Integrability and complexity in quantum spin chains

Ben Craps

with Marine De Clerck, Oleg Evnin and Philip Hacker, 2305.00037

see also BC, MDC, OE, PH and Maxim Pavlov, 2202.13924

Complexity: Between Field Theory and Gravity, Madrid, May 22, 2023

Complexity of time evolution

- Circuit complexity: minimal number of "simple gates" to perform quantum computation.
- Geometric complexity (Nielsen): length of shortest path in manifold of unitaries between identity operator and operator of interest:

[Nielsen 2005,...]

$$U(0) = \mathbf{I}\,, \qquad U(t) = U_{\text{target}}$$

- Metric needed. (Too) simple possibility: bi-invariant metric $ds^2_{\text{bi-inv}} = \text{Tr}[dU^\dagger dU]$

Introduce velocity V and expand it in a basis of generators:

$$\frac{dU(t)}{dt} = -iV(t)U(t) \qquad V = \sum_i V^i T_i \qquad \text{Tr}[T_i T_j] = \delta_{ij}$$

Then $\mathcal{C}_{\text{bi-inv}}(t) = \min \int_0^t dt' \left(\text{Tr}[V(t')^2]\right)^{1/2} = \min \int_0^t dt' \left(\sum V_i^2\right)^{1/2}$

Complexity of time evolution

- Introduce "complexity metric" by splitting tangent space into "easy/local" and "hard/nonlocal" directions,

$$V = V_e + V_h \quad \text{with} \quad V_e \equiv V^\alpha T_\alpha, \quad V_h \equiv V^{\dot\alpha} T_{\dot\alpha}$$

  and assigning a "cost factor" μ to the hard directions:

$$\mathcal{C}(t) = \min \int_0^t dt' \left[\mathrm{Tr}[V_e(t')^2] + \mu \, \mathrm{Tr}[V_h(t')^2] \right]^{1/2} = \min \int_0^t dt' \left[\sum_\alpha (V^\alpha)^2 + \mu \sum_{\dot\alpha} \left(V^{\dot\alpha}\right)^2 \right]^{1/2}$$

  Will choose $\mu = D \equiv \mathrm{dim}\mathcal{H}$. Hamiltonian is always considered easy/local.

- Goal: characterize complexity of time evolution operator for integrable/chaotic models.

[Balasubramanian, DeCross, Kar, Parrikar 2019; Balasubramanian, DeCross, Kar, Li, Parrikar 2021]

Outline

1. A bound on complexity
2. Complexity in quantum resonant systems
3. Integrability and complexity in quantum spin chains
4. Conclusions and outlook

Complexity in bi-invariant metric

- Geodesics in bi-invariant metric have constant velocity V: $U(t) = e^{iVt}$

- Geodesics connecting $U(0) = \mathbf{I}$ and $U(t) = e^{-iHt}$:

$U(t') = e^{-iH't'}$ with $e^{-iH't} = e^{-iHt}$, so (if no degeneracies)

$$H' = \sum_n \left(E_n - \frac{2\pi}{t} k_n\right) |n\rangle\langle n| \quad \text{with} \quad k_n \in \mathbb{Z}$$

- Complexity $\mathcal{C}(t) = \left[\min_{k_n \in \mathbb{Z}} \sum_n \left(E_n t - 2\pi k_n\right)^2 \right]^{1/2}$

- Solve by choosing k_n such that $-\pi < tE_n - 2\pi k_n \leq \pi$

[Balasubramanian, DeCross, Kar, Li, Parrikar 2021]

Early-time complexity

$$\mathcal{C}(t) = \left[\min_{k_n \in \mathbb{Z}} \sum_n \left(E_n t - 2\pi k_n \right)^2 \right]^{1/2}$$

For early times, the shortest geodesic path is given by time evolution: $U(t') = e^{-iHt'}$

Early-time complexity: $\mathcal{C}(t) = t \left(\mathrm{Tr}[H^2] \right)^{1/2} = t \left(\sum_n E_n^2 \right)^{1/2}$

To compare different models, perform energy shift and scaling to obtain

$\mathrm{Tr} H = \sum_n E_n = 0$ and $\mathrm{Tr}[H^2] = \sum_n E_n^2 = 1$

Universal early-time growth $\mathcal{C}(t) = t$. Continues to hold with complexity metric.

Note: $\mathcal{C}_{\mathrm{Nielsen}}(t) = \mathcal{C}(t)/\sqrt{D}$

Late-time bi-invariant complexity

$$\mathcal{C}(t) = t\ \min\left(\sum_n {E'_n}^2\right)^{1/2} = \left[\min_{k_n \in \mathbb{Z}} \sum_n \left(E_n t - 2\pi k_n\right)^2\right]^{1/2}$$

For later times, shorter geodesics can be found by choosing nonzero k_n

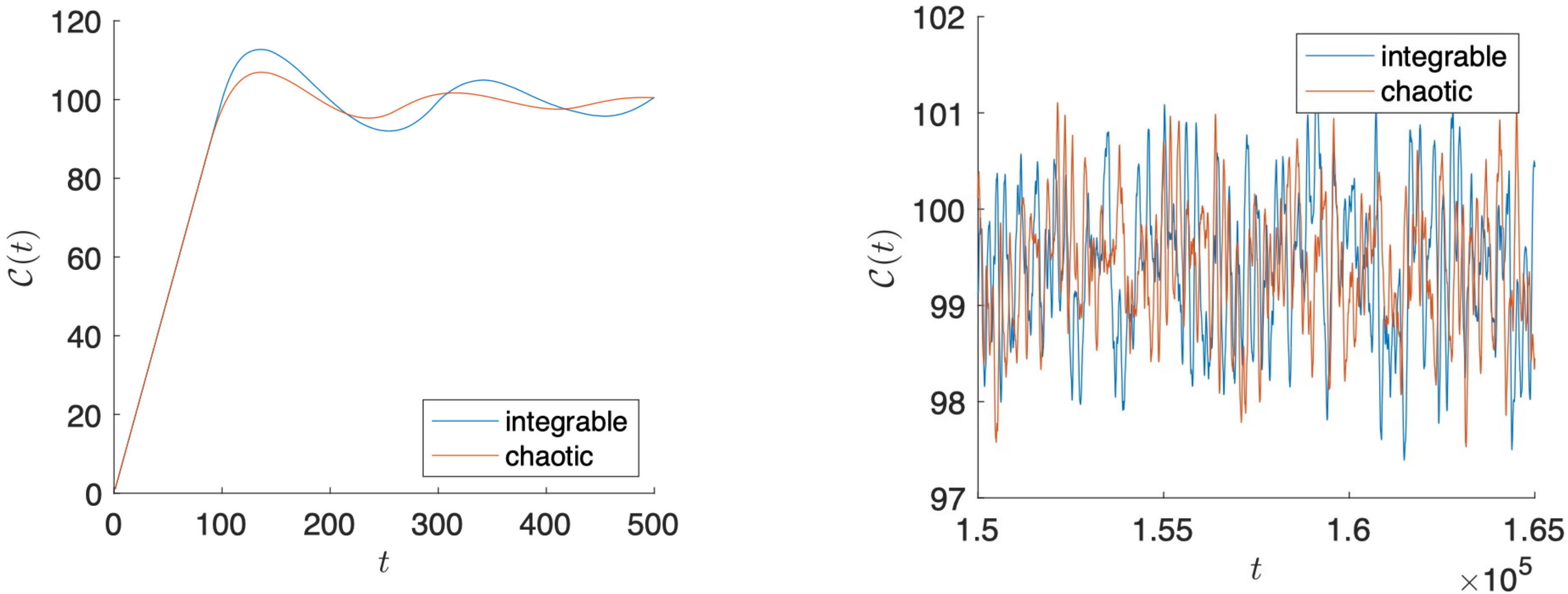
At late times, one observes saturation. Intuition: generic point along line $E_n t$ is as close to the lattice $2\pi\mathbb{Z}^D$ as a generic point in space, and the latter distribution of distances concentrates on the value $\pi(D/3)^{1/2}$ at large D. Saturation height and variance agree with numerics.

Bi-invariant plateau heights do not distinguish chaotic and integrable dynamics: too universal for our purposes. (More subtle distinctions may exist.)

(We ignore Poincaré recurrences at extremely late times, typically exponential in $D = \dim \mathcal{H}$.)

Complexity in bi-invariant metric

Bi-invariant complexity only depends on energy spectra. Compare random energy levels ("integrable") with eigenvalues of random matrices ("chaotic"): very similar.



[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

Complexity at $\mu > 1$: upper bound

Variational method: minimizing over all possible curves (exact solution) is too hard, so let us minimize over bi-invariant geodesics instead. This gives a rigorous upper bound on Nielsen complexity.

Bi-invariant geodesics are generically not geodesics of the complexity metric. This is fine.

Finding the bi-invariant geodesic with shortest length (as measured using the complexity metric) is also too hard, but a suboptimal solution also provides a rigorous upper bound.

Question: is this bound useful?

Will show that it manages to separate chaotic from integrable time evolution.

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

Complexity at $\mu > 1$: upper bound

Minimize over bi-invariant geodesics:

$$U_{\vec{k}}(t') = e^{-iH'_{\vec{k}}t'} \quad \text{with} \quad H'_{\vec{k}} = \sum_n \left(E_n - \frac{2\pi}{t}k_n\right)|n\rangle\langle n|$$

Decompose projectors on energy eigenstates into easy and hard directions:

$$|n\rangle\langle n| = c_n^{\alpha}T_{\alpha} + c_n^{\dot{\alpha}}T_{\dot{\alpha}} \qquad c_n^i = \mathrm{Tr}\big(|n\rangle\langle n|T_i\big) = \langle n|T_i|n\rangle$$

$$||H'_{\vec{k}}||_{\mu}^2 = \sum_{\alpha}\Big[\sum_n \Big(E_n - \frac{2\pi k_n}{t}\Big)c_n^{\alpha}\Big]^2 + \mu\sum_{\dot{\alpha}}\Big[\sum_n \Big(E_n - \frac{2\pi k_n}{t}\Big)c_n^{\dot{\alpha}}\Big]^2$$

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

Introduce Q matrix to rewrite complexity bound

$$||H'_{\vec{k}}||^2_\mu = \sum_\alpha \Big[\sum_n \Big(E_n - \frac{2\pi k_n}{t}\Big) c_n^\alpha\Big]^2 + \mu \sum_{\dot\alpha} \Big[\sum_n \Big(E_n - \frac{2\pi k_n}{t}\Big) c_n^{\dot\alpha}\Big]^2$$

$$= \sum_{mn} \Big(E_n - \frac{2\pi k_n}{t}\Big)\Big(\delta_{nm} + (\mu - 1)Q_{nm}\Big)\Big(E_m - \frac{2\pi k_m}{t}\Big)$$

with

$$Q_{mn} \equiv \sum_{\dot\alpha} \langle n|T_{\dot\alpha}|n\rangle\langle m|T_{\dot\alpha}|m\rangle = \delta_{mn} - \sum_\alpha \langle n|T_\alpha|n\rangle\langle m|T_\alpha|m\rangle$$

$$\mathcal{C}_{\text{bound}}(t) = \min_{\vec{k}\in\mathbb{Z}^D} \Big\{ \sum_{mn} (E_n t - 2\pi k_n)\left[\delta_{nm} + (\mu - 1)Q_{nm}\right](E_m t - 2\pi k_m) \Big\}^{1/2}$$

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

Properties of the Q matrix

$$Q_{mn} \equiv \sum_{\dot{\alpha}} \langle n|T_{\dot{\alpha}}|n\rangle\langle m|T_{\dot{\alpha}}|m\rangle = \delta_{mn} - \sum_{\alpha} \langle n|T_{\alpha}|n\rangle\langle m|T_{\alpha}|m\rangle$$

- Eigenvalues between 0 and 1: $\displaystyle\sum_{mn} Q_{mn} y_m y_n = \sum_{\dot{\alpha}} \left(\sum_n \langle n|T_{\dot{\alpha}}|n\rangle y_n \right)^2$

- The vector of energy eigenvalues is a zero mode:

$$\sum_{m=1}^{D} Q_{nm} E_m = \sum_{\dot{\alpha}} \sum_{m=1}^{D} \langle n|T_{\dot{\alpha}}|n\rangle E_m \langle m|T_{\dot{\alpha}}|m\rangle = \sum_{\dot{\alpha}} \langle n|T_{\dot{\alpha}}|n\rangle \mathrm{Tr}(T_{\dot{\alpha}} H) = 0$$

- Similarly, any zero mode of Q corresponds to a purely local conservation law:

$$0 = \sum_{mn} Q_{mn} c_m c_n = \sum_{\dot{\alpha}} \left(\sum_n \langle n|T_{\dot{\alpha}}|n\rangle c_n \right)^2 = \sum_{\dot{\alpha}} \left(\mathrm{Tr}\left[T_{\dot{\alpha}} \sum_n c_n |n\rangle\langle n| \right] \right)^2$$

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

Bound on complexity as Closest Vector Problem

$$\mathcal{C}_{\text{bound}}(t) = \min_{\vec{k}\in\mathbb{Z}^D} \left\{ \sum_{mn} (E_n t - 2\pi k_n) \left[\delta_{nm} + (\mu - 1)Q_{nm}\right] (E_m t - 2\pi k_m) \right\}^{1/2}$$

Minimization problem on hypercubic lattice in space with nontrivial metric
$\delta_{nm} + (\mu - 1)Q_{nm}$

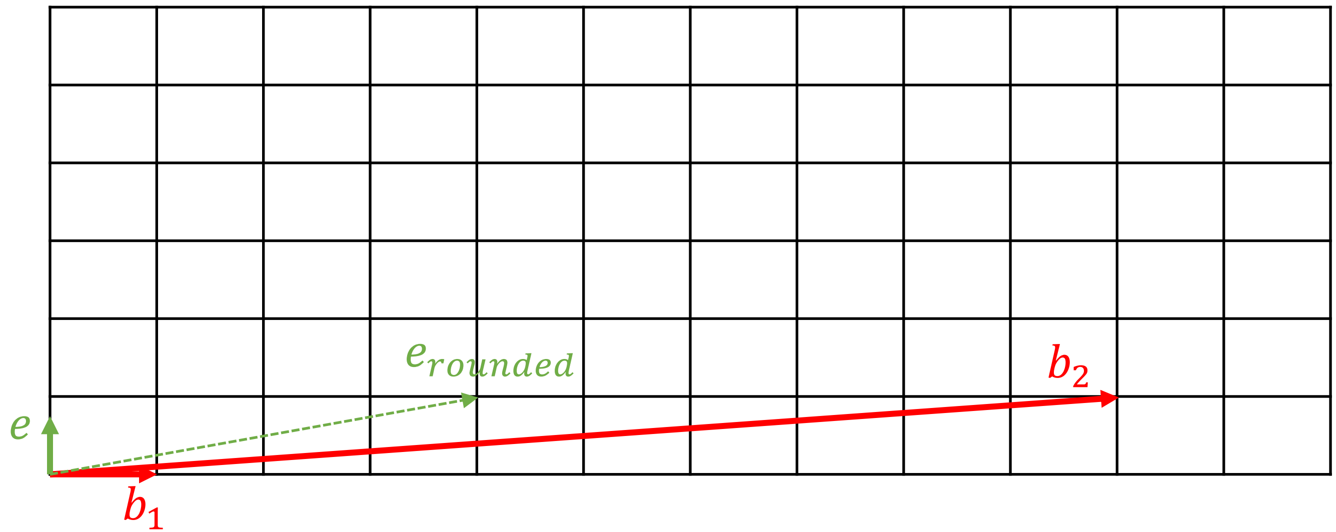
Equivalent to minimization problem on non-hypercubic lattice in flat space.

“Closest Vector Problem”

Very difficult to solve exactly (time exponential in D -- lattice cryptography!),
but approximate solutions can be found in polynomial time.

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

Closest Vector Problem: rounding does not work



$e = 0.6\, b_2 - 6\, b_1$

$e_{rounded} = b_2 - 6\, b_1$

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

Closest Vector Problem: approximation methods

- LLL algorithm: transforms lattice basis into more orthogonal one, with rounder, less elongated unit cell – "basis reduction". [Lenstra, Lenstra, Lovász 1982]
- Babai's nearest plane algorithm: consider family of lattice hyperplanes spanned by $D-1$ basis vectors and project orthogonally on nearest one. Repeat in one dimension less, etc. [Babai 1986]
- Greedy algorithm: check whether complexity bound can be improved by moving in the direction of any of the D basis vectors; then move in direction of maximal gain. [Park, Boyd 2015]

All these algorithms run in a time polynomial in D.

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

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Quantum resonant systems

- Hamiltonian $H = \frac{1}{2} \sum_{\substack{n,m,k,l=0, \\ n+m=k+l}}^{\infty} C_{nmkl}\, a_n^\dagger a_m^\dagger a_k a_l$ $\qquad [a_n, a_m^\dagger] = \delta_{nm}$

- Hilbert space generated by $|\eta_0, \eta_1, \ldots\rangle = \prod_{i=0}^{\infty} \frac{(a_i^\dagger)^{\eta_i}}{\sqrt{\eta_i!}} |0,0,0,\ldots\rangle$

- Obvious conserved charges: $N = \sum_{n=0}^{\infty} a_n^\dagger a_n \qquad M = \sum_{n=0}^{\infty} n\, a_n^\dagger a_n$

- Hilbert space is infinite-dimensional, but Hamiltonian is block-diagonal. Each (N, M) block is finite-dimensional and contains those states with

$$N = \sum_{i=0}^{\infty} \eta_i \qquad M = \sum_{i=0}^{\infty} i\eta_i$$

[Evnin, Piensuk 2019]

Quantum resonant systems

$$H = \frac{1}{2} \sum_{\substack{n,m,k,l=0,\\ n+m=k+l}}^{\infty} C_{nmkl}\, a_n^\dagger a_m^\dagger a_k a_l \qquad [a_n, a_m^\dagger] = \delta_{nm}$$

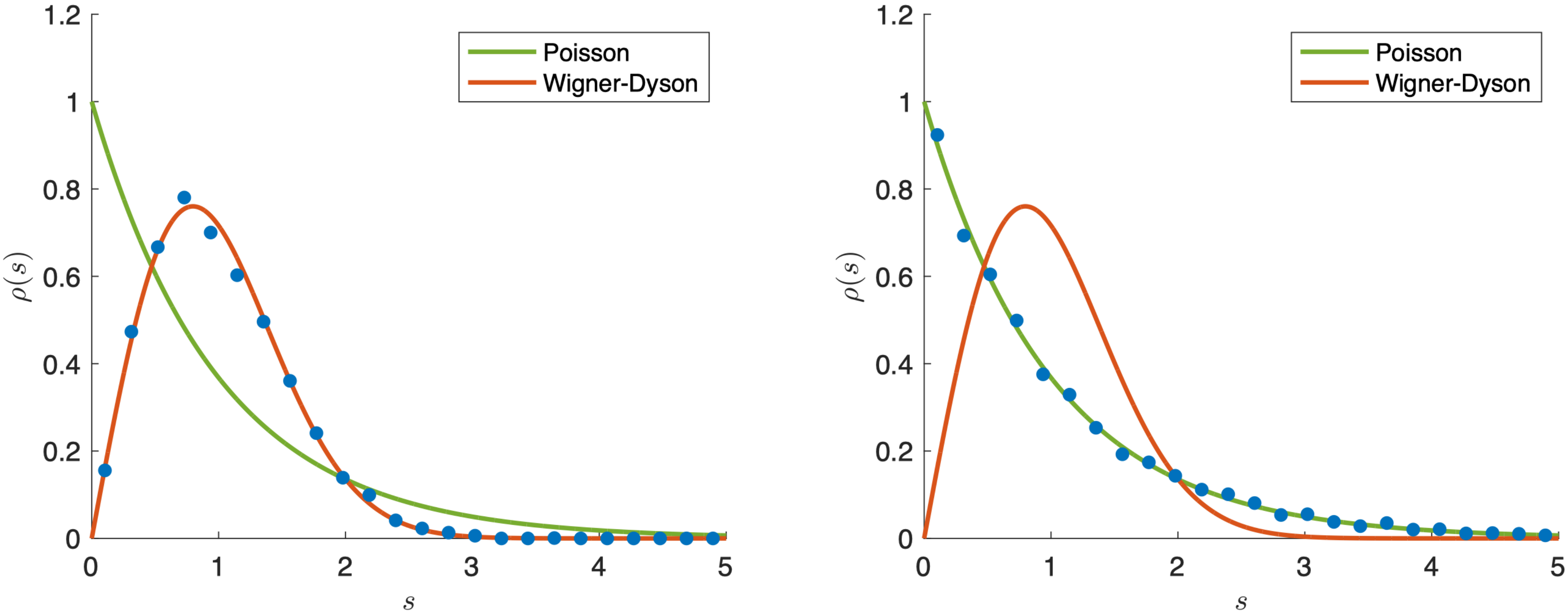
- Classical resonant systems appear as controlled approximations to weakly nonlinear dynamics.

- Depending on choice of "interaction coefficients" C_{nmkl} a variety of dynamical behavior is possible (turbulence, recurrences, integrability, invariant submanifolds, chaos,...).

- Example of "generic" integrable model ("truncated Szegö˝"): [Biasi, Evnin 2020]
  $C_{nmkl} = 0$ if all indices nonzero, $C_{nmkl} = 1$ if at least one index is zero.

- Chaotic for C_{nmkl} randomly drawn from normal distribution.

Chaotic and integrable quantum resonant systems



random

truncated Szegö

$(N = 30, M = 30)$

$D = 5604$

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

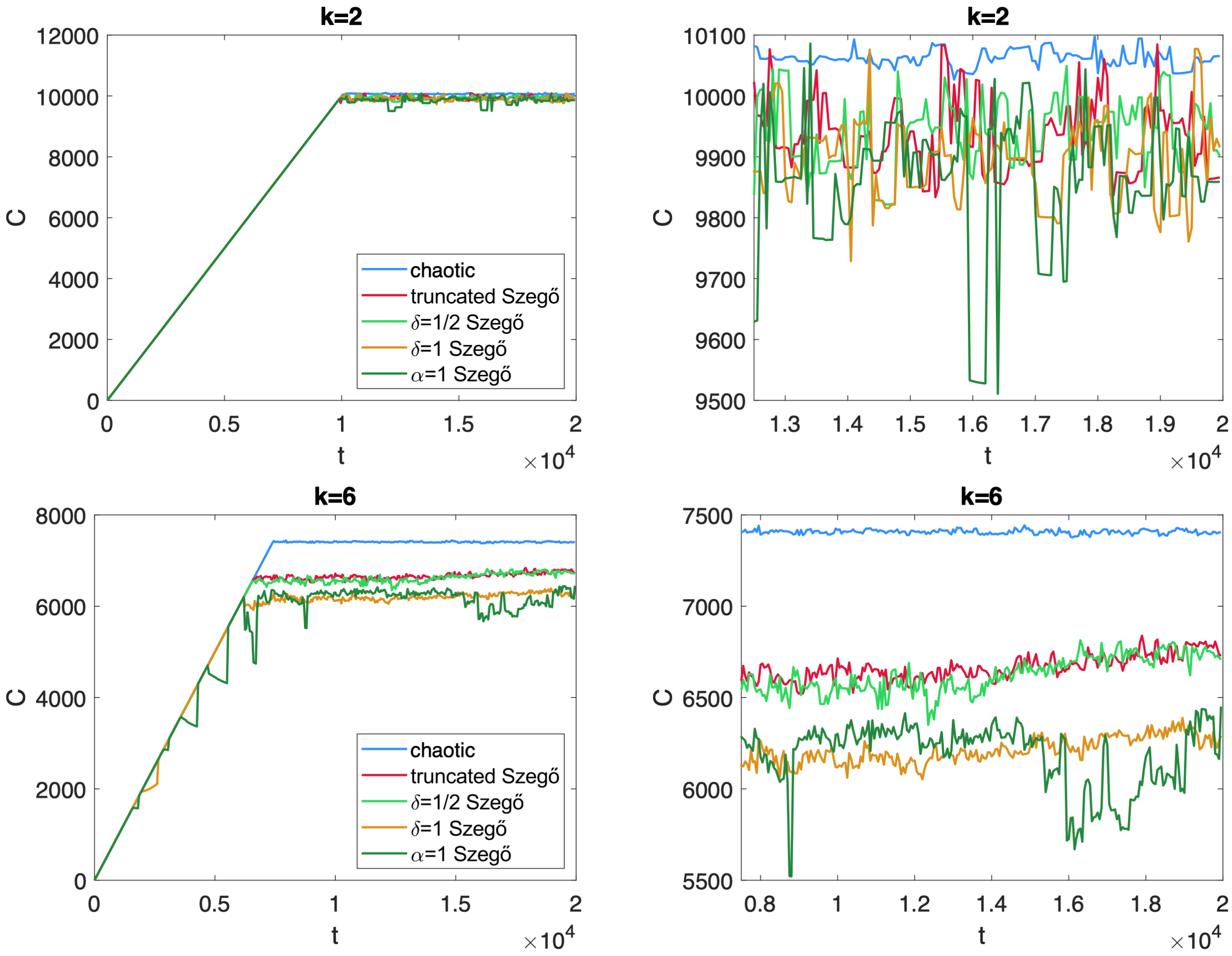
Complexity in quantum resonant systems

$$H = \frac{1}{2} \sum_{\substack{n,m,k,l=0,\\ n+m=k+l}}^{\infty} C_{nmkl}\, a_n^\dagger a_m^\dagger a_k a_l \qquad [a_n, a_m^\dagger] = \delta_{nm}$$

- Hamiltonian changes at most 2 modes in a state in an (N, M) block. Generalize this idea to define a “locality degree” k.
- There is a noticeable difference in complexity plateau height between integrable (truncated Szegö) and chaotic models (10-20%).

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

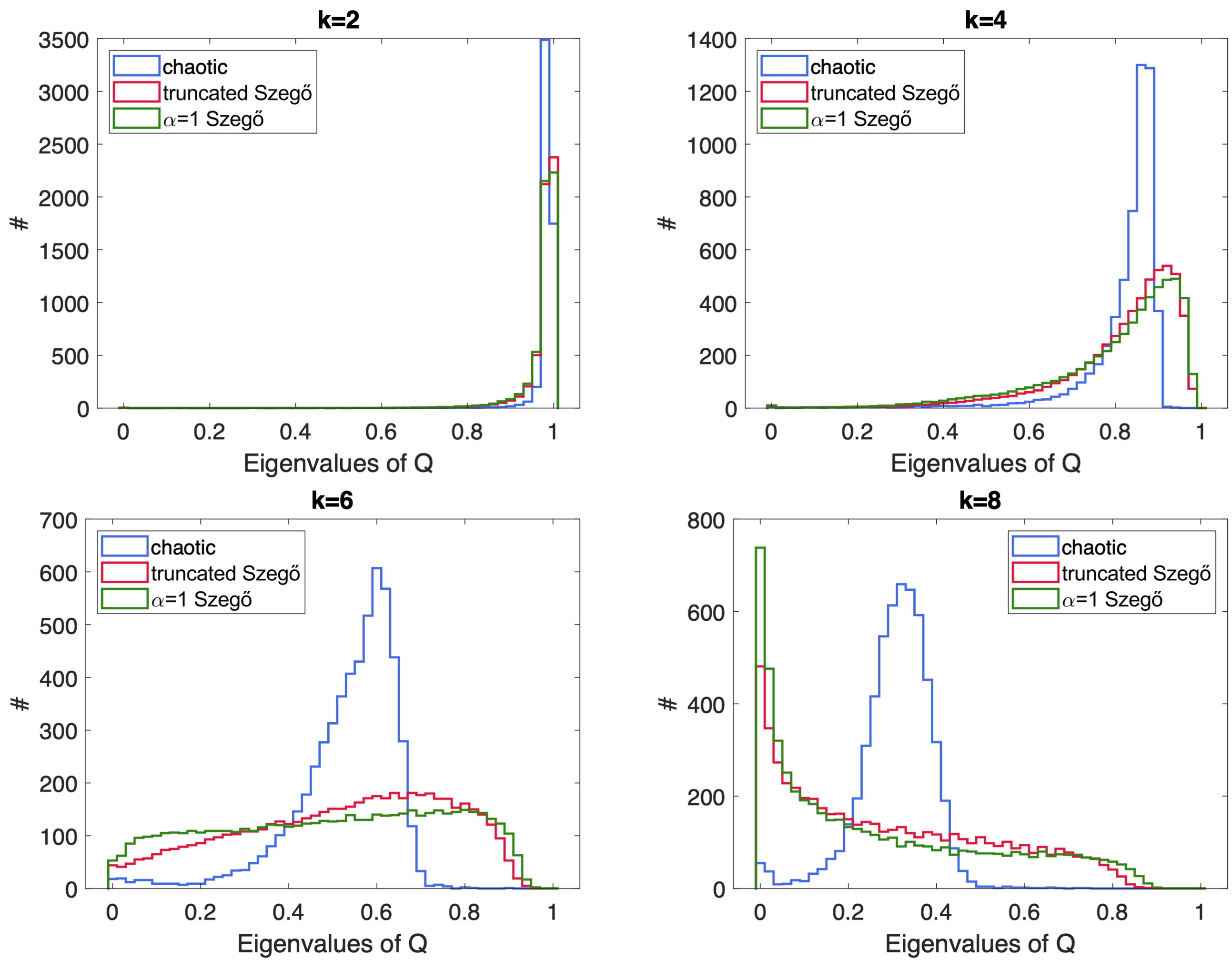
Integrable models have lower complexity



$$(N = 30, M = 30)$$
$$D = 5604$$

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

Distribution of Q eigenvalues



Integrable models (red/green) possess tower of conserved charges of increasing locality degree.

Complexity plateau height correlates with the number of small Q eigenvalues.

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

$$(N = 30, M = 30)$$
$$D = 5604$$

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Chaotic models: saturation value from RMT

$$\mathcal{C}_{\text{bound}}(t) = \min_{\vec{k}\in\mathbb{Z}^D} \left\{ \sum_{mn} (E_n t - 2\pi k_n) \left[\delta_{nm} + (\mu - 1) Q_{nm}\right] (E_m t - 2\pi k_m) \right\}^{1/2}$$

$$Q_{mn} \equiv \sum_{\dot{\alpha}} \langle n|T_{\dot{\alpha}}|n\rangle\langle m|T_{\dot{\alpha}}|m\rangle = \delta_{mn} - \sum_{\alpha} \langle n|T_{\alpha}|n\rangle\langle m|T_{\alpha}|m\rangle$$

Assume that energy eigenvectors are columns of random unitary matrix (GUE)

For large D: RMT estimate for mean Q-eigenvalue is $\langle\bar{\lambda}\rangle = 1 - \dfrac{N_{loc}}{D^2}$

RMT estimate for variance: $\langle \mathrm{Var}(\lambda)\rangle \approx \dfrac{N_{loc}}{D^3}$

In large D limit with fixed N_{loc} or fixed N_{loc}/D^2 : Q-eigenvalues concentrate about the mean, so Q approaches multiple of unit matrix

$$\mathcal{C}_{\text{average estimate}} = \frac{\pi}{\sqrt{3}} \left(\sum_{n=1}^{D} (1 + (\mu - 1)\lambda_n) \right)^{1/2} \approx \pi \sqrt{\frac{D\mu\bar{\lambda}}{3}}$$

[BC, De Clerck, Evnin, Hacker 2023]

How does integrability lower complexity?

$$Q_{mn} \equiv \sum_{\dot{\alpha}} \langle n|T_{\dot{\alpha}}|n\rangle\langle m|T_{\dot{\alpha}}|m\rangle = \delta_{mn} - \sum_{\alpha} \langle n|T_{\alpha}|n\rangle\langle m|T_{\alpha}|m\rangle$$

Hypothesis: In integrable models, complexity reduction is due to null eigenvalues of Q

Null eigenvalues of Q correspond to local conservation laws, characteristic of integrable systems → direct relation between complexity reduction and integrable structures

Will test this in detail for integrable spin chains

[BC, De Clerck, Evnin, Hacker 2023]

Spin chain Hamiltonians

Mixed-field Ising: $$H_{\text{Ising}} = -\sum_j [S_z^{(j)} S_z^{(j+1)} + h_x S_x^{(j)} + h_z S_z^{(j)}]$$

Heisenberg with magnetic field:

$$H_{XYZ} = \sum_j [J_x S_x^{(j)} S_x^{(j+1)} + J_y S_y^{(j)} S_y^{(j+1)} + J_z S_z^{(j)} S_z^{(j+1)} - h_z S_z^{(j)}]$$

Both are integrable if $h_z = 0$

Definitions of ”easy” operators for spin chains

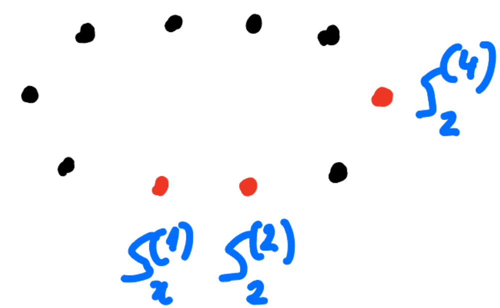
Unitaries are generated by strings of Pauli operators

Locality degrees:

- k_{op} : number of sites on which generator acts = number of Pauli operators
- k_{sp} : length of lattice region on which generator acts nontrivially

Example: $S_x^{(1)} S_z^{(2)} S_z^{(4)}$ has $k_{op} = 3$

and $k_{sp} = 4$



[BC, De Clerck, Evnin, Hacker 2023]

Integrable spin chain: towers of local conserved charges

E.g. transverse-field Ising: $$H_{\text{Ising}} = -\sum_j [S_z^{(j)} S_z^{(j+1)} + h_x S_x^{(j)}]$$

$$I_1 = \sum_j \left(S_y^{(j)} S_z^{(j+1)} - S_z^{(j)} S_y^{(j+1)} \right)$$

$$I_2 = \sum_j \left(S_z^{j} S_x^{(j+1)} S_z^{(j+2)} - h_x S_y^{(j)} S_y^{(j+1)} - h_x S_z^{(j)} S_z^{(j+1)} - S_x^{(j)} \right)$$

$$I_{2l-1} = \sum_j \left(S_y^{(j)} S_x^{(j+1)} S_x^{(j+2)} \ldots S_x^{(j+l-1)} S_z^{(j+l)} - S_z^{(j)} S_x^{(j+1)} S_x^{(j+2)} \ldots S_x^{(j+l-1)} S_y^{(j+l)} \right)$$

$$I_{2l} = \sum_j \left(S_z^{(j)} S_x^{(j+1)} S_x^{(j+2)} \ldots S_x^{(j+l)} S_z^{(j+l+1)} - h_x S_y^{(j)} S_x^{(j+1)} S_x^{(j+2)} \ldots S_x^{(j+l-1)} S_y^{(j+l)} \right.$$
$$\left. - h_x S_z^{(j)} S_x^{(j+1)} S_x^{(j+2)} \ldots S_x^{(j+l-1)} S_z^{(j+l)} + S_y^{(j)} S_x^{(j+1)} S_x^{(j+2)} \ldots S_x^{(j+l-2)} S_y^{(j+l-1)} \right)$$

Bound on complexity for degenerate energy spectra

- Geodesics in bi-invariant metric have constant velocity V: $U(t) = e^{iVt}$
- Geodesics connecting $U(0) = \mathbf{I}$ and $U(t) = e^{-iHt}$:

$U(t') = e^{-iH't'}$ with $e^{-iH't} = e^{-iHt}$, so (if no degeneracies)

$$H' = \sum_n \left(E_n - \frac{2\pi}{t}k_n\right)|n\rangle\langle n| \quad \text{with} \quad k_n \in \mathbb{Z}$$

- Degeneracies: choice of eigenbasis labeled by continuous angles
- Solution: use energy eigenbasis with largest number of zero Q-eigenvalues

[BC, De Clerck, Evnin, Hacker 2023]

Testing the complexity reduction hypothesis

For Ising and Heisenberg XYZ Hamiltonians: study correlation between complexity reduction and various choices of the set of local/easy operators.

Predictions:

- adding non-Q-kernel easy operators does not change complexity reduction much
- declaring a local conservation law to become nonlocal does increases the complexity plateau significantly

[BC, De Clerck, Evnin, Hacker 2023]

Additional non-Q-kernel "easy" operators: Ising

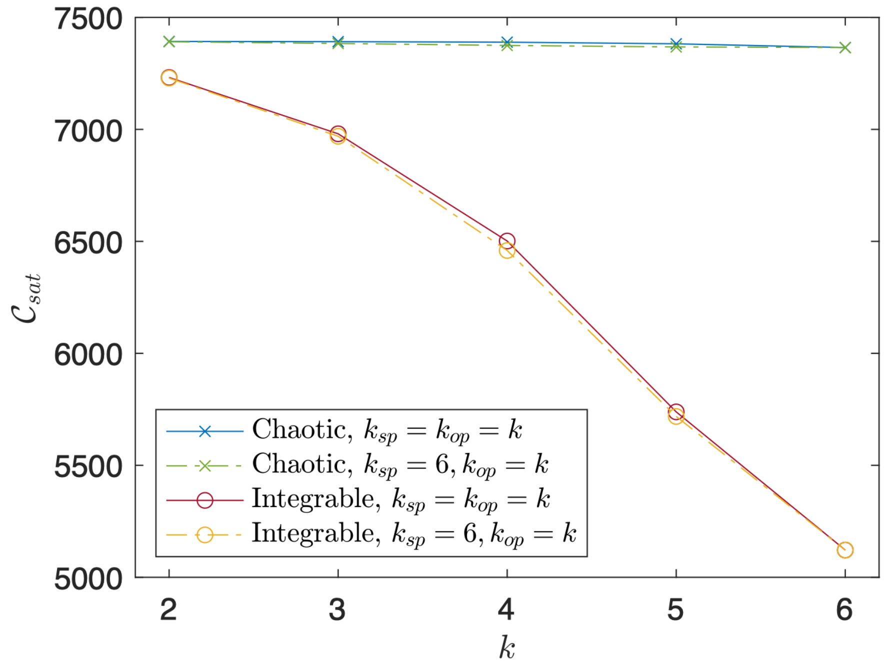


Figure 4: The saturation value of the complexity of the integrable $(h_x, h_z) = (-1.05, 0)$ and chaotic $(h_x, h_z) = (-1.05, 0.5)$ Ising model with $L = 12$ sites as a function of a locality threshold specified by k. The dashed line corresponds to $k_{sp} = 6$ fixed and varying k_{op}, while for the solid line we vary both locality degrees $k_{sp} = k_{op}$.

[BC, De Clerck, Evnin, Hacker 2023]

Additional non-Q-kernel "easy" operators: XYZ

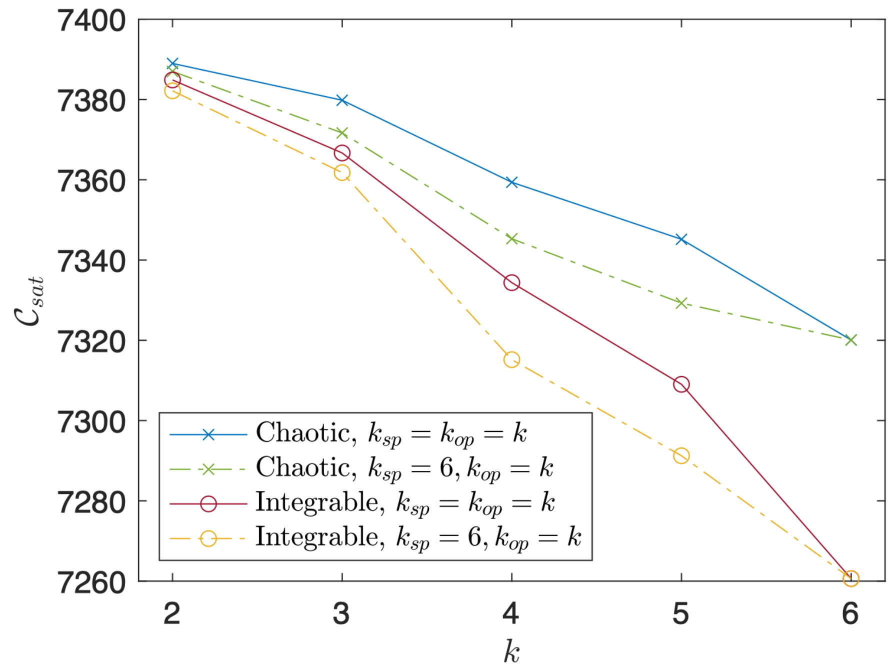
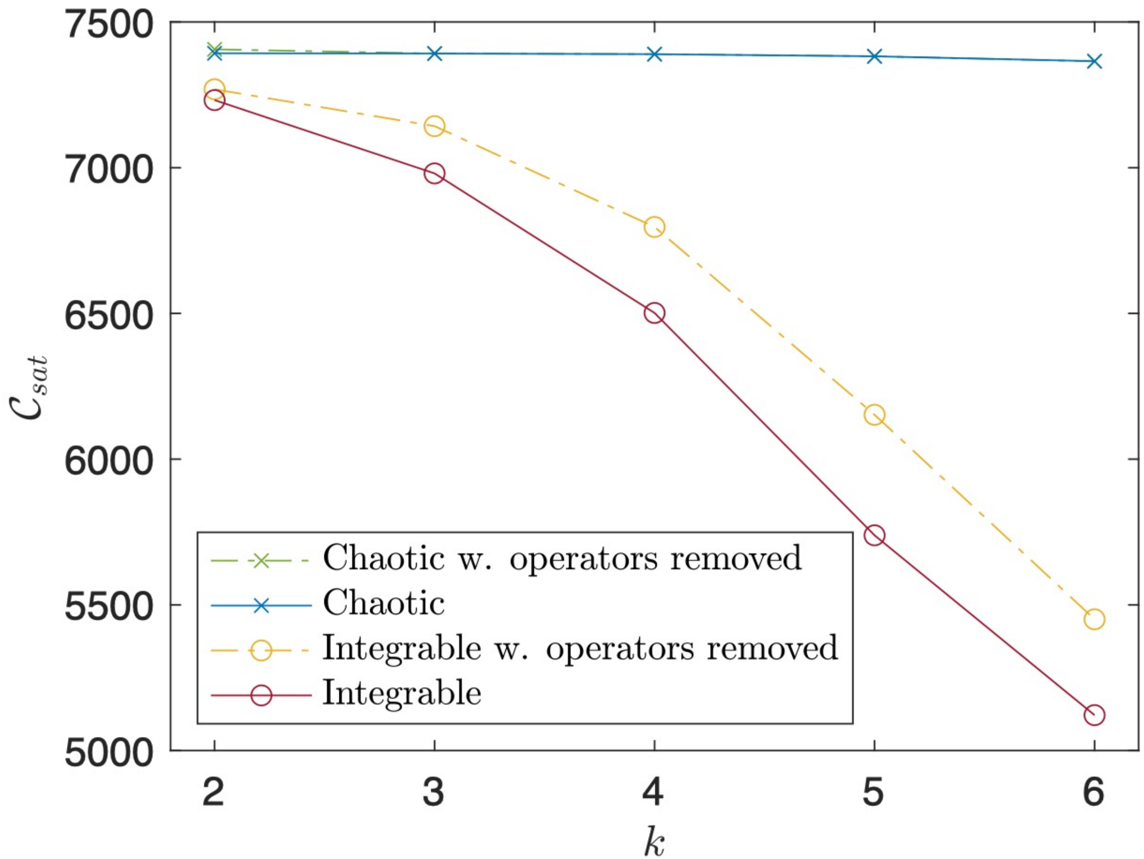


Figure 6: The saturation values of the complexity of the integrable XYZ model (4.14) and the chaotic Hamiltonian with magnetic field (4.19) for $L = 12$. For the coupling constants we used the numbers $(J_x, J_y, J_z) = (-0.35, 0.5, -0.1)$ in both cases and $h_z = 0.8$ for the chaotic Hamiltonian.

[BC, De Clerck, Evnin, Hacker 2023]

Slightly decreasing the kernel of Q: Ising



[BC, De Clerck, Evnin, Hacker 2023]

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Conclusions

- Reduced upper bound on Nielsen complexity to Closest Vector Problem.
- Our bound separates integrable and chaotic systems.
- Key quantity: Q matrix. Determines metric on lattice. Zero modes correspond to local conservation laws.

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

- Extended bound on complexity to degenerate energy spectra.
- RMT estimate of saturation value for chaotic systems.
- Hypothesis: complexity reduction in integrable models is due to null eigenvalues of Q.
- Elaborate tests on spin chains confirm hypothesis.

[BC, De Clerck, Evnin, Hacker 2023]

Outlook

- Complexity plateau height from average distance to lattice: extend to integrable systems?
- Are there simple improvements of our bound? Paths with piecewise constant velocities?
- To which geodesics do our simple curves flow?
- Limit of large penalty factor μ ?

SYK models

$$\{\psi_i, \psi_j\} = \delta_{ij}$$

$$i, j = 1, \dots N$$

- Free SYK:

$$H^{\text{free}} = i \sum_{i,j=1}^{N} J_{ij} \psi^i \psi^j = \sum_{p=1}^{N/2} \omega_p J_3^{(p)}$$

$$\{E_n\} = \left\{\pm\omega_1 \pm \omega_2 \pm \cdots \pm \omega_{N/2}\right\}$$

- Interacting integrable SYK:

$$H^{\text{int}} = H^{\text{free}} + \epsilon \sum_{1 \le p < p' \le N/2} M_{pp'} J_3^{(p)} J_3^{(p')}$$

- Chaotic SYK:

$$H^{\text{chaotic}} = H^{\text{free}} + \epsilon \sum_{1 \le i < j < k < l \le N} J_{ijkl} \psi^i \psi^j \psi^k \psi^l$$

Complexity in SYK models

- Define “local operators” as those with “locality degree” at most k:

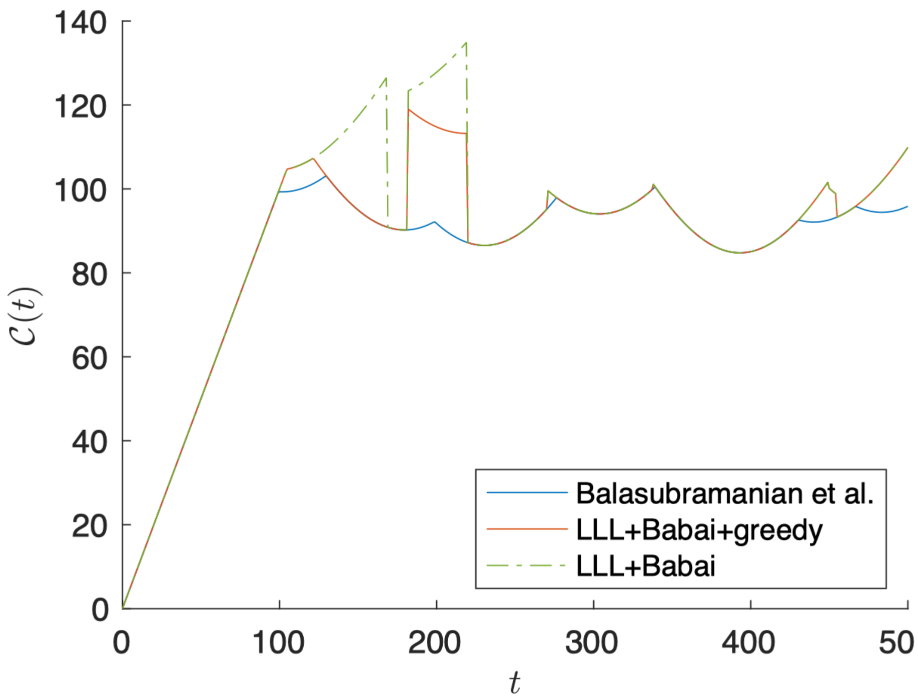
$$\{\psi_{i_1}\psi_{i_2}\dots\psi_{i_n}|n \leq k\}$$

- For free and integrable SYK, significant complexity reduction was found using model-specific methods in [Balasubramanian, DeCross, Kar, Li, Parrikar 2021] . This analysis relied on many quadratic and quartic conservation laws (corresponding to many zero modes of Q). Lattice vectors in these directions can be easily used to reduce complexity.

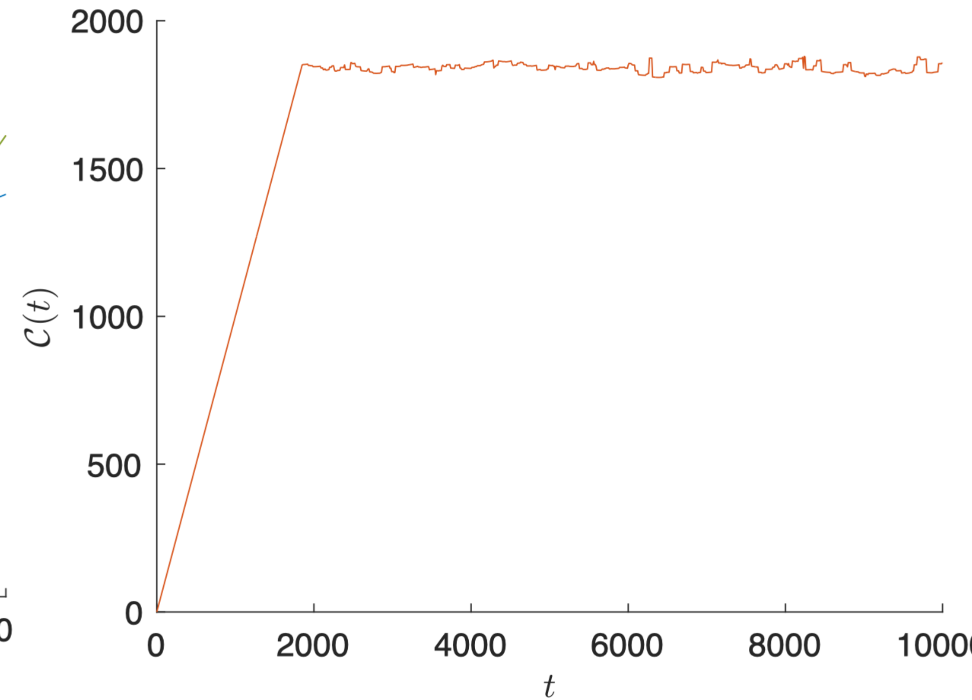
- Our automated method essentially recovers the results for free SYK, significantly improves those for integrable SYK, and provides results for chaotic SYK.

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

Complexity in SYK models



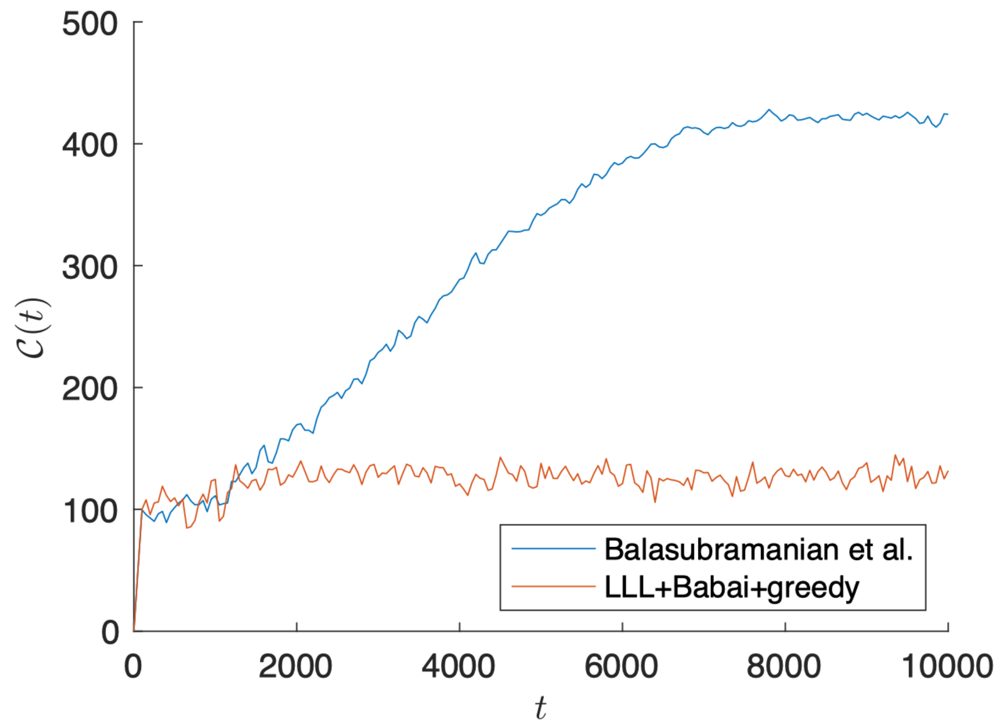
Free SYK



Chaotic SYK

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

Complexity in SYK models



Integrable SYK

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

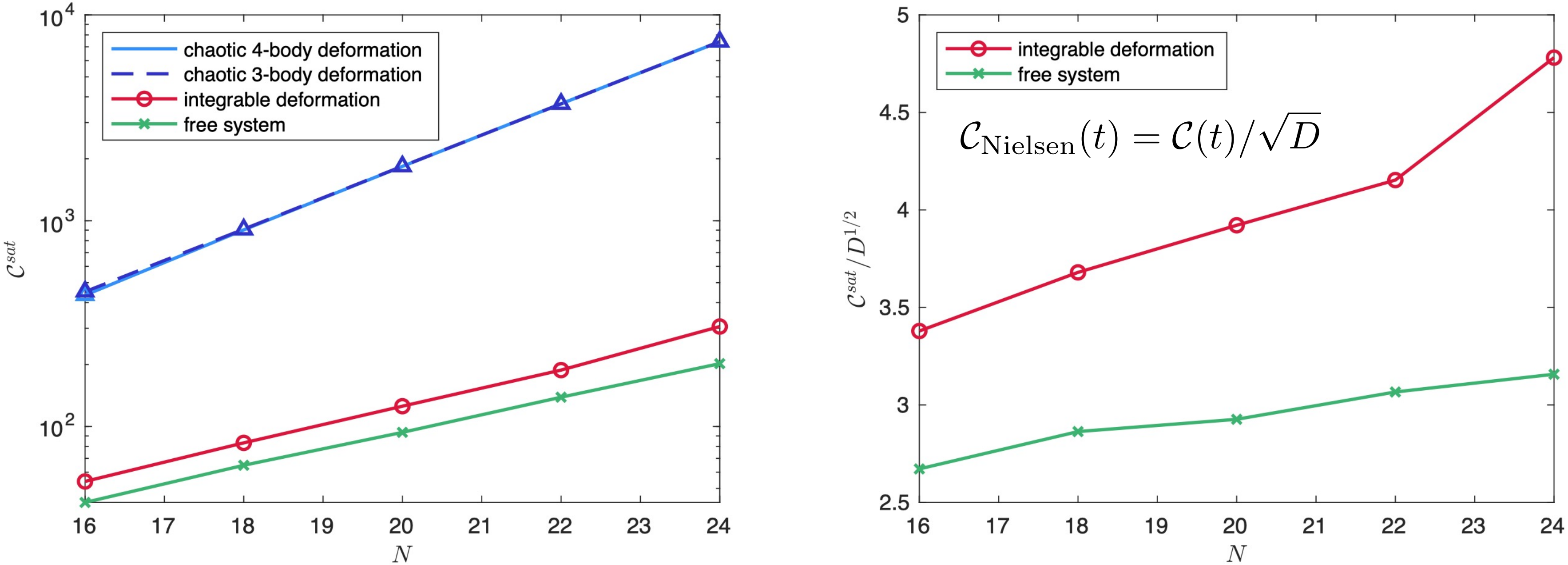


Figure 4: Plots of the saturation value of complexity (left) and the saturation value of complexity divided by $\sqrt{D}$ (right) against N for different systems, locality degree $k = 4$ and for the range $N = 16, 18, \ldots, 24$. On the left, we display, in addition to the integrable and free systems, 3 realizations of each chaotic Hamiltonian using triangles and the spread amongst the chaotic runs can be observed to be very small. The blue lines connect their mean value at every D. On the right, we illustrate the slower growth of the integrable models by dividing the complexity curve by $\sqrt{D}$, which removes the exponential dependence on N. This normalization matches the complexity convention used in [19, 20], as explained in footnote 7. For all deformations, we set $\epsilon = 1$.

[BC, De Clerck, Evnin, Hacker, Pavlov 2022]

SYK is special

- Integrable SYK is not a generic integrable system (unusually many quadratic and quartic conservation laws, which furthermore have integer spectra).
- Fermionic models have no conventional semiclassical limit, which is a drawback when discussing chaos and integrability.

Complexity and black hole physics

Complexity is increasingly present in discussions of black holes:

- volume behind horizon,…
  [Susskind 2014,…]
- difficulty of reconstructing BH interior, protection of causality
  [Harlow, Hayden 2013; Aaronson 2016; Kim, Tang, Preskill 2020,…]
- difficulty of distinguishing BH microstates
  [Balasubramanian, de Boer, Jejjala, Simón 2005,…]

Lots of work on proposed gravitational duals. Need tools for computing complexity in QM/QFT.

What causes black holes to be complex? Chaotic time-evolution? Does complexity distinguish chaotic from integrable evolution?

[Brown, Susskind, Zhao 2016; Balasubramanian, DeCross, Kar, Li, Parrikar; Rabinovici, Sánchez-Garrido, Shir, Sonner;…]