Towards Lattice Calculations of Double Parton Distributions

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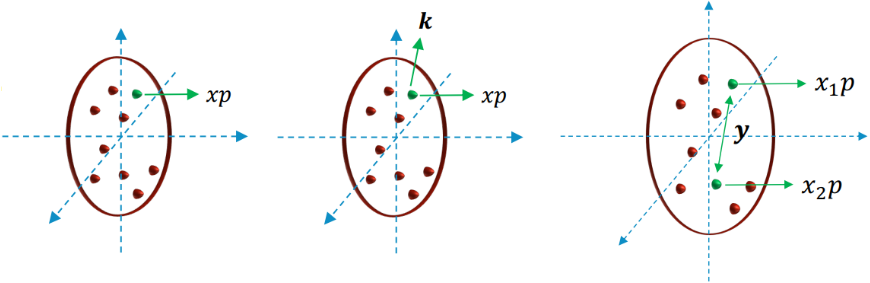
MPI Workshop 2022
November 14th

Progress in extracting PDFs and TMDs from lattice calculations

- We aim to extend this success to double parton distributions (DPDs)



PDFs TMDs DPDs

Figure adapted from seminar by J. Gaunt

Outline

- Lightspeed introduction to double parton distributions
  - Factorization and definitions
- Lightcone correlators on the lattice
  - Crash course in lattice QCD
  - What is the problem?
- The quasi-PDF approach
  - A clever work-around for PDFs
- Extension to double parton distributions
  - What do we need?
  - Our conjecture
- Outlook: how do we proceed from here?

Lightspeed introducion to double parton scattering

Introduction to DPDs - Factorization

- Cross section of DPS process factorizes as Hard $\otimes$ DPDs $\otimes$ Soft.

$$\mathrm{d}\sigma^{\mathsf{DPS}} = \left(\frac{4\pi\alpha^2 Q_q^2}{3N_c s}\right)\frac{1}{q_1^2 q_2^2}\int \mathrm{d}^2\mathbf{b}_\perp$$
$$\times \Big\{ \left[{}^1F_{qq}\,{}^1F_{\bar{q}\bar{q}} + {}^1F_{\Delta q\Delta q}\,{}^1F_{\Delta\bar{q}\Delta\bar{q}} + {}^1F_{q\bar{q}}\,{}^1F_{\bar{q}q} + {}^1F_{\Delta q\Delta\bar{q}}\,{}^1F_{\Delta\bar{q}\Delta q}\right]{}^{11}S$$
$$+ \tfrac{2N_c}{C_F}\left[{}^8F_{qq}\,{}^8F_{\bar{q}\bar{q}} + {}^8F_{\Delta q\Delta q}\,{}^8F_{\Delta\bar{q}\Delta\bar{q}} + {}^8F_{q\bar{q}}\,{}^8F_{\bar{q}q} + {}^8F_{\Delta q\Delta\bar{q}}\,{}^8F_{\Delta\bar{q}\Delta q}\right]{}^{88}S$$
$$+ \text{interference terms}\Big\}$$

- Many different color and spin structures

Manohar, Waalewijn (2012)
Gaunt (2014)
Diehl, Gaunt, Ostermeier, Ploessl, Schäfer (2015)

Introduction to DPDs - Definitions

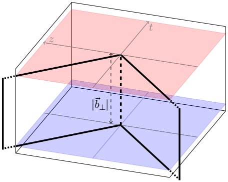
- DPDs can be expressed as hadronic lightcone correlators. For F_{qq}:

$$
\begin{aligned}
{}^{R}F_{a_1 a_2} = -\pi P^+ \int \frac{db_1^-}{2\pi}\frac{db_2^-}{2\pi}\frac{db_3^-}{2\pi}\; & e^{-ix_1 P^+ b_1^-} e^{-ix_2 P^+ b_2^-} e^{ix_1 P^+ b_3^-} \\
& \times \langle P|\, T^\dagger \Big[\bar\psi_n(0^+, b_1^-, \mathbf{b}_\perp)\Gamma_{a_1} R_1\Big]_i \Big[\bar\psi_n(b_2^-)\Gamma_{a_2} R_2\Big]_j \\
& \times T\Big[\psi_n(0^+, b_3^-, \mathbf{b}_\perp)\Big]_i \Big[\psi_n(0)\Big]_j \,|P\rangle
\end{aligned}
$$

  - different spin and color structures allowed
- Soft factors can be written as vacuum matrix elements of Wilson loops

$$
{}^{11}S = 1\ , \qquad {}^{88}S = \frac{1}{2N_c C_F}\,\langle 0|\,\mathrm{tr}\big[\mathcal{S}\big]\mathrm{tr}\big[\mathcal{S}^\dagger\big]\,|0\rangle - \frac{1}{2N_c C_F}
$$

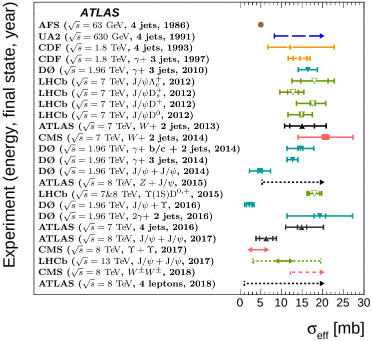
$\mathcal{S}$:



Diehl, Ostermeier, Schäfer (2011)
Manohar, Waalewijn (2012)
Diehl, Nagar (2019)

What is known about DPDs?

- Poorly constrained by experiment: only measurement on σ_{eff}
- Lowest few moments from the lattice, higher moments are difficult



ATLAS Collaboration (2019)
Bali, Diehl, Gläßle, Schäfer, Zimmermann (2021)

Goal

Calculate full x-dependence of DPDs from first principles using lattice QCD

Lightcone correlators on the lattice

How do we calculate matrix elements on the lattice?

- Path integral formulation of QFT is central in lattice QCD

$$\langle \phi(x_1)\phi(x_2)\rangle = \int \mathcal{D}\phi \; e^{iS[\phi]} \; \phi(x_1)\phi(x_2)$$

- Discretized space-time $\rightarrow$ discrete path integral

$$\int \mathcal{D}\phi \rightarrow \int \prod_i d\phi_i$$

  - Infinite dimensional integral becomes a very-large-dimensional integral

- Calculate the integral using Monte-Carlo methods
  - Random sampling of field configurations with $e^{iS[\phi]}$ acting as weight

The problem with having $e^{iS[\phi]}$ as the weight

- Monte-Carlo simulation should converge to the exact result
  - This is only the case for a real and positive weight
- For complex weight the convergence is extremely slow (NP-hard)

How does lattice QCD deal with the sign problem?

- Wick rotation gives a positive real weight

$$t \to it_E \qquad \& \qquad e^{iS[\phi]} \to e^{-S_E[\phi]}$$

- But only allowed if matrix element is time-independent

$$\langle \phi(x_1)\phi(x_2) \rangle = \int \mathcal{D}\phi \; e^{-S_E[\phi]} \; \phi(x_1)\phi(x_2)$$

Calculating PDFs from first principles

Let us first focus on the easier case: PDFs

$$f_q(x) = \int \frac{dy^-}{4\pi}\, e^{ixP^+y^-}\, \langle P|\bar{\psi}(y^-)\gamma^+ W(y^-,0)\psi(0)|P\rangle$$

- Proton state with momentum $P = (\infty^+, 0^-, \mathbf{0}_\perp)$
- Quark fields with light-like separation
- Wilson line for gauge invariance

Light-like separated quark fields $\Rightarrow$ time-dependent matrix element

The sign problem

PDFs cannot be calculated on the lattice due to the inability of this method to calculate time-dependent matrix elements

The quasi-PDF approach

Quasi-PDF approach: avoid the sign problem

Consider a completely different object:

$$\tilde{f}(x, P^z) = \int \frac{dz}{4\pi}\, e^{-ixP^z z}\, \langle P|\, \bar{\psi}(z)\gamma^z W(z,0)\psi(0)\,|P\rangle$$

How is the quasi-PDF different?

- Quark fields separated only along the z-axis
- Equal-time correlator $\rightarrow$ time-independent $\rightarrow$ Lattice!
- Can be calculated on the lattice for large but finite P^z
- quasi-PDF is not Lorentz invariant, depends on P^z

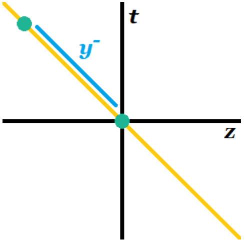
What is the relevance of this?

How is the quasi-PDF related to the physical lightcone-PDF?

Ji (2013)

Lightcone-PDF

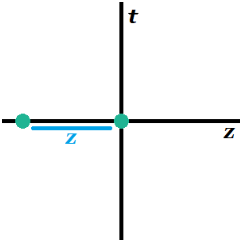
$$\langle P | \bar{\psi}(y^-)\gamma^+ W(y^-, 0)\psi(0) | P\rangle$$



- Light-like separated fields
- Infinite P^+ proton state
- Lorentz invariant
  - $f(x)$ independent of P^+

Quasi-PDF

$$\langle P | \bar{\psi}(z)\gamma^z W(z, 0)\psi(0) | P\rangle$$



- Space-like separated fields
- Finite P^z proton state
- Not Lorentz invariant
  - $\tilde{f}(x, P^z)$ depends on P^z

Boost the quasi-PDF

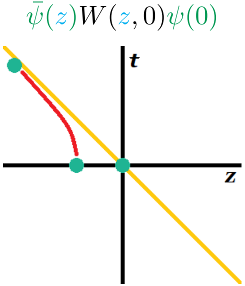
$$\frac{z}{\sqrt{2}}(-1^+, 1^-, \mathbf{0}_\perp) \to \frac{z}{2}(-e^{-y_B}, e^{y_B}, \mathbf{0}_\perp)$$

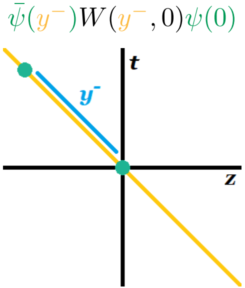
Some observations:

- Approaches lightcone-PDF asymptotically
- Boosting $\tilde{f}(x, P^z)$ is varying P^z

Unfortunately

$$\lim_{P^z\to\infty} \tilde{f}(x, P^z) \neq f(x)$$





Lightcone vs. Quasi: round 2

Lightcone-PDF

- Constructed for factorization formulas
- Factorization theorems require $P^z \to \infty$ from the start
- Renormalization happens at the end

Quasi-PDF

- Constructed for lattice calculations
- Lattice calculations can only simulate at finite P^z (at most $\sim 1/a$)
- Renormalization must happen before $P^z \to \infty$

A tale of two limits

Lightcone- and quasi-PDF differ in the order in which renormalization and the $P^z \to \infty$ limit happen

The final formula

Not equal, but difference is perturbative

$$\tilde{f}(x, P^z, \mu) = \int \frac{dy}{y}\, \mathcal{C}\left(\frac{x}{y}, \frac{\mu}{yP^z}\right) f(y, \mu) + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda^2_{\mathrm{QCD}}}{(xP^z)^2}\right)$$

Recipe for calculating PDFs from first principles:

- Calculate quasi-PDF on the lattice at large P^z
- Calculate matching kernel in perturbation theory
- Extract lightcone-PDF from the above relation

Achievment unlocked

We can now calculate the physical lightcone-PDF from the lattice calculable quasi-PDF and a perturbative matching kernel

Ji (2013)
Ma, Qiu (2014)
Izubuchi, Ji, Jin, Stewart, Zhao (2018)

Extension to DPDs

What do we need?

- (1) Quasi-DPD
  - That can be calculated on the lattice
- (2) Matching relation
  - Relationship between the quasi- and lightcone-DPD
- (3) Perturbative matching kernel
  - Consistency check: does IR behaviour match up?

- Define a quasi-DPD
  - Replace light-cone correlator by equal-time correlator

$$
{}^{R}\tilde{F}_{a_1 a_2} = -\pi P^+ \int \frac{db_1^z}{2\pi}\frac{db_2^z}{2\pi}\frac{db_3^z}{2\pi}\; e^{ix_1 P^z b_1^z} e^{ix_2 P^z b_2^z} e^{-ix_1 P^z b_3^z}
\times \langle P|\, T^\dagger \Big[\bar{\psi}_z(0, \mathbf{b}_\perp, b_1^z)\tilde{\Gamma}_{a_1} R_1\Big]_i \Big[\bar{\psi}_z(b_2^z)\tilde{\Gamma}_{a_2} R_2\Big]_j
\times T\Big[\psi_n(0, \mathbf{b}_\perp, b_3^z)\Big]_i \Big[\psi_n(0)\Big]_j \,|P\rangle
$$

- Define quasi DPS soft function
  - Boost one of the staples of Wilson loop $\mathcal{S}$ to lie on the z-axis

$$
{}^{88}\tilde{S} = \frac{1}{2N_c C_F}\,\langle 0|\, \mathrm{tr}\big[\tilde{\mathcal{S}}\big]\mathrm{tr}\big[\tilde{\mathcal{S}}^\dagger\big]\,|0\rangle - \frac{1}{2N_c C_F}
$$

  - Two sets of oppositely directed staples → not straightforward for lattice

(2) Matching relation - where to start?

Lessons from the past

Matching relation encodes the difference in UV and large rapidity (LR) behaviour

For matching, we need to know the UV and LR behaviour of DPDs

- How do DPDs behave in the ultraviolet regime?
  - Convolution and mixing
- What is the large rapidity behaviour of DPDs?
  - Regulate rapidity divergences and divide by $\sqrt{S}$
  - Rapidity scale dependence described by Collins-Soper kernel

$$\frac{d}{d\log\zeta}\,{}^{R}F^{\mathsf{sub}}_{a_1 a_2} = \tfrac{1}{2}\,{}^{R}\gamma_{\zeta}(b_{\perp},\mu)\;{}^{R}F^{\mathsf{sub}}_{a_1 a_2} \qquad\qquad \zeta = 4x_1 x_2 (P^{+})^2$$

Manohar, Waalewijn (2012)
Diehl, Nagar (2014)
Diehl, Gaunt (2016)
Diehl, Gaunt, Ploessl, Schäfer (2019)
Diehl, Gaunt, Ploessl (2021)

(2) Matching relation - conjecture

Based on PDF and TMD cases (proven), and UV and LR behaviour:

Quasi-DPD = perturbative kernel ⊗ rapidity evolution ⊗ Physical-DPD

For those who want to see more details:

$$
\begin{aligned}
&{}^{R}\tilde{F}^{\mathsf{sub}}_{a_1 a_2}(x_1, x_2, b_\perp, \mu, \tilde{\zeta}, P^z) \\
&= \int \frac{\mathrm{d}y_1}{y_1} \frac{\mathrm{d}y_2}{y_2} \, {}^{RR'}C_{a_1 a_2, a_1' a_2'} \left(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{(x_1 P^z)^2}{\mu^2}, \frac{(x_2 P^z)^2}{\mu^2}, \frac{\tilde{\zeta}}{\mu^2} \right) \\
&\qquad \times \exp\left[\frac{1}{2} \, {}^{R'}\gamma_\zeta(b_\perp, \mu) \log\left(\frac{\tilde{\zeta}}{\zeta} \right) \right] {}^{R'}F^{\mathsf{sub}}_{a_1' a_2'}(x_1, x_2, b_\perp, \mu, \zeta) \\
&\quad + \text{mixing with single PDFs} \\
&\quad + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{1}{x_i b_\perp P^z}, \frac{\Lambda^2_{\mathsf{QCD}}}{(x_i P^z)^2} \right)
\end{aligned}
$$

(2) Matching relation - consistency check

No proof yet . . . , but we do have a consistency check

$$
\begin{aligned}
&{}^{R}\tilde{F}^{\mathsf{sub}}_{a_1 a_2}(x_1, x_2, b_\perp, \mu, \tilde{\zeta}, P^z) \\
&= \int \frac{\mathrm{d}y_1}{y_1}\frac{\mathrm{d}y_2}{y_2}\, {}^{RR'}C_{a_1 a_2, a_1' a_2'}\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{(x_1 P^z)^2}{\mu^2}, \frac{(x_2 P^z)^2}{\mu^2}, \frac{\tilde{\zeta}}{\mu^2}\right) \\
&\qquad \times \exp\left[\frac{1}{2}\ {}^{R'}\gamma_\zeta(b_\perp, \mu)\log\left(\frac{\tilde{\zeta}}{\zeta}\right)\right] {}^{R'}F^{\mathsf{sub}}_{a_1' a_2'}(y_1, y_2, b_\perp, \mu, \zeta)
\end{aligned}
$$

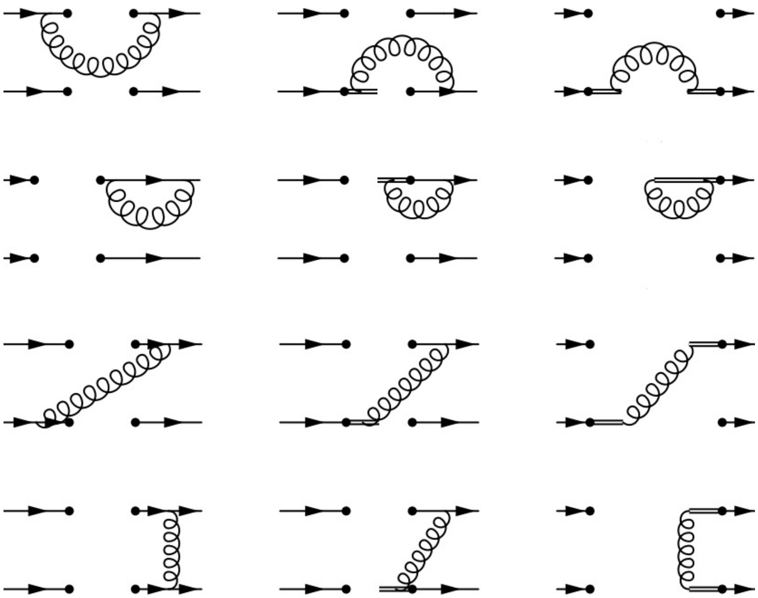
What is the logic here?

- Quasi and lightcone must share the same IR behaviour
- Transverse separation $b_\perp$ acts as an infrared scale

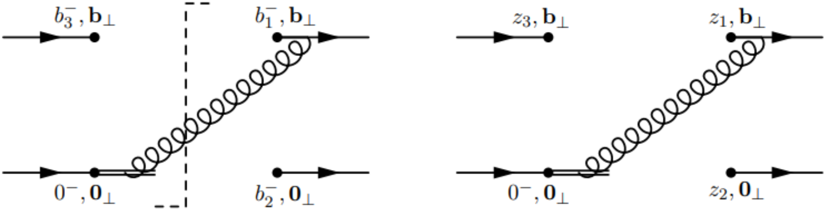
Order by order consistency check

Matching kernel should not be sensitive to infrared scales

(3) Calculating the kernel: All diagrams



(3) Calculating the kernel: Example



What do we find along the way?

$$\Delta^{\text{example}} = -4\pi\delta(1-x_1)\delta(1-x_2)\left[\frac{1}{\epsilon_{\text{ir}}} + \log\left(\frac{\mu^2\mathbf{b}_\perp^2}{b_0^2}\right)\right]\left[2 + \log\left(\frac{\delta^2}{p_1^+ p_2^+}\right)\right]$$
$$+ \text{ other terms}$$

Infrared poles and logs spotted

But our formula passes the test!

After combining all diagrams, soft factor subtraction and including the CS kernel, all IR poles and logs drop out of the matching kernel

Results

What did we find?

- No mixing of color and spin structures at one-loop
  - Generalisation to higher orders expected, but not proven

- Matching kernel of color-summed DPD greatly simplifies

$$ {}^{1}C_{a_1 a_2}\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{\mu}{|y_1|P^z}, \frac{\mu}{|y_2|P^z}\right) = \mathcal{C}_{a_1}\left(\frac{x_1}{y_1}, \frac{\mu}{|y_1|P^z}\right)\mathcal{C}_{a_2}\left(\frac{x_2}{y_2}, \frac{\mu}{|y_2|P^z}\right) $$

  - True to higher orders? Maybe OPE can tell

- Color-correlated kernel also simplifies

$$ {}^{8}C^{(1)}_{a_1 a_2} = \left(1 - \frac{N_c}{2C_F}\right){}^{1}C^{(1)}_{a_1 a_2} + \delta\big(1 - \tfrac{x_1}{y_1}\big)\delta\big(1 - \tfrac{x_2}{y_2}\big) $$

$$ \times N_c\left[2\log\left(\frac{\tilde{\zeta}}{\mu^2}\right) - \tfrac{1}{2}\log^2\left(\frac{(2y_1 P^z)^2}{\mu^2}\right) - \tfrac{1}{2}\log^2\left(\frac{(2y_2 P^z)^2}{\mu^2}\right) - \frac{5}{2} + \frac{\pi^2}{6}\right] $$

But most importantly . . .

One-loop Matching kernel passes consistency check

Perturbative nature of matching kernel consistent with one-loop result

How do we proceed from here?

- Proof of matching relation
  - Proof is available for PDF and TMD cases
  - Generalization to DPDs is desired
- Lattice calculable soft function
  - Difficulties in constructing lattice calculable soft function (two opposite light-like staples)
- Putting it on the lattice
  - Mixing and renormalization
- Extending the matching relation
  - Mixing with flavors and mixing with single PDFs
  - Extension to include interference DPDs

Conclusions

Conclusion

Achievement unlocked: formulating DPDs on the lattice

We are paving the way for lattice calculations of double parton distributions

- Lattice calculations of PDFs possible using quasi-PDF approach
- Conjectured a matching relation for the double parton case
  - quasi-DPD $=$ perturbative kernel $\otimes$ rapidity evolution $\otimes$ physical-DPD
- Passed the one-loop consistency check: no infrared logs
- Some interesting findings:
  - No mixing between color- and spin structures
  - Relation to single-PDF matching kernel
  - Relation between color-summed/correlated kernels

Thank you for your attention!

Achievement unlocked: formulating DPDs on the lattice

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Backup slides

Color structures

- Color-summed DPD ${}^1F_{qq}$:

$$
\begin{aligned}
{}^1F_{qq} = -\pi P^+ \int \frac{db_1^-}{2\pi}\frac{db_2^-}{2\pi}\frac{db_3^-}{2\pi}\; e^{-ix_1P^+b_1^-}e^{-ix_2P^+b_2^-}e^{ix_1P^+b_3^-} \\
\times \langle P|\, T^\dagger\Big[\bar\psi(0^+,b_1^-,\mathbf{b}_\perp)\gamma^+W[b_1\leftarrow b_3]\Big]_i\Big[\bar\psi(b_2^-)\gamma^+W[b_2\leftarrow 0]\Big]_j \\
\times T\Big[\psi(0^+,b_3^-,\mathbf{b}_\perp)\Big]_i\Big[\psi(0)\Big]_j\,|P\rangle
\end{aligned}
$$

- Color-correlated DPD ${}^8F_{qq}$:

$$
\begin{aligned}
{}^8\tilde F_{qq} = -\frac{T_F}{N}\,{}^1\tilde F_{a_1a_2} - \pi P^+ \int \frac{\mathrm{d}b_1^-}{2\pi}\frac{\mathrm{d}b_2^-}{2\pi}\frac{\mathrm{d}b_3^-}{2\pi}\; e^{-ix_1P^+b_1^-}e^{-ix_2P^+b_2^-}e^{ix_1P^+b_3^-} \\
\times \langle P|\,\Big[\bar\psi(b_1)\mathcal{W}_\sqsupset[b_1\leftarrow 0]\gamma^+\Big]_{i\alpha}\Big[\bar\psi(b_2)\mathcal{W}_\sqsupset[b_2\leftarrow b_3]\gamma^+\Big]_{j\beta} \\
\times \psi_{j\alpha}(b_3)\psi_{i\beta}(0)\,|P\rangle\,,
\end{aligned}
$$

Asymptotic freedom comes to the rescue

Asymptotic freedom: strength of coupling decreases at high energy

- $P^z \to \infty$ only affects large z-direction momentum modes
- $\Lambda \to \infty$ only affects large (all directions) momentum modes

Large momentum modes

Order of these limits only affect the large-momentum modes of the field theory

Asymptotic freedom

Difference in order of $P^z \to \infty$ and $\Lambda \to \infty$ limits can be studied in perturbation theory

Summary of quasi-PDF approach

- Lightcone-PDF is defined as hadronic lightcone correlator
  - Cannot be calculated on the lattice directly
- Quasi-PDF has same definition, but with space-like separated fields
  - Can be calculated on the lattice
- Boosting the quasi-PDF takes you closer to lightcone-PDF
- The two differ by an order of non-commuting limits
  - $P^z \to \infty$ and $\Lambda \to \infty$
- Asymptotic freedom of QCD guarantees that difference in order of limits is perturbative
- Quasi-PDFs can be matched perturbatively onto physical lightcone-PDFs

Power corrections to the matching relation

$$\tilde{f}(x, P^z, \mu) = \int \frac{dy}{y}\, \mathcal{C}\left(\frac{x}{y}, \frac{\mu}{yP^z}\right) f(y, \mu) + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{(xP^z)^2}\right)$$

Power corrections originate from OPE proof

- Difference in causal structure (light-like vs. space-like)
- Neglecting higher twist terms

Proof of matching relation

Beautiful application of the Operator Product Expansion (OPE)

$$\lim_{x \to y} \mathcal{O}_i(x)\mathcal{O}_j(y) \sim \sum_k C_{ijk}(x-y)\mathcal{O}_k(y)$$

Apply OPE to quasi-PDF

$$\tilde{f}(x, P^z) = \int \frac{dz}{4\pi} e^{-ixP^z z} \langle P| \tilde{\mathcal{O}}(z,\mu) |P\rangle$$

Outline of the proof

Expand $\tilde{\mathcal{O}}(z,\mu)$ in terms of leading twist operators $\mathcal{O}^{z\mu_1\cdots\mu_n}$

$\Downarrow$

Relate matrix elements of $\mathcal{O}^{z\mu_1\cdots\mu_n}$ to moments of lightcone-PDF

$\Downarrow$

Take the limits $P^z \gg M$ and $P^z \gg \Lambda_{\text{QCD}}$

$\Downarrow$

Write the quasi-PDF in terms of the lightcone-PDF

Mixing with single-PDFs