{O}_{n-1})$$

$$|\mathcal{O}_n) = b_n^{-1}\left|\tilde{\mathcal{O}}_n\right)$$

Parker et al, 2019; Viswanath & Muller, 2008

Time-dependence of an Operator

Any Operator can be expanded in this basis:

$$|\mathcal{O}(t)) = \sum_n i^n \phi_n(t) |\mathcal{O}_n)$$

Operator wave-function

Heisenberg equation of motion:

$$\partial_t \phi_n(t) = b_n \phi_{n-1}(t) - b_{n+1}\phi_{n+1}(t) + i a_n \phi_n(t)$$

Boundary condition: $\phi_n(t=0) = \delta_{n0}$

$$\phi_0(t) = (\mathcal{O}(0)|\mathcal{O}(t))$$

A physical input = Auto-correlation

Parker et al, 2019; Viswanath & Muller, 2008

Operator Complexity

Definition of Krylov complexity:

$$K_{\mathcal{O}}(t) = \sum_n n \left|\phi_n(t)\right|^2$$

Average position of the wave function along the chain

Parker et al, 2019; Viswanath & Muller, 2008

Correspondingly, there exists a Krylov operator

$$\hat{K}_{\mathcal{O}}(t) = \sum_n n \left|\mathcal{O}_n\right)\left(\mathcal{O}_n\right|$$

Krylov operator expectation value

$$K_{\mathcal{O}}(t) = \left(\mathcal{O}(t)\right| \hat{K}_{\mathcal{O}}(t) \left|\mathcal{O}(t)\right)$$

A devil inside the inner product detail

There exists an infinite family of inner products:

$$(\mathcal{O}_1|\mathcal{O}_2)^g_\beta = \int_0^\beta g(\lambda)\mathrm{Tr}\left(\rho e^{\lambda H}\mathcal{O}_1^\dagger e^{-\lambda H}\mathcal{O}_2\right)$$

A commonly used inner product: $g(\lambda) = \delta(\lambda - \beta/2)$

Wightman inner product:

$$(\mathcal{O}_1|\mathcal{O}_2)^g_\beta = \mathrm{Tr}\left(\rho^{1/2}\mathcal{O}_1^\dagger\rho^{1/2}\mathcal{O}_2\right) \qquad \rho = e^{-\beta H}$$

Auto-correlation=Thermal correlator

$$(\mathcal{O}(0)|\mathcal{O}(t)) = \mathrm{Tr}\left(\rho\mathcal{O}(0)^\dagger\mathcal{O}\left(t + \frac{i\beta}{2}\right)\right)$$

What is interesting with this?

Some Observations:

$$b_n \sim \mathcal{O}(1)$$ Free theory

$$b_n \sim \mathcal{O}(\sqrt{n})$$ Integrable (interacting) theory

$$b_n \sim \mathcal{O}(n)$$ Chaotic systems

Operator growth hypothesis: Parker et al, 2019

Chaotic systems display a linear growth in Lanczos coefficients

E.g. SYK-model, other lattice models

Good and interesting diagnostic in discrete quantum systems

What happens in continuum QFTs? Barbon et al., 2019

Continuum version: QFT

A nice starting point is to consider CFTs

For any primary operator, the thermal 2-point function is fixed!

The resulting auto-correlation: $$C_{\mathcal{O}}(\tau) = \frac{1}{\cos\left(\frac{\pi\tau}{\beta}\right)^{2\Delta}}$$

Dymarsky et al, 19-20

Generic contact singularities at: $\tau = \pm\frac{\pi}{2}$

The contact singularity guarantees a linear Lanczos coefficient

This implies an exponentially growing Krylov complexity

A more general choice

Recall that all results so far depend on the Wightman function as a choice for the inner product

A trivial generalization:

$$(\mathcal{O}_1|\mathcal{O}_2) = \mathrm{Tr}\left(\mathcal{O}_1^\dagger \rho_1 \mathcal{O}_2 \rho_2\right) = (\mathcal{O}_2|\mathcal{O}_1)^*$$

Arbitrary pure state density matrix

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Obviously depends on the "state"

This satisfies all the desired properties of an inner product

Reduces to the familiar Wightman function: $\rho_1 = \rho^{1/2}\,, \quad \rho_2 = \rho^{1/2}\,, \quad \rho = e^{-\beta H}$

A standard correlator: $\rho_1 = I\,, \quad \rho_2 = \rho$

Exploring the general inner product

Choose a pure state created by a heavy primary in CFT (2-dim) $\rho = |\psi\rangle \langle\psi|$

State-operator correspondence makes it a 4-point function in vacuum

Let us define the CFT on a cylinder: $S_L \times R$

Allows us to define an in and an out state

$$|\psi_{\text{in}}\rangle = \lim_{t_0 \to \infty} \mathcal{O}_{\text{H}}(t_0)|0\rangle, \quad \langle\psi_{\text{out}}| = \lim_{t_0 \to \infty} \langle 0|\mathcal{O}_{\text{H}}(t_0)$$

Heavy primary states, which are pure states

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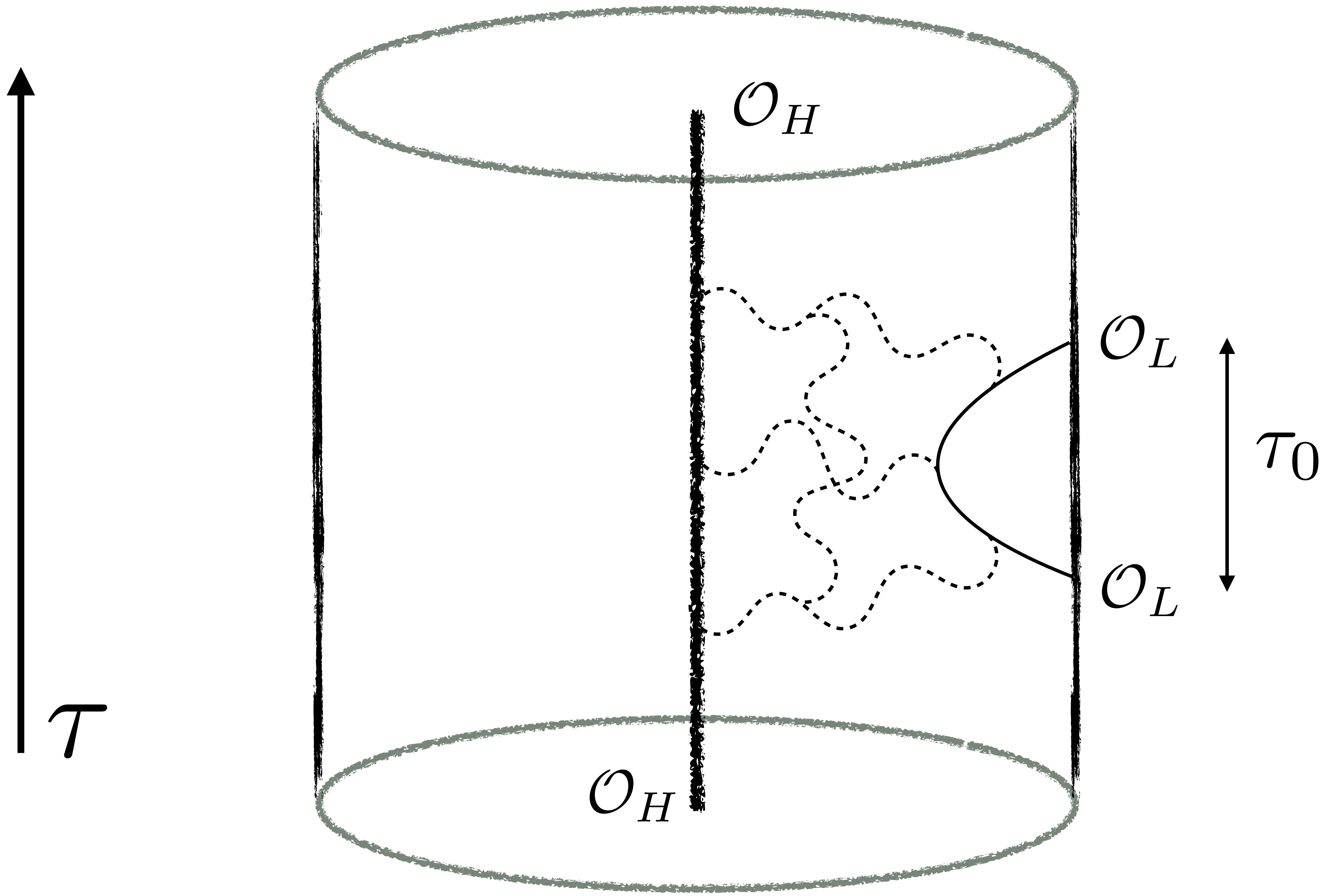
A pictorial representation



Fitzpatrick-Kaplan et. al.

Exploring the general inner product

Look at a two point function of another primary operator in this state

$$C_\psi(\tau) = \langle \mathcal{O}(\tau)\mathcal{O}(0)\rangle = \frac{\langle \psi_{\text{out}}|\mathcal{O}(\tau)\mathcal{O}(0)|\psi_{\text{in}}\rangle}{\langle \psi_{\text{out}}|\psi_{\text{in}}\rangle}$$

This depends on the details of the CFT, therefore highly dynamical in nature

Where can we explicitly calculate this?

Free field theory: trivial

Interacting CFT: minimal models, eg Ising CFT

Interacting CFT: Large-c CFT/Holographic CFT

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Results: First Pass

Large-c CFT: Use HHLL correlators

The Auto-Correlation Function

$$C_\psi(t) = \left(\frac{\alpha\gamma}{\sinh\left[\alpha\gamma\left(\tau_0 + it\right)\right]}\right)^{2\Delta_L} \qquad \gamma = \sqrt{1 - \frac{12\Delta_H}{c}}$$

UV-separation of the two LL-operators

Auto-correlation becomes thermal correlator $\Delta_H > \frac{c}{12}$

This is like an Eigenstate Thermalization Hypothesis kind of behaviour

We are not assuming ETH here!

Irrespective of how heavy the state is, the Lanczos growth is always linear

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Results: Second Pass

Complexity growth knows about the heaviness of the state

$$K_L(t) = 2\Delta_L \text{csch}^2(\tau_0)\sin^2(t) \;,$$
$$K_H(t) = 2\Delta_L \csc^2(\tau_0)\sinh^2(t) \;.$$

Not heavy enough: $\Delta_H < \frac{c}{12}$

Bounded oscillatory behaviour

Heavy enough: $\Delta_H > \frac{c}{12}$

Exponentially growing behaviour

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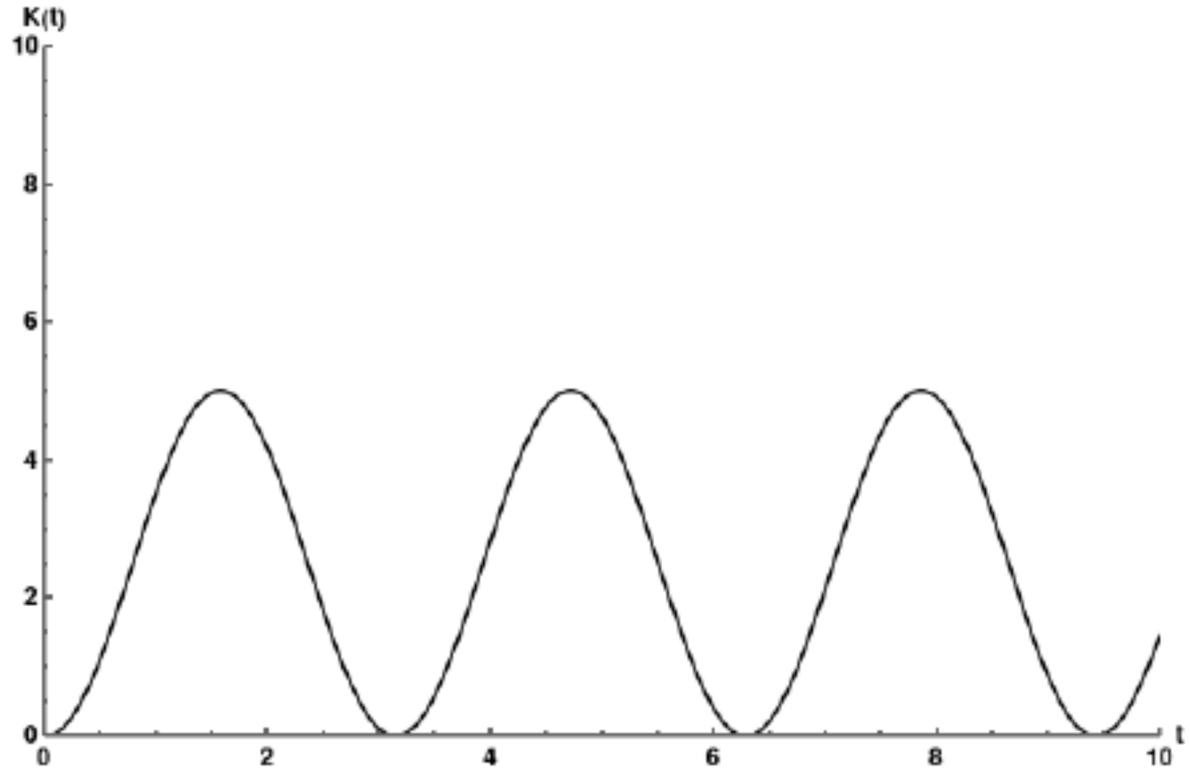
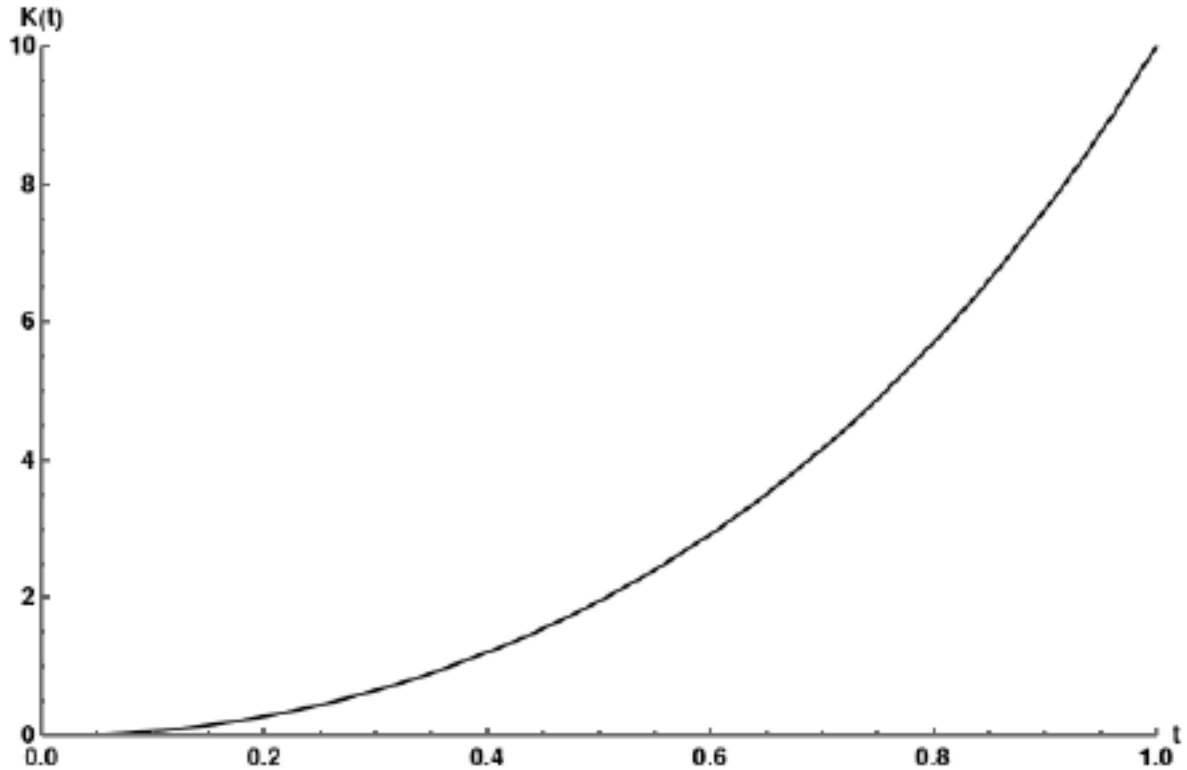
K-complexity knows about the Hawking-Page like transition

Transition between Area-law entanglement & Volume-law entanglement

Results: Pictorial & Summary



(a) K-Compexity $K_L(t)$ vs t for LLLL .



(b) K-Compexity $K_H(t)$ vs t for HLLH.

Reminiscent of a Hawking-Page like transition: K-complexity captures

K-complexity distinguishes, but it is a state-dependent statement

Particularly, K-complexity is sensitive to the entanglement structure of the state

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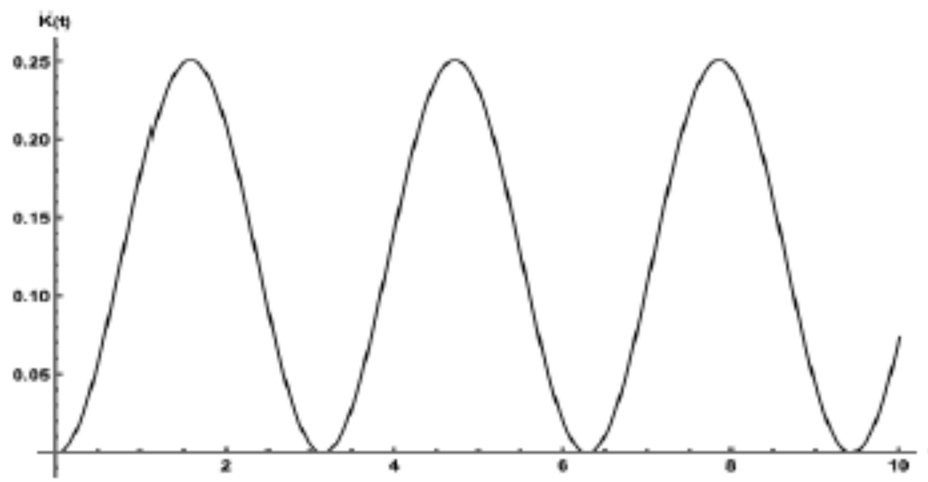
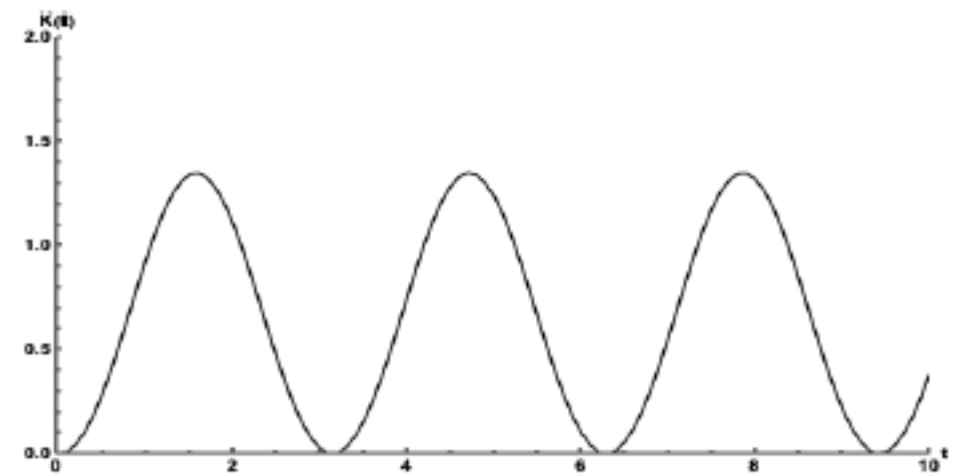
Results: More examples & pictures

A similar analyses can be carried out for the Ising CFT



(a) K-complexity for a light operator (σ) in the light state $|\sigma\rangle$ in the Ising model.



(b) K-complexity for a light operator (σ) in the heavy state $|\epsilon\rangle$ in the Ising model

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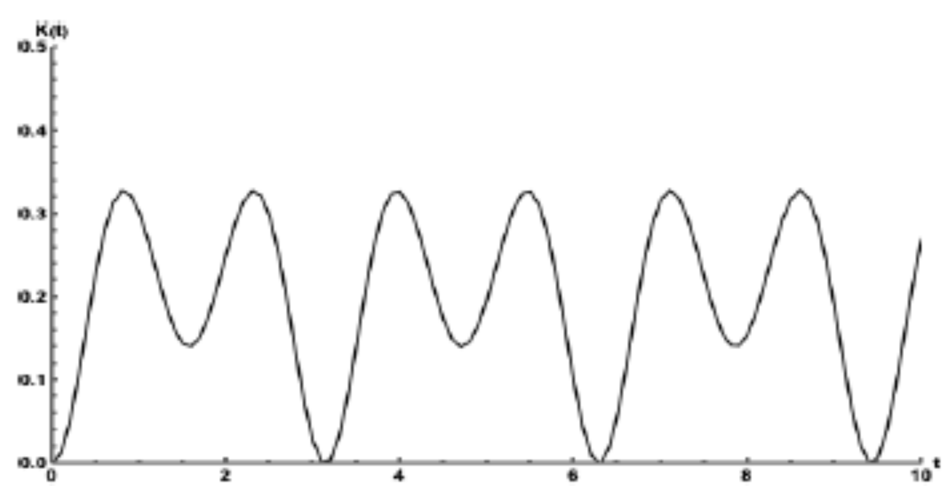

Figure: K-complexity for the ϵ operator in $|\epsilon\rangle$ state in the Ising model.

Also, for a free CFT: $K(t) \sim \sin^{-2}(t)$

Comments & Discussions

Linear growth in Lanczos coefficients (does not) translate into an exponential growth of K-complexity

A non-trivial effect of: $a_n \neq 0$

A two-point function no longer imposes a non-trivial relation on a four-point function

E.g. no universal relation: $\lambda_{\rm L} \leq \lambda_{\rm K}$

This universality is not desired: no distinction between an integrable and a chaotic dynamics

On the other hand, K-complexity knows more than we think

Accessible through an appropriate state-dependence

This dependence further raises interesting questions

Comments & Discussions

2-dim CFT, with a large c: Transition between exponential growth & bounded Behaviour

A reminiscent of a Hawking-Page transition

K-complexity knows about the entanglement feature of the state: Area-law vs Volume-law

This universality is not desired: no distinction between an integrable and a chaotic dynamics

Hypothesis: K-complexity is highly sensitive to the entanglement structure of the state

Free CFT & Ising CFT results are consistent with the hypothesis

But a better check will be in a system that can interpolate between an integrable state and a chaotic state

Of particular interest: Scar states, Thermo-Field Double State, ...

More Future

There are related physical aspects

How does K-complexity depend on global charges?

What, if any, is the precise connection between OTOCs & K-complexity?

Broader connection with ergodicity, entanglement and typicality?

A precise Holographic dual?

Does the kinematic aspect of von-Neumann algebra play any role here?

Thank You!