Magnetic scattering: pairwise little group and pairwise helicity

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Outline

- Introduction - the weird properties of the e-g system

- Multi-particle representations of the Poincare group: pairwise little group and pairwise helicity

- Pairwise spinor-helicity variable

- Constructing the magnetic S-matrix, 3-point

- 2→2 electric-magnetic scattering

- Scattering of GUT monopoles

- Dressed states and pairwise helicity, Dirac quantization from Berry phase

Introduction: the weird properties of the monopole-charge system

- J.J. Thompson (1904)

$$\vec{J}^{\,\text{field}} = \frac{1}{4\pi} \int d^3x \; \vec{x} \times \left(\vec{E} \times \vec{B} \right) = -eg\,\hat{r} \;\equiv -q\hat{r}$$

- Another derivation of Dirac quantization
- For dyons:

$$\vec{J}^{\,\text{field}} = \sum q_{ij}\, \hat{r}_{ij}$$

- Zwanziger-Schwinger quantization

$$q_{ij} = e_i\, g_j - e_j\, g_i = \frac{n}{2}$$

- Relativistic (Zwanziger): $M^{\nu\rho}_{\text{field};\pm} = \pm \sum_{i>j} q_{ij} \dfrac{\epsilon^{\nu\rho\alpha\beta}\, p_{i\alpha}\, p_{j\beta}}{\sqrt{(p_i \cdot p_j)^2 - m_i^2\, m_j^2}}$

Angular momentum

- Note the ± sign - origin is t/|t| in asymptotic expression. Non-rel. limit: $\vec{J}_{\pm}^{\,\mathrm{field}} = \pm \sum q_{ij}\, \hat{p}_{ij}$.
- Expression for in/out states differs by sign…
- Consequences far reaching:
    - Conserved angular momentum different from that of free theory
    - Asymptotic states do not factorize into one-particle states
    - No crossing symmetry for S-matrix

The NRQM lesson

- Hamiltonian: $H = -\frac{1}{2m}\left(\vec{\nabla} - ie\vec{A}\right)^2 + V(r) = -\frac{1}{2m}\vec{D}^2 + V(r)$

- Monopole background $A_\phi = \frac{\pm g}{r\sin\theta}\left(1 \mp \cos\theta\right)$

- Naive $\vec{L} = -i\vec{r} \times \vec{D}$ does NOT satisfy $[L_i, L_j] = i\epsilon_{ijk}L_k$

- Correct expression: $\vec{L} = -i\vec{r} \times \vec{D} - eg\hat{r} = m\vec{r} \times \dot{\vec{r}} - eg\hat{r}$

- Contribution from angular momentum in field shows up here as well

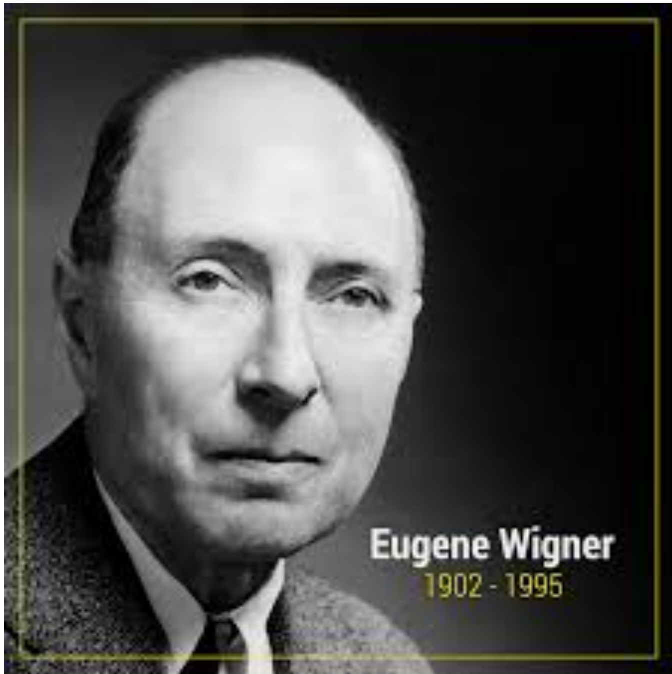
Multi-particle respresentations of the Poincare group

- Need to understand the effect of the extra angular momentum piece on the two-particle states

- Reminder: one particle states of Poincare

- Classification by Wigner (1939)

- For every $p^2 = m^2$ choose a reference momentum $k = (m, 0, 0, 0)$ or $k = (E, 0, 0, E)$ for massive vs massless particles



Multi-particle respresentations of the Poincare group

- Arbitrary momentum along $p^2 = m^2$ will be boost of reference momentum $p = L_p k.$

- States defined by momentum eigenstates

$$P^\mu |p;\sigma\rangle = p^\mu |p;\sigma\rangle$$

- Definition of generic state $|p\,;\,\sigma\rangle \equiv U(L_p)\,|k\,;\,\sigma\rangle$

- Momentum of $U(\Lambda)|p,\sigma\rangle$ is Λp due to defining relation

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, a_2 + \Lambda_2 a_1)$$

Multi-particle respresentations of the Poincare group

- By definition $W = L^{-1}_{\Lambda p}\Lambda L_p$ leave k unchanged

-these form the LITTLE GROUP (LG) of the particle

- Then $U^{-1}(L_{\Lambda p})U(\Lambda)U(L_p)$ must be just a representation of the LG on the reference states:

$$U(W)|k;\sigma\rangle = D_{\sigma\sigma'}(W)|k;\sigma'\rangle$$

- Where D is a representation of the LG - for massive particles SO(3) ~ SU(2) characterized by a spin s

- For massless particles strictly speaking it is E_2=ISO(2) 2d Euclidean group, but in practice just SO(2) ~ U(1) rotations around the z-axis

Multi-particle respresentations of the Poincare group

- General form of representation:

$$U(\Lambda)\,|p\,;\,\sigma\rangle = D_{\sigma'\sigma}(W)\,|\Lambda p\,;\,\sigma'\rangle$$

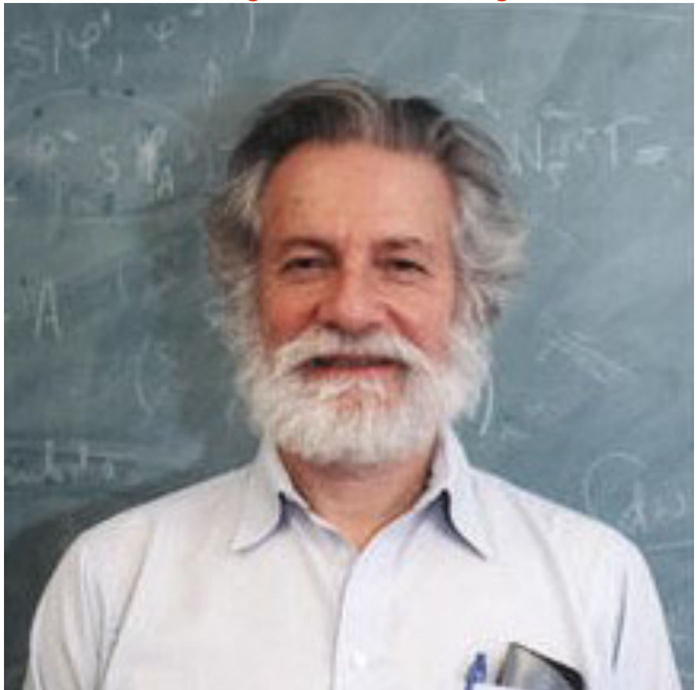
- Very intuitive: find frame with biggest symmetry, that symmetry is LG, and general case will be a combination of boosting into special frame, do the symmetry transformation in the special frame and then boost back.

- What happens for multi-particle states? Usual assumption they are just direct products of 1-particle states $|p_1, p_2, \ldots, p_n; \sigma_1, \sigma_2, \ldots, \sigma_n\rangle$

Multi-particle respresentations of the Poincare group

- However, Zwanziger in 1972 noticed: for 2 particles there is another ``special frame” - the center of momentum frame! In that frame momenta back-to-back

- There could be another symmetry transformation for A PAIR of particles



Daniel Zwanziger

Multi-particle respresentations of the Poincare group

- Repeat the Wigner story for 2 particles $|p_1, p_2\rangle$

- Choose as reference pair the COM frame

$$
\begin{aligned}
(k_1)_\mu &= (E_1^c, 0, 0, +p_c) \\
(k_2)_\mu &= (E_2^c, 0, 0, -p_c)
\end{aligned}
$$

$$
p_c = \sqrt{\frac{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}{s}} \quad , \quad E_{1,2}^c = \sqrt{m_{1,2}^2 + p_c^2}
$$

- Can get to arbitrary pair of momenta via boost from reference pair

$$
p_1 = L^{12}_{p_1 p_2} k_1 , \; p_2 = L^{12}_{p_1 p_2} k_2
$$

Multi-particle respresentations of the Poincare group

- In the COM frame there is a remaining symmetry - an SO(2) ~ U(1) rotation around the z axis

- This pairwise LG is independent from and in addition to the single particle LG's

- Clear for spinless particles (Zwanziger's derivation)

- Definition $$|\, p_1, p_2 \;;\; q_{12} \,\rangle \equiv U\left(L_p\right) |\, k_1, k_2 \;;\; q_{12} \,\rangle$$

- Like for single particles

$$U(\Lambda)\,|\, p_1, p_2 \;;\; q_{12} \,\rangle = U\left(L_{\Lambda p}\right)\, U\left(L_{\Lambda p}^{-1} \Lambda L_p\right) |\, k_1, k_2 \;;\; q_{12} \,\rangle$$
$$= U\left(L_{\Lambda p}\right)\, U\left(W_{k_1,k_2}\right) |\, k_1, k_2 \;;\; q_{12} \,\rangle$$

Multi-particle respresentations of the Poincare group

- Get the usual LG rotation but now from the pairwise LG

$$W_{k_1,k_2}(p_1,p_2,\Lambda) \equiv L^{-1}_{\Lambda p}\Lambda L_p = R_z\left[\phi(p_1,p_2,\Lambda)\right]$$

- Overall effect will be a phase ``pairwise helicity”

$$U(\Lambda)\,|\,p_1,p_2\;;\;q_{12}\,\rangle = e^{iq_{12}\phi(p_1,p_2,\Lambda)}\;|\,\Lambda p_1,\Lambda p_2\;;\;q_{12}\,\rangle$$

- What is q_{12}? Take spinless states in COM frame

$$J_z\;|\,k_1,k_2\;;\;q_{12}\,\rangle = q_{12}\,|\,k_1,k_2\;;\;q_{12}\,\rangle$$

- To reproduce effect of angular momentum from field

$$q_{ij} = e_i g_j - e_j g_i$$

Multi-particle respresentations of the Poincare group

- The pairwise little group is really SO(2) ~ U(1) and NOT E_2 - since the masses in general are not equal and $E \neq p_c$

- We get a true U(1) helicity-type phase even for massive particles

- Any higher little group (triple, quadruple etc) is trivial, so do not expect additional possible phases or symmetries

- Provides a new derivation of Zwanziger-Schwinger quantization $$e^{i4\pi q_{12}} = 1 \;\;\Rightarrow\;\; q_{12} \equiv e_1\, g_2 - e_2\, g_1 = \frac{n}{2},\; n \in \mathbb{Z}$$

Multi-particle respresentations of the Poincare group

- How about the general case with spin?

- Can construct representation by first considering

$$P_1 \times P_2 \times \tilde{P}_{12}$$

Three copies of the Poincare group, where the third copy is itself already a diagonal subgroup of $\tilde{P}_1 \times \tilde{P}_2$ acting on a pair of momenta $(\tilde{p}_1, \tilde{p}_2)$ for now distinct from p_1 and p_2

- The states we will be considering are

$$|p_1, p_2, (\tilde{p}_1, \tilde{p}_2)\,;\,\sigma\,\rangle \equiv |p_1\,;\,\sigma_1\rangle \otimes |p_2\,;\,\sigma_2\rangle \otimes |(\tilde{p}_1, \tilde{p}_2)\,;\,q_{12}\rangle$$

Multi-particle respresentations of the Poincare group

- Clearly we can now play the same game with each of those copies of the Poincare group as for single particle/spinless two particle states - define reference momenta and Lorentz boosts:

$$p_1 = L^1_{p_1} k_1, \quad p_2 = L^2_{p_2} k_2,$$

$$(\tilde{p}_1 \ , \ \tilde{p}_2) = \left(\tilde{L}^{12}_{\tilde{p}_1, \tilde{p}_2} \tilde{k}_1 \ , \ \tilde{L}^{12}_{\tilde{p}_1, \tilde{p}_2} \tilde{k}_2 \right)$$

- Definition of general state

$$|p_1, p_2, (\tilde{p}_1, \tilde{p}_2) ; \sigma\rangle \equiv \left(U(L^1_{p_1}) \ |k_1 ; \sigma_1\rangle \right) \otimes$$
$$\left(U(L^2_{p_2}) \ |k_2 ; \sigma_2\rangle \right) \otimes \left(U(\tilde{L}^{12}_{\tilde{p}_1, \tilde{p}_2}) \ |(\tilde{k}_1, \tilde{k}_2) ; q_{12}\rangle \right)$$

Multi-particle respresentations of the Poincare group

- Action of general Lorentz transformation

$$\Lambda \equiv \left(\Lambda_1, \Lambda_2, \tilde{\Lambda}_{12}\right) \in P_1 \times P_2 \times \tilde{P}_{12}$$

$$U(\Lambda)\,|p_1, p_2, (\tilde{p}_1, \tilde{p}_2), \sigma\rangle = \left(D_{\sigma_1'\sigma_1}(W_1)\,|\Lambda_1\, p_1\,;\, \sigma_1'\rangle\right) \otimes \left(D_{\sigma_2'\sigma_2}(W_2)\,|\Lambda_2\, p_2\,;\, \sigma_2'\rangle\right) \otimes$$
$$\left(U(\tilde{L}_{\tilde{\Lambda}_{12}\tilde{p}_1,\,\tilde{\Lambda}_{12}\tilde{p}_2})\,U(\tilde{W}_{12})\,|(\tilde{p}_1, \tilde{p}_2)\,;\, q_{12}\rangle\right)$$

- With the usual LG transformations

$$\begin{aligned} W_i &\equiv \left(L^i_{\Lambda_i p_i}\right)^{-1} \Lambda_i L^i_{p_i} \\ \tilde{W}_{12} &\equiv \tilde{L}^{-1}_{\tilde{\Lambda}_{12}\,\tilde{p}_1,\,\tilde{\Lambda}_{12}\tilde{p}_2}\,\tilde{\Lambda}_{12}\,\tilde{L}_{\tilde{p}_1,\,\tilde{p}_2} \end{aligned}$$

- Full transformation:

$$U(\Lambda)\,|p_1, p_2, (\tilde{p}_1, \tilde{p}_2)\,;\, \sigma\rangle = e^{iq_{12}\tilde{\phi}_{12}}\;\cdot$$

$$D_{\sigma_1'\sigma_1}(W_1)\,D_{\sigma_2'\sigma_2}(W_2)\,|\Lambda_1\, p_1, \Lambda_2\, p_2, (\tilde{\Lambda}_{12}\,\tilde{p}_1, \tilde{\Lambda}_{12}\,\tilde{p}_2)\,;\, \sigma\rangle$$

Multi-particle respresentations of the Poincare group

- This is clearly a proper unitary representation of $P_1 \times P_2 \times P_{12}$

- Now we can project onto physical states $p_1 = \tilde{p}_1, p_2 = \tilde{p}_2$ and $\Lambda_i = \tilde{\Lambda}_{12} \equiv \Lambda$ diagonal subgroup (physical LT's)

- Representation on physical states:

$$U(\Lambda)\,|p_1, p_2\,;\,\sigma_1, \sigma_2\,;\,q_{12}\rangle =$$

$$e^{iq_{12}\tilde{\phi}_{12}}\,D_{\sigma_1'\sigma_1}(W_1)\,D_{\sigma_2'\sigma_2}(W_2)\,|\Lambda p_1, \Lambda p_2\,;\,\sigma_1', \sigma_2'\,;\,q_{12}\rangle$$

- Clearly projection allowed since $p_1, p_2, (p_1, p_2) \to \Lambda p_1, \Lambda p_2, (\Lambda p_1, \Lambda p_2)$ stays within the physical momenta

Multi-particle respresentations of the Poincare group

- For $q_{12}=0$ reproduces usual direct-product 2-particle states

- For $j_1=j_2=0$ we get Zwanziger's states

- Easy to generalize to n particles - start with 2^n-1 Poincare groups $P_1 \times \ldots \times P_n \times P_{12} \times \ldots \times P_{n-1,n} \times$
$P_{123} \times \ldots \times P_{n-2,n-1,n} \times \ldots \times P_{123\ldots n}$

- However all k≥3 LG's are trivial - so general state

$$|p_1, \ldots, p_n\,;\,(\tilde{p}_1, \tilde{p}_2), \ldots, (\tilde{p}_{n-2}, \tilde{p}_n), (\tilde{p}_{n-1}, \tilde{p}_n)\,;\,\sigma\rangle$$

Multi-particle respresentations of the Poincare group

- After projection onto physical states get general representations

$$U(\Lambda)\,|p_1,\ldots,p_n\,;\,\sigma_1,\ldots,\sigma_n\,;\,q_{12},\ldots,q_{n-1,n}\rangle = \prod_{i>j} e^{iq_{ij}\phi_{ij}} \prod_i D_{\sigma_i\sigma_i'}(W_i)\,|\Lambda p_1,\ldots,\Lambda p_n,\,;\,\sigma_1',\ldots,\sigma_n'\,;\,q_{12},\ldots q_{n-1,n}\rangle$$

- A pairwise helicity for every pair of particles, in addition for each spin and mass.

- For charge/monopole system $q_{ij} = e_i g_j - e_j g_i$

- For $G \rightarrow U(1)^n$ will get n fundamental monopoles, and the pairwise helicity will be $q_{ij} = \vec{H}_i \cdot \vec{\alpha}_j - \vec{H}_j \cdot \vec{\alpha}_i$

  H Cartan generators, α simple roots

Asymptotic states

- One of the rare examples where free Hamiltonian H_0 has different conserved angular momentum from H

$$\left[H, \vec{J}\right] = \left[H_0, \vec{J}_0\right] = 0, \quad \vec{J} \neq \vec{J}_0$$

- Usually in/out states - eigenstates of H as t→±∞ approach free states. Here they don't

- In/out states will be represented differently

$$U(\Lambda)\,|p_1,\ldots,p_n\,;\pm\rangle = \prod_i \mathcal{D}(W_i)\,|\Lambda p_1,\ldots,\Lambda p_n\,;\pm\rangle\, e^{\pm i\,\Sigma}$$

- \+ is in state - is out state, and $\Sigma \equiv \sum_{i>j}^{n} q_{ij}\,\phi(p_i,p_j,\Lambda)$

- Origin of +- sign $\vec{J}_{\pm}^{\,\mathrm{field}} = \pm\sum q_{ij}\,\hat{p}_{ij}$

Transformation of the S-matrix

- Overlap of asymptotic states - LT:

$$S\left(p_1',\ldots,p_m'\,|\,p_1,\ldots,p_n\right) \;\equiv\; \langle p_1',\ldots,p_m';-\,|\,p_1,\ldots,p_n;+\rangle$$

$$= \left\langle p_1',\ldots,p_m';-\,|U(\Lambda)^\dagger\, U(\Lambda)|\,p_1,\ldots,p_n;+\right\rangle$$

$$= e^{i(\Sigma_+ + \Sigma_-)} \prod_{i=1}^{m} \mathcal{D}(W_i)^\dagger \prod_{j=1}^{n} \mathcal{D}(W_j),\; S\left(\Lambda\,p_1',\ldots,\Lambda\,p_m'\,|\,\Lambda\,p_1,\ldots,\Lambda\,p_n\right)$$

$$\Sigma_+ \equiv \sum_{i>j}^{n} q_{ij}\,\phi(p_i,p_j,\Lambda) \quad , \quad \Sigma_- \equiv \sum_{i>j}^{m} q_{ij}\,\phi(p_i',p_j',\Lambda)$$

- Transformation of S-matrix (crossing sym. violation)

$$S\left(\Lambda\,p_1',\ldots,\Lambda\,p_m'\,|\,\Lambda\,p_1,\ldots,\Lambda\,p_n\right) \;=$$

$$e^{-i\,(\Sigma_+ + \Sigma_-)} \prod_{i=1}^{m} \mathcal{D}(W_i) \prod_{j=1}^{n} \mathcal{D}(W_j)^\dagger \;\; S\left(p_1',\ldots,p_m'\,|\,p_1,\ldots,p_n\right)$$

- Need objects that saturate U(1) phase for S-matrix!

The standard spinor-helicity variables

- We use spinor-helicity variables $|p_i\rangle_\alpha \; [p_i|_{\dot\alpha}$ to construct scattering amplitudes/S-matrices

- Their transformation

$$\Lambda_\alpha{}^\beta \, |p_i\rangle_\beta \;=\; e^{+\frac{i}{2}\phi(p_i,\Lambda)} \; |\Lambda p_i\rangle_\beta \,, \qquad [p_i|_{\dot\beta} \, \tilde\Lambda^{\dot\beta}{}_{\dot\alpha} \;=\; e^{-\frac{i}{2}\phi(p_i,\Lambda)} \; [\Lambda p_i|_{\dot\alpha}$$

- Under U(1) massless LG. Abbreviation $|i\rangle_\alpha \equiv |p_i\rangle_\alpha$ $[i|_{\dot\alpha} \equiv [p_i|_{\dot\alpha}$

- For massive particles use $|\mathbf{i}\rangle^I{}_\alpha$ I is SU(2) LG index

	$U(1)_i$	$SU(2)_i$	$U(1)_{ij}$
Required weight	h_i	$\mathbf{S}_i$	$-q_{ij}$
$\|i\rangle_\alpha \,,\ [i\|_{\dot\alpha}$	$-\frac{1}{2} \,,\ \frac{1}{2}$	—	—
$\langle\mathbf{i}\|^{I;\alpha}$	—	$\square$	—

Pairwise momenta

- We need the analog of the spinor-helicity to saturate the pairwise helicity

- Since it is a true U(1) transformation - expect massless momentum made out of pair of momenta

- Pairwise reference null momenta (``flat momenta") in COM frame

$$\left(k_{ij}^{\flat\pm}\right)_\mu = p_c\,(1,0,0,\pm 1)$$

- In any other frame can boost it

$$p_{ij}^{\flat+} = \frac{1}{E_i^c + E_j^c}\left[\left(E_j^c + p_c\right)p_i - \left(E_i^c - p_c\right)p_j\right]$$

$$p_{ij}^{\flat-} = \frac{1}{E_i^c + E_j^c}\left[\left(E_i^c + p_c\right)p_j - \left(E_j^c - p_c\right)p_i\right]$$

Pairwise momentum

- Properties: $k_{ij}^{\flat\pm} \cdot k_{ij}^{\flat\pm} = 0$ and $k_{ij}^{\flat +} \cdot k_{ij}^{\flat -} = 2p_c^2$
- Limits: $m_i \rightarrow 0$ $\quad p_{ij}^{\flat +} \rightarrow p_i$ and $p_{ij}^{\flat -}$ parity conjug.

Pairwise spinor-helicity variable

- To find spinor-helicity variable that has the right U(1) pairwise LG phase just consider the spinor-helicity variable corresponding to the pairwise momenta.

- Note: since linear combination $L_p \, k_{ij}^{\flat\pm} \;=\; p_{ij}^{\flat\pm}$

- Reference pairwise spinor-helicity

$$\left|k_{ij}^{\flat +}\right\rangle_\alpha = \sqrt{2\,p_c}\begin{pmatrix}1\\0\end{pmatrix} \quad , \quad \left|k_{ij}^{\flat -}\right\rangle_\alpha = \sqrt{2\,p_c}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\left[k_{ij}^{\flat +}\right|_{\dot\alpha} = \sqrt{2\,p_c}\,(1\quad 0) \quad , \quad \left[k_{ij}^{\flat -}\right|_{\dot\alpha} = \sqrt{2\,p_c}\,(0\quad 1)$$

- Square root of momentum $k_{ij}^{\flat\pm}\cdot\sigma_{\alpha\dot\alpha} \;=\; \left|k_{ij}^{\flat\pm}\right\rangle_\alpha \left[k_{ij}^{\flat\pm}\right|_{\dot\alpha}$

Pairwise spinor-helicity variable

- Definition of general pairwise spinor-helicity variables $\left|p_{ij}^{\flat\pm}\right\rangle_\alpha = (\mathcal{L}_p)_\alpha^{\ \beta} \left|k_{ij}^{\flat\pm}\right\rangle_\beta \quad , \quad \left[p_{ij}^{\flat\pm}\right|_{\dot\alpha} = \left[k_{ij}^{\flat\pm}\right|_{\dot\beta} \left(\tilde{\mathcal{L}}_p\right)^{\dot\beta}_{\ \dot\alpha}$

- By construction can easily go through another round of ``Wigner trick" to show

$$\Lambda_\alpha^{\ \beta} \left|p_{ij}^{\flat\pm}\right\rangle_\beta = e^{\pm\frac{i}{2}\phi(p_i,p_j,\Lambda)} \left|\Lambda p_{ij}^{\flat\pm}\right\rangle_\alpha \quad , \quad \left[p_{ij}^{\flat\pm}\right|_{\dot\beta} \tilde{\Lambda}^{\dot\beta}_{\ \dot\alpha} = e^{\mp\frac{i}{2}\phi(p_i,p_j,\Lambda)} \left[\Lambda p_{ij}^{\flat\pm}\right|_{\dot\alpha}$$

- Pairwise spinors have right covariant transformation under pairwise LG

- Note $\left|p_{ij}^{\flat+}\right\rangle_\alpha$ and $\left|p_{ij}^{\flat-}\right\rangle_\beta$ have opposite pairwise helicities

Limits of pairwise spinor-helicity

- In the $m_i \to 0$ limit

$$\left| p_{ij}^{\flat +} \right\rangle_\alpha = |\, i \,\rangle_\alpha \quad , \quad \left[p_{ij}^{\flat +} \right|_{\dot\alpha} = [\, i \,|_{\dot\alpha}$$
$$\left| p_{ij}^{\flat -} \right\rangle_\alpha = \sqrt{2p_c}\, |\hat\eta_i\rangle_\alpha \quad , \quad \left[p_{ij}^{\flat -} \right|_{\dot\alpha} = \sqrt{2p_c}\, [\hat\eta_i|_{\dot\alpha}$$

- $|i\rangle_\alpha$, $[i|_{\dot\alpha}$ standard spinor-helicities, $|\hat\eta_i\rangle_\alpha$, $[\hat\eta_i|_{\dot\alpha}$ the parity conjugates

- These will imply selection rules in the $m_i \to 0$ since the following contractions vanish

$$\left[p_{ij}^{\flat +}\, i \right] = \left\langle i\, p_{ij}^{\flat +} \right\rangle = \left[\hat\eta_i\, p_{ij}^{\flat -} \right] = \left\langle p_{ij}^{\flat -}\, \hat\eta_i \right\rangle = 0$$
$$\left[p_{ij}^{\flat -}\, i \right] = \left\langle i\, p_{ij}^{\flat -} \right\rangle = \left[\hat\eta_i\, p_{ij}^{\flat +} \right] = \left\langle p_{ij}^{\flat +}\, \hat\eta_i \right\rangle = 2p_c$$

Constructing the magnetic S-matrix

- We have seen: general transformation of S-matrix

$$S\left(\Lambda p_1', \ldots, \Lambda p_m' \,|\, \Lambda p_1, \ldots, \Lambda p_n\right) \;=\; e^{-i\,(\Sigma_- + \Sigma_+)} \prod_{i=1}^{m} \mathcal{D}(W_i) \prod_{j=1}^{n} \mathcal{D}(W_j)^\dagger \;\; S\left(p_1', \ldots, p_m' \,|\, p_1, \ldots, p_n\right)$$

- Implies weird twist - forward scattering not allowed - does not have right PLG property. So usual construction in terms of scattering amplitude

$$S_{\alpha\beta} \;=\; \delta(\alpha - \beta) \;-\; 2i\pi\delta^{(4)}(p_\alpha - p_\beta)\,\mathcal{A}_{\alpha\beta}$$

does not make sense. Rather than trying to adjust this formula will just directly construct S-matrix elements always

The out-out formalism

- So far we have made distinction of in and out states - very reasonable for magnetic scattering since we have no crossing symmetry

- However all of scattering amplitude literature assumes all particles outgoing… Would like to not have to rewrite all of those to compare to our new results… So force ourselves to use out-out

- While no crossing symmetry, can still do a crossing transformation and transform an in state to an out state via

$$
\begin{aligned}
\text{particle} &\leftrightarrow \text{antiparticle} \\
\text{incoming} &\leftrightarrow \text{outgoing} \\
\text{helicity } h &\leftrightarrow -h \\
p^\mu &\leftrightarrow -p^\mu
\end{aligned}
$$

The out-out formalism

• This does NOT assume/imply crossing symmetry. We will always stay in the kinematic regime where some of the particles actually have negative energies, implying those were really incoming particles.

• Note q_{ij} does not flip sign - it is quadratic in momenta.

• Note q_{ij} still only calculated for states that would be both in states or both out states (ie. now according to the sign of the energies)

Constructing the S-matrix

- The full set of rules:

	$U(1)_i$	$SU(2)_i$	$U(1)_{ij}$
Required weight	h_i	$\mathbf{S}_i$	$-q_{ij}$
$\vert i\rangle_\alpha\,,\ [i\vert_{\dot\alpha}$	$-\frac{1}{2}\,,\ \frac{1}{2}$	—	—
$\langle\mathbf{i}\vert^{I;\alpha}$	—	$\square$	—
$\left\vert p_{ij}^{\flat+}\right\rangle_\alpha\,,\ \left[p_{ij}^{\flat+}\right\vert_{\dot\alpha}$	—	—	$-\frac{1}{2}\,,\ \frac{1}{2}$
$\left\vert p_{ij}^{\flat-}\right\rangle_\alpha\,,\ \left[p_{ij}^{\flat-}\right\vert_{\dot\alpha}$	—	—	$\frac{1}{2}\,,\ -\frac{1}{2}$

- To satisfy the scaling of the S-matrix

$$S\left(\omega^{-1}|i\rangle, \omega|i]\right) = \omega^{2h_i} S\left(|i\rangle, |i]\right), \quad \text{for } \forall i$$

$$S\left(\omega^{-1}|p_{ij}^{\flat+}\rangle, \omega|p_{ij}^{\flat+}], \omega|p_{ij}^{\flat-}\rangle, \omega^{-1}|p_{ij}^{\flat-}]\right) = \omega^{-2q_{ij}} S\left(|p_{ij}^{\flat+}\rangle, |p_{ij}^{\flat+}], |p_{ij}^{\flat-}\rangle, |p_{ij}^{\flat-}]\right) \text{for } \forall \text{ pair } \{i,j\}$$

- Will allow us to fix all angular dependence of magnetic scattering. Everything non-perturbative

Simple example 1. Massive fermion decaying to massive fermion + massless scalar, q=-1

- $S\left(\mathbf{1}^{s=1/2}\,|\,\mathbf{2}^{s=1/2},3^{0}\right)_{q_{23}=-1} \sim \left\langle p_{23}^{\flat-}\,\mathbf{1}\right\rangle\left\langle p_{23}^{\flat-}\,\mathbf{2}\right\rangle$

- Other allowed combinations $\left[p_{23}^{\flat+}\mathbf{1}\right]\left[p_{23}^{\flat+}\mathbf{2}\right]$, $\left[p_{23}^{\flat+}\mathbf{1}\right]\left\langle p_{23}^{\flat-}\mathbf{2}\right\rangle$ and $\left\langle p_{23}^{\flat-}\mathbf{1}\right\rangle\left[p_{23}^{\flat+}\mathbf{2}\right]$ equivalent by Dirac equation

$$p_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}I} = m\lambda_{\alpha}^{I}$$

	$U(1)_i$	$SU(2)_i$	$U(1)_{ij}$
Required weight	h_i	$\mathbf{S}_i$	$-q_{ij}$
$\|i\rangle_\alpha$, $[i\|_{\dot{\alpha}}$	$-\frac{1}{2}$, $\frac{1}{2}$	$-$	$-$
$\langle\mathbf{i}\|^{I;\alpha}$	$-$	$\square$	$-$
$\left\|p_{ij}^{\flat+}\right\rangle_\alpha$, $\left[p_{ij}^{\flat+}\right\|_{\dot{\alpha}}$	$-$	$-$	$-\frac{1}{2}$, $\frac{1}{2}$
$\left\|p_{ij}^{\flat-}\right\rangle_\alpha$, $\left[p_{ij}^{\flat-}\right\|_{\dot{\alpha}}$	$-$	$-$	$\frac{1}{2}$, $-\frac{1}{2}$

Simple example 2. Massive scalar decaying to massive scalar + massless vector, q=-1

- $$S\left(\mathbf{1}^{s=0}\,|\,\mathbf{2}^{s=0},3^{+1}\right)_{q_{23}=-1}\sim\left[p_{23}^{\flat+}\,3\right]^2\sim\left\langle p_{23}^{\flat-}|2|3\right]^2$$

- No way to write $S\left(\mathbf{1}^{s=0}\,|\,\mathbf{2}^{s=0},3^{-1}\right)_{q_{23}=-1}$ - case of more general selection rule

	$U(1)_i$	$SU(2)_i$	$U(1)_{ij}$
Required weight	h_i	$\mathbf{S}_i$	$-q_{ij}$
$\|i\rangle_\alpha\,,\ [i\|_{\dot\alpha}$	$-\frac{1}{2}\,,\ \frac{1}{2}$	–	–
$\langle\mathbf{i}\|^{I;\alpha}$	–	$\square$	–
$\left\|p_{ij}^{\flat+}\right\rangle_\alpha,\ \left[p_{ij}^{\flat+}\right\|_{\dot\alpha}$	–	–	$-\frac{1}{2}\,,\ \frac{1}{2}$
$\left\|p_{ij}^{\flat-}\right\rangle_\alpha,\ \left[p_{ij}^{\flat-}\right\|_{\dot\alpha}$	–	–	$\frac{1}{2}\,,\ -\frac{1}{2}$

Simple example 3. Massive vector decaying to two massless fermions, q=-2

- $$S\left(\mathbf{1}^{s=1}\,|\,2^{-1/2},3^{+1/2}\right)_{q_{23}=-2} \sim \left\langle 2p_{23}^{\flat -}\right\rangle \left[p_{23}^{\flat +}\,3\right] \left\langle \mathbf{1}\,p_{23}^{\flat -}\right\rangle^2$$

- Opposite helicity vanishes since $\left\langle p_{23}^{\flat -}3\right\rangle = \left[p_{23}^{\flat +}2\right] = 0$
  Another implication of the selection rules

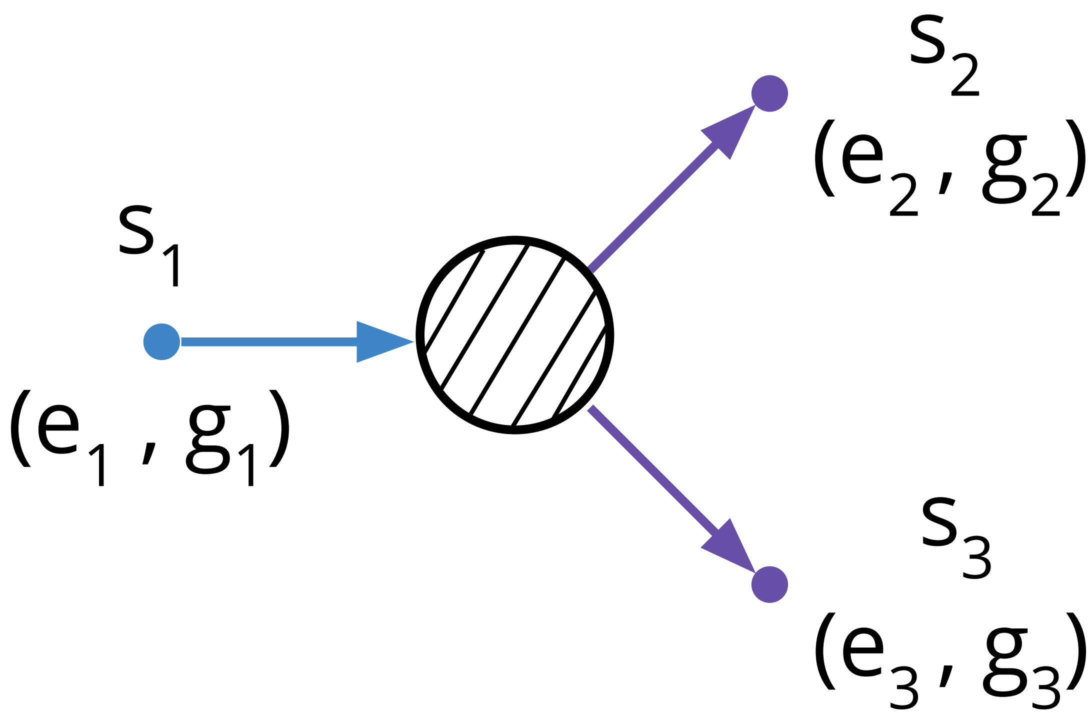
	$U(1)_i$	$SU(2)_i$	$U(1)_{ij}$
Required weight	h_i	$\mathbf{S}_i$	$-q_{ij}$
$\lvert i\rangle_\alpha\,,\ [i\rvert_{\dot\alpha}$	$-\frac{1}{2}\,,\ \frac{1}{2}$	–	–
$\langle\mathbf{i}\rvert^{I;\alpha}$	–	$\square$	–
$\left\lvert p_{ij}^{\flat +}\right\rangle_\alpha\,,\ \left[p_{ij}^{\flat +}\right\rvert_{\dot\alpha}$	–	–	$-\frac{1}{2}\,,\ \frac{1}{2}$
$\left\lvert p_{ij}^{\flat -}\right\rangle_\alpha\,,\ \left[p_{ij}^{\flat -}\right\rvert_{\dot\alpha}$	–	–	$\frac{1}{2}\,,\ -\frac{1}{2}$

Simple example 4. Massive vector decaying to two massless fermions, q=-1

- $$S\left(\mathbf{1}^{s=1}\,|\,2^{-1/2},3^{-1/2}\right)_{q_{23}=-1} \sim \left\langle 2 p_{23}^{\flat -}\right\rangle \left\langle p_{23}^{\flat +}\, 3\right\rangle \left\langle \mathbf{1}\, p_{23}^{\flat -}\right\rangle^2$$
- $h_2 = -h_3 = 1/2$ vanishes since $\left[p_{23}^{\flat -} 3\right] = 0$
- Note in this example number of pairwise spinors is NOT $2q_{23}$ since we needed 4 spinors for the particles

	$U(1)_i$	$SU(2)_i$	$U(1)_{ij}$
Required weight	h_i	$\mathbf{S}_i$	-q_{ij}
$\vert i\rangle_\alpha\,,\ [i\vert_{\dot\alpha}$	-$\frac{1}{2}$, $\frac{1}{2}$	–	–
$\langle \mathbf{i}\vert^{I;\alpha}$	–	$\square$	–
$\left\vert p_{ij}^{\flat +}\right\rangle_\alpha\,,\ \left[p_{ij}^{\flat +}\right\vert_{\dot\alpha}$	–	–	-$\frac{1}{2}$, $\frac{1}{2}$
$\left\vert p_{ij}^{\flat -}\right\rangle_\alpha\,,\ \left[p_{ij}^{\flat -}\right\vert_{\dot\alpha}$	–	–	$\frac{1}{2}$, -$\frac{1}{2}$

The general 3-point S-matrix
1. Incoming massive to two outgoing massive



$$q_{23} \equiv e_2\, g_3 - e_3\, g_2$$

The general 3-point S-matrix

1. Incoming massive to two outgoing massive

- For the massive part need:

$$\left(\langle\mathbf{1}|^{2s_1}\right)^{\{\alpha_1\ldots\alpha_{2s_1}\}}\left(\langle\mathbf{2}|^{2s_2}\right)^{\{\beta_1\ldots\beta_{2s_2}\}}\left(\langle\mathbf{3}|^{2s_3}\right)^{\{\gamma_1\ldots\gamma_{2s_3}\}}$$

- In total have $\hat{s} = s_1 + s_2 + s_3$ spinors - need same number of pairwise spinors $|w\rangle_\alpha \equiv \left|p_{23}^{\flat -}\right\rangle_\alpha$ and $|r\rangle_\alpha \equiv \left|p_{23}^{\flat +}\right\rangle_\alpha$

- Pairwise helicity needs to add up to q_{23} so use

$$S^q_{\{\alpha_1,\ldots,\alpha_{2s_1}\}\{\beta_1,\ldots,\beta_{2s_2}\}\{\gamma_1,\ldots,\gamma_{2s_3}\}} = \sum_{i=1}^{C} a_i \left(|w\rangle^{\hat{s}-q}\,|r\rangle^{\hat{s}+q}\right)_{\{\alpha_1,\ldots,\alpha_{2s_1}\}\{\beta_1,\ldots,\beta_{2s_2}\}\{\gamma_1,\ldots,\gamma_{2s_3}\}}$$

- Selection rule: $|q| \leq \hat{s}$

- For q=0 recover usual amplitudes expressions

The general 3-point S-matrix 2. Incoming massive, outgoing massive + massless, unequal mass

- Massive part: $\left(\langle\mathbf{1}|^{2s_1}\right)^{\{\alpha_1\ldots\alpha_{2s_1}\}}\left(\langle\mathbf{2}|^{2s_2}\right)^{\{\beta_1\ldots\beta_{2s_2}\}}$

- Massless part regular spinors $(|u\rangle_\alpha\,,|v\rangle_\alpha) = (|3\rangle_\alpha\,,|\,2\,|3]_\alpha)$

  pairwise spinors $(|w\rangle_\alpha\,,|r\rangle_\alpha) = \left(\left|p_{23}^{\flat-}\right\rangle_\alpha,\left|p_{23}^{\flat+}\right\rangle_\alpha\right)$

- Most general massless part:

$$S^{h,q,\text{ unequal}}_{\{\alpha_1,\ldots,\alpha_{2s_1}\}\{\beta_1,\ldots,\beta_{2s_2}\}} = \sum_{i=1}^{C}\sum_{j,k} a_{ijk}\,\langle ur\rangle^{\max(j+k,0)}\,\langle vw\rangle^{\max(-j-k,0)}\left(|u\rangle^{\frac{\hat{s}}{2}-h-j}\,|v\rangle^{\frac{\hat{s}}{2}+h+k}\,|w\rangle^{\frac{\hat{s}}{2}-q+j}\,|r\rangle^{\frac{\hat{s}}{2}+q-k}\right)_{\{\alpha_1,\ldots,\alpha_{2s_1}\}\{\beta_1,\ldots,\beta_{2s_2}\}}$$

- In sums $-\frac{\hat{s}}{2}+q\leq j\leq\frac{\hat{s}}{2}-h$ and $-\frac{\hat{s}}{2}-h\leq k\leq\frac{\hat{s}}{2}+q$

- Selection rule: $|h+q|\leq\hat{s}$

  eg. $s_1 = s_2 = 0 \;\rightarrow\; h = -q$

The general 3-point S-matrix
3. Incoming massive, outgoing massive + massless, equal mass

- Subtlety: in this case $|u\rangle \propto |v\rangle$ and $|w\rangle \propto |r\rangle$

- Method of Nima et al. ``x-factor”

$$m\, x\, |u\rangle \;=\; |v\rangle \qquad\qquad \langle ur\rangle^2\, x\, |w\rangle \;\sim\; |r\rangle$$

$$S^{h,q,\mathrm{equl}}_{\{\alpha_1\ldots\alpha_{2s_1}\}\{\beta_1\ldots\beta_{2s_2}\}} = \sum_{i=1}^{C}\sum_{j}\sum_{k=-j}^{j} x^{h+q+j}\,\langle ur\rangle^{\max[2q+j-k,0]}\,\langle vw\rangle^{\max[-2q-j+k,0]} \left(|u\rangle^{j+k}\,|w\rangle^{j-k}\,\epsilon^{\hat{s}-j}\right)_{\{\alpha_1\ldots\alpha_{2s_1}\}\{\beta_1\ldots\beta_{2s_2}\}},$$

- Power of x can be negative - no selection rule

The general 3-point S-matrix
4. Incoming massive, two outgoing massless

- Massive part: $\left(\langle\mathbf{1}|^{2s}\right)^{\{\alpha_1\ldots\alpha_{2s}\}}$

- Massless part from regular spinors $|u\rangle_\alpha = |2\rangle_\alpha\,,\ |v\rangle_\alpha = |3\rangle_\alpha$ and pairwise spinors $|w\rangle_\alpha = \left|p_{23}^{\flat -}\right\rangle_\alpha$ and $|r\rangle_\alpha = \left|p_{23}^{\flat +}\right\rangle_\alpha$

- General expression:

$$S^q_{\{\alpha_1,\ldots,\alpha_{2s}\}} = \sum_{ij} a_{ij} \left(|u\rangle^{s/2-i-\Delta} \ |v\rangle^{s/2-j+\Delta} \ |w\rangle^{s/2+j-q} \ |r\rangle^{s/2+i+q}\right)_{\{\alpha_1,\ldots,\alpha_{2s}\}} \cdot$$

$$[uv]^{\max[\Sigma+(s-i-j)/2\,,0]} \ \langle uv\rangle^{\max[-\Sigma-(s+i+j)/2\,,0]} \ (\langle uw\rangle\,[vr])^{\frac{1}{2}\max[i-j\,,0]} \ ([uw]\,\langle vr\rangle)^{\frac{1}{2}\max[j-i\,,0]}$$

- With $\Sigma = h_2 + h_3,\ \Delta = h_2 - h_3$

$$-s/2 - q \le i \le s/2 - \Delta \ \text{ and } \ -s/2 + q \le j \le s/2 + \Delta$$

- Selection rule: $|\Delta - q| \le s$.

The general 3-point S-matrix
4. Incoming massive, two outgoing massless

- Agrees with usual selection rule for q=0

$$s = 0 \quad \to \quad h_2 = h_3 = 0$$

$$s = 1 \quad \to \quad |h_2 - h_3| \leq 1 \quad \to \quad |h_2| = |h_3| \leq 1/2$$ massless h >1/2 can't couple to current

$$s = 2 \quad \to \quad |h_2 - h_3| \leq 2 \quad \to \quad |h_2| = |h_3| \leq 1$$ massless h>1 can't couple to stress tensor

- For magnetic case even more restrictive q=±1/2

$$s = 0 \quad \to \quad \text{forbidden}$$

$$s = 1 \quad \to \quad |h_2 - h_3 \mp 1/2| \leq 1 \quad \to \quad |h_2| = |h_3| = 0 \quad \text{or} \quad h_2 = -h_3 = \pm 1/2$$

$$s = 2 \quad \to \quad |h_2 - h_3 \mp 1/2| \leq 2 \quad \to \quad |h_2| = |h_3| \leq 1/2 \quad \text{or} \quad h_2 = -h_3 = \pm 1\,.$$

- More restrictive because $h_2 = -h_3 = -qs$ option not allowed

General 2→2 scattering

- Just kinematics can not fully fix the S-matrix - some dynamical input will be needed
- However we can always do partial wave decomp. as in NRQM - fully Lorentz and LG invariant way
- Will see kinematics fixes everything up to phase shifts like in QM
- Lowest partial wave will be completely fixed → famous helicity flip of Kazama, Yang, Goldhaber
- Higher partial waves monopole spherical harmonics appear naturally as expected from Wu & Yang

Partial wave expansion for magnetic case

- Expansion in the eigenbasis of Casimir operator

$$W^\mu \equiv \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\,P^\nu\,M^{\rho\sigma}$$

- Pauli-Lubanski operator, eigenvalues of W^2 are $-P^2\,J\,(J+1)$ J is total angular momentum

- Representation in spinor-helicity space (Witten):

$$(\sigma_\mu)_{\alpha\dot\alpha}\,P^\mu \equiv P_{\alpha\dot\alpha} = \sum_i |i\rangle_\alpha\,[i|_{\dot\alpha}$$

$$(\sigma_{\mu\nu})_{\alpha\beta}\,M^{\mu\nu} \equiv M_{\alpha\beta} = i\sum_i |i\rangle_{\{\alpha}\,\frac{\partial}{\partial\,\langle i|^{\beta\}}}$$

$$(\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta}\,M^{\mu\nu} \equiv \tilde M_{\dot\alpha\dot\beta} = i\sum_i [i|_{\{\dot\alpha}\,\frac{\partial}{\partial\,|i]^{\dot\beta\}}}$$

Partial wave expansion for magnetic case

- Expression of Casimir (Shu et al. 2020):

$$W^2 = \frac{P^2}{8}\left[\mathrm{Tr}\left(M^2\right) + \mathrm{Tr}\left(\tilde{M}^2\right)\right] - \frac{1}{4}\mathrm{Tr}\left(M\,P\,\tilde{M}\,P^{\mathrm{T}}\right)$$

- Generalization to magnetic case:

$$(\sigma_{\mu\nu})_{\alpha\beta}\,M^{\mu\nu} \equiv M_{\alpha\beta} = i\left[\sum_i |i\rangle_{\{\alpha}\frac{\partial}{\partial\,\langle i|^{\beta\}}} + \sum_{i>j,\pm}\left|p_{ij}^{\flat\pm}\right\rangle_{\{\alpha}\frac{\partial}{\partial\left\langle p_{ij}^{\flat\pm}\right|^{\beta\}}}\right]$$

$$(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}}\,M^{\mu\nu} \equiv \tilde{M}_{\dot{\alpha}\dot{\beta}} = i\left[\sum_i [i|_{\{\dot{\alpha}}\frac{\partial}{\partial\,|i]^{\dot{\beta}\}}} + \sum_{i>j,\pm}\left[p_{ij}^{\flat\pm}\right|_{\{\dot{\alpha}}\frac{\partial}{\partial\left|p_{ij}^{\flat\pm}\right]^{\dot{\beta}\}}}\right]$$

- Can show $W^2\,\langle 12\rangle = W^2\left\langle p_{12}^{\flat\pm}2\right\rangle = W^2\left\langle p_{12}^{\flat\pm}1\right\rangle = W^2\left\langle p_{12}^{\flat\pm}p_{12}^{\flat\mp}\right\rangle = 0$

$$W^2\left|1\right\rangle_{\{\alpha}\left|p_{12}^{\flat-}\right\rangle_{\beta\}} = -\,s\,1(1+1)\left|1\right\rangle_{\{\alpha}\left|p_{12}^{\flat-}\right\rangle_{\beta\}}$$

Partial wave expansion for magnetic case

- Eigenfunctions of W^2 symmetrized products of ordinary and pairwise spinors

$$W^2 \left(f\, \Pi \,|s_k\rangle\right)_{\{\alpha_1 \ldots \alpha_J\}} = -sJ(J+1) \left(f\, \Pi \,|s_k\rangle\right)_{\{\alpha_1 \ldots \alpha_J\}}$$

- Partial wave decomposition:

$$S_{12\to 34} \; = \; \mathcal{N} \sum_J (2J+1)\, \mathcal{M}^J(p_c)\, \mathcal{B}^J$$

- The $\mathcal{B}^J$ are basis amplitudes

$$W^2 \; \mathcal{B}^J \; = \; -\, s\, J\, (J+1)\; \mathcal{B}^J$$

- $\mathcal{B}^J$ contain all angular dependence

Partial wave expansion for magnetic case

- $\mathcal{M}^J(p_c)$ are reduced matrix elements - contain information on dynamics $W_{12}^2\,\mathcal{M}^J(p_c) \;=\; W_{34}^2\,\mathcal{M}^J(p_c) \;=\; 0$

- $\mathcal{N} \equiv \sqrt{8\pi s}$ normalization factor

- Shu et al. '20: $\mathcal{B}^J \;=\; C^{J;\,\mathrm{in}}_{\{\alpha_1,\ldots,\alpha_{2j}\}}\, C^{J;\,\mathrm{out};\;\{\alpha_1,\ldots,\alpha_{2j}\}}$

$$W_{12}^2\; C^{J;\,\mathrm{in}}_{\{\alpha_1,\ldots,\alpha_{2J}\}} \;=\; -\,s\,J\,(J+1)\,C^{J;\,\mathrm{in}}_{\{\alpha_1,\ldots,\alpha_{2J}\}}$$

$$W_{34}^2\; C^{J;\,\mathrm{out};\;\{\alpha_1,\ldots,\alpha_{2J}\}} \;=\; -\,s\,J\,(J+1)\,C^{J;\,\mathrm{out};\;\{\alpha_1,\ldots,\alpha_{2J}\}}$$

- The $C^{J;\,\mathrm{in/out}}$ are generalized Clebsch-Gordan tensors, completely fixed by group theory.

Partial wave expansion for magnetic case

- The $C^{J;\,\mathrm{in/out}}$ depend on the spinors of the in/out states, saturate the LG and pairwise LG quantum numbers of the S-matrix

- They can be read off from the 1+2→J and J→3+4 S-matrix constructions by peeling off the spinors corresponding to intermediate J state

- Example: scalar charge+monopole → J, q=-1

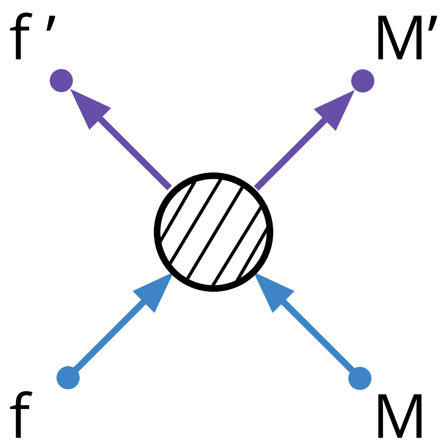
$$S\left(1^0,2^0\,|\,\mathbf{3}^J\right)_{q_{12}=-1} \;=\; a\left\langle \mathbf{3}\,p_{12}^{\flat -}\right\rangle^{J+1}\left\langle \mathbf{3}\,p_{12}^{\flat +}\right\rangle^{J-1}$$

- Only one contraction in this case:

$$\left(C^{J;\,\mathrm{in}}_{0,0,-1}\right)_{\{\alpha_1,\ldots,\alpha_{2J}\}} \;=\; \left(\left|p_{12}^{\flat -}\right\rangle^{J+1}\left|p_{12}^{\flat +}\right\rangle^{J-1}\right)_{\{\alpha_1,\ldots,\alpha_{2J}\}}$$

Fermion charge+scalar monopole scattering

- Let's apply our results to the most famous example: scattering f+M→f+M, arbitrary q



- C^J is extracted from 3 massive 3pt S-matrix

- Selection rule: $|q| \leq \hat{s}$

Fermion charge+scalar monopole scattering

- Apply selection rule:

$$\hat{s} = \frac{1}{2} + 0 + J \geq |q| \quad \rightarrow \quad J \geq |q| - \frac{1}{2}$$

- Lowest partial wave amplitude depends on q - as expected from NRQM

- Extract the J=|q|-1/2 lowest partial wave basis spinors

- The form of the 3pt S-matrix for q>0:

$$S^{\text{3-pt,in}}_{q>0} \;=\; a \left\langle \mathbf{f}\, p^{\flat +}_{fM} \right\rangle \left\langle \mathbf{J}\, p^{\flat +}_{fM} \right\rangle^{2|q|-1}$$

Fermion charge+scalar monopole scattering

- Stripping away the J spinors:

$$C_{q>0}^{|q|-1/2;\,\mathrm{in}} = \left\langle \mathbf{f}\, p_{fM}^{\flat+} \right\rangle \left(\left| p_{fM}^{\flat+} \right\rangle^{2|q|-1} \right)_{\{\alpha_1,\ldots,\alpha_{2|q|-1}\}}$$

- Similarly for the out state. Contracting get basis spinors:

$$\mathcal{B}_{q>0}^{|q|-1/2} = \frac{\left\langle \mathbf{f}\, p_{fM}^{\flat+} \right\rangle \left\langle \mathbf{f}'\, p_{f'M'}^{\flat+} \right\rangle}{4p_c^2} \left(\frac{\left\langle p_{fM}^{\flat+} p_{f'M'}^{\flat+} \right\rangle}{2p_c} \right)^{2|q|-1}$$

- Similar for q<0:

$$\mathcal{B}_{q<0}^{|q|-1/2} = \frac{\left\langle \mathbf{f}\, p_{fM}^{\flat-} \right\rangle \left\langle \mathbf{f}'\, p_{f'M'}^{\flat-} \right\rangle}{4p_c^2} \left(\frac{\left\langle p_{fM}^{\flat-} p_{f'M'}^{\flat-} \right\rangle}{2p_c} \right)^{2|q|-1}$$

Fermion charge+scalar monopole scattering - the massless limit

- To see physics contained consider massless limit

- This is the case when we expect only helicity flip amplitudes (Kazama et al)

- In principle 4 allowed processes by quantum numbers

$$\begin{array}{llll}
\text{Helicity non-flip :} & f + M \to f + M & , & \bar{f}^{\dagger} + M \to \bar{f}^{\dagger} + M \\
\text{Helicity flip :} & f + M \to \bar{f}^{\dagger} + M & , & \bar{f}^{\dagger} + M \to f + M \, .
\end{array}$$

- $f, \bar{f}$ LH fermions

Fermion charge+scalar monopole scattering - the massless limit

- Going from massive to massless ("unbolding")

$$\langle \mathbf{1}|^{\alpha}$$

$h_1 = -½$ $\qquad$ $h_1 = ½$

$$\langle 1|^{\alpha} \qquad\qquad \sim \langle \hat{\eta}_1|^{\alpha}$$ P-conjugate of ⟨ 1 | $^{\alpha}$

- Start with $\bar{f}^{\dagger} + M \to f \; + M$ helicity flip (in out-out formalism both fermions -1/2 helicity)

$$\mathcal{B}^{|q|-\frac{1}{2}} = \frac{\left\langle f\, p^{\flat\pm}_{fM}\right\rangle \left\langle f'\, p^{\flat\pm}_{f'M'}\right\rangle}{4p_c^2} \left(\frac{\left\langle p^{\flat\pm}_{fM} p^{\flat\pm}_{f'M'}\right\rangle}{2p_c} \right)^{2|q|-1} \quad \text{for } \operatorname{sgn}(q) = \pm 1$$

- Vanishes for q>0 since $\quad \left\langle f\, p^{\flat +}_{fM}\right\rangle = \left\langle f'\, p^{\flat +}_{f'M'}\right\rangle = 0$

- Non-vanishing for q<0

Fermion charge+scalar monopole scattering - the massless limit

- Intuitive explanation: field contribution to angular momentum q - has eigenvalues q,q+1,q+2,…

- For RH incoming fermion minimal z-component of total angular momentum q+1/2

- But we are looking at lowest J=|q|-1/2 - doesn't have q+1/2 z-component…

- Similarly for q<0 we only get the $f + M \rightarrow \bar{f}^{\dagger} + M$ helicity flip process non-vanishing.

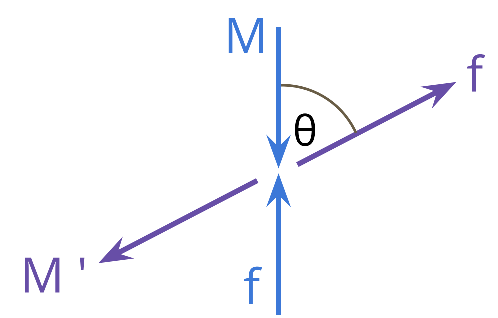
Fermion charge+scalar monopole scattering - the massless limit

- For the helicity non-flip processes all amplitudes vanish: either incoming or outgoing fermion can not be part of J=|q|-1/2 multiplet

- Using the explicit expressions for the spinors we find the helicity flipping amplitudes $\mathcal{N} \equiv \sqrt{8\pi s}$

$$S^{|q|-\frac{1}{2}}_{f\to \bar{f}^\dagger} = \mathcal{N}\ 2\,|q|\,\mathcal{M}^{|q|-\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}\left[\sin\left(\frac{\theta}{2}\right)\right]^{2|q|-1} \quad \text{for } q>0$$

$$S^{|q|-\frac{1}{2}}_{\bar{f}^\dagger\to f} = \mathcal{N}\ 2\,|q|\,\mathcal{M}^{|q|-\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}\left[\sin\left(\frac{\theta}{2}\right)\right]^{2|q|-1} \quad \text{for } q<0$$



- $\mathcal{M}^{|q|-\frac{1}{2}}_{\mp\frac{1}{2},\pm\frac{1}{2}}$ are angle independent constants - will see other channels do not contribute so unitarity fixes them!

$$\left|\mathcal{M}^{|q|-\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}\right| = \left|\mathcal{M}^{|q|-\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}\right| = 1$$

- Exactly Kazama et al. result!

Higher partial waves

- For massive particles follow our rules

$$\mathcal{B}^J \sim \sum_{\sigma}\sum_{\sigma'} a_\sigma a'_{\sigma'} \frac{\left\langle \mathbf{f}\, p^{\flat\sigma}_{fM}\right\rangle \left\langle \mathbf{f}'\, p^{\flat\sigma'}_{f'M'}\right\rangle}{4p_c^2}\, \tilde{\mathcal{B}}^J(-q_\sigma, -q_{\sigma'})$$

where $\sigma, \sigma' = \pm$, $q_\pm = q \pm \frac{1}{2}$ and

$$\tilde{\mathcal{B}}^J(\Delta,\Delta') = \frac{1}{(2p_c)^{2J}} \left(\left\langle p^{\flat-}_{fM}\right|^{J+\Delta} \left\langle p^{\flat+}_{fM}\right|^{J-\Delta} \right)^{\{\alpha_1,\ldots,\alpha_{2J}\}} \left(\left| p^{\flat-}_{f'M'}\right\rangle^{J+\Delta'} \left| p^{\flat+}_{f'M'}\right\rangle^{J-\Delta'} \right)_{\{\alpha_1,\ldots,\alpha_{2J}\}}$$

- In COM frame can show $\tilde{\mathcal{B}}^J(\Delta,\Delta') = (-1)^{J-\Delta'} \mathcal{D}^{J*}_{-\Delta,\Delta'}(\Omega_c)$

with $\mathcal{D}^J_{-\Delta,\Delta'}(\Omega) \equiv \mathcal{D}^J_{-\Delta,\Delta'}(\phi,\theta,-\phi) = e^{i\phi(\Delta+\Delta')}\, d^J_{-\Delta,\Delta'}(\theta)$ Wigner matrix $d^J_{m,m'}(\theta) = \langle J,m|\exp(-i\theta J_y)|J,m'\rangle$

- Exactly the ``monopole harmonics" of Wu & Yang:

$$\mathcal{D}^{l*}_{q,m}(\Omega) = \sqrt{\frac{4\pi}{2l+1}}\; {}_qY_{l,m}(-\Omega)$$

Higher partial waves - massless limit

- In massless limit get a compact result

$$S^J_{h_{\rm in}\to h_{\rm out}} \;=\; \mathcal{N}\;(2J+1)\;\mathcal{M}^J_{-h_{\rm in},h_{\rm out}}\;\mathcal{D}^{J*}_{q-h_{\rm in},-q+h_{\rm out}}\;(\Omega_c)$$

in out-out convention,
h_{in}=1/2 (-1/2) for LH (RH) for incoming fermion
h_{out}=-1/2 (1/2) for LH (RH) for outgoing fermion

- The $\mathcal{M}^J_{-h_{\rm in},h_{\rm out}}$ are dynamics dependent phase shifts

- Take them from Kazama et al detailed NRQM calculation

$$\mathcal{M}^J_{\pm\frac{1}{2},\pm\frac{1}{2}} = e^{-i\pi\mu}, \quad \mu = \sqrt{\left(J+\tfrac{1}{2}\right)^2 - q^2}.$$

Higher partial waves - massless limit

- Partial wave unitarity implies

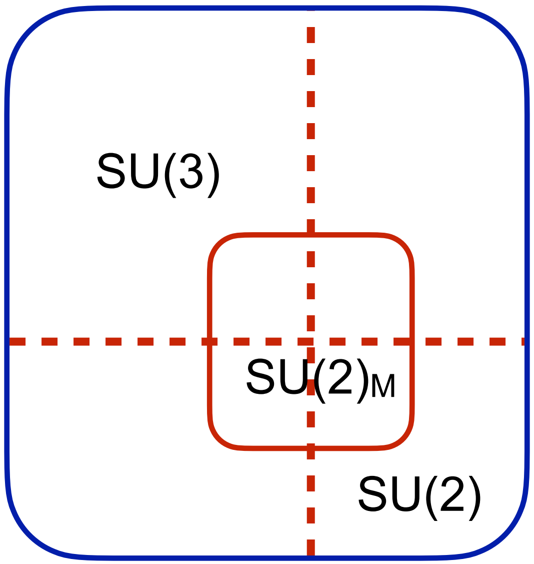
$$\left|\mathcal{M}^J_{\pm\frac{1}{2},\mp\frac{1}{2}}\right|^2 \;=\; 1 - \left|\mathcal{M}^J_{\pm\frac{1}{2},\pm\frac{1}{2}}\right|^2 \;=\; 0$$

- All higher J partial waves have zero helicity flip - only J=|q|-1/2 lowest non-zero. Justifies calculation of the helicity flip amplitude

Scattering on GUT monopoles

- GUT SU(5)→SU(3)xSU(2)xU(1)/Z_6 via adjoint Higgs VEV

- 't Hooft-Polyakov monopole embedded into SU(5)



$$T^3_M = Q_{EM} - \frac{1}{\sqrt{3}}\lambda_8$$

- g_M=-1 to match the notation of Rubakov

Scattering on GUT monopoles

- Decomposition of SM fermions unusual under this SU(2):

$$\bar{\mathbf{5}} = (\bar{d}^{\,1}, \bar{d}^{\,2}, \bar{d}^{\,3}, e^-, \nu_e) \qquad \mathbf{10} = \begin{pmatrix} 0 & \bar{u}^{\,3} & -\bar{u}^{\,2} & u^1 & d^1 \\ -\bar{u}^{\,3} & 0 & \bar{u}^{\,1} & u^2 & d^2 \\ \bar{u}^{\,2} & -\bar{u}^{\,1} & 0 & u^3 & d^3 \\ -u^1 & -u^2 & -u^3 & 0 & \bar{e} \\ -d^1 & -d^2 & -d^3 & -\bar{e} & 0 \end{pmatrix}$$

- Will give 4 doublets - the rest are singlets

$$\begin{pmatrix} e \\ -\bar{d}^3 \end{pmatrix}, \begin{pmatrix} \bar{u}^1 \\ u^2 \end{pmatrix}, \begin{pmatrix} -\bar{u}^2 \\ u^1 \end{pmatrix}, \begin{pmatrix} d^3 \\ \bar{e} \end{pmatrix} \quad \begin{array}{cc} e_M & q = e_M g_m \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} \end{array}$$

- Will give SU(4) horizontal symmetry (exchange of 4 doublets - identical for interaction with monopole)

The Rubakov-Callan amplitude

- Scattering amplitudes have to obey SM gauge conservation + SU(4) symmetry + LG + pairwise LG

- The Rubakov-Callan amplitude:

$$u^1 + u^2 + M$$

- Focus on s-wave incoming states (that can reach the core of the monopole) $J_{u1} = J_{u2} = 0$

- Incoming part of amplitude: $\left[u^1 \, p^{\flat -}_{u^1, M}\right] \left[u^2 \, p^{\flat -}_{u^2, M}\right]$

- Pairwise helicity -1/2, ordinary helicity +1/2 in all outgoing convention

The Rubakov-Callan amplitude

• Outgoing state? Could it be the same (forward scattering)?

$$\left\langle u^1\, p^{\flat+}_{u^1,M} \right\rangle \left\langle u^2\, p^{\flat+}_{u^2,M} \right\rangle$$

• Would be the candidate amplitude - needed to flip single particle helicity due to all outgoing convention.

• But $\left\langle i\, p^{\flat+}_{iM} \right\rangle = \left[i\, p^{\flat+}_{iM} \right] = 0$ because for massless fermions the pairwise momentum = ordinary mom.

• No forward scattering!

The Rubakov-Callan amplitude

- Only possible final state:

$$\left[\bar{e}^{\dagger}\, p^{\flat-}_{\bar{e}^{\dagger},M}\right]\left[\bar{d}^{3\dagger}\, p^{\flat-}_{\bar{d}^{3\dagger},M}\right]$$

- Helicity flipping, but still J=0 states
- All quantum numbers conserved

	u^1	$+$	u^2	$\rightarrow$	$e^{\dagger}$	$+$	$d^{3\dagger}$
λ_3	1		-1		0		0
$\sqrt{3}\,\lambda_8$	1		1		0		2
T_L^3	½		½		½		½
Y	⅙		⅙		½		- ⅙

$$\mathcal{A}_{\text{Rubakov-Callan}} \propto \left[u^1\, p^{\flat-}_{u^1,M}\right]\left[u^2\, p^{\flat-}_{u^2,M}\right]\left[\bar{e}^{\dagger}\, p^{\flat-}_{\bar{e}^{\dagger},M}\right]\left[\bar{d}^{3\dagger}\, p^{\flat-}_{\bar{d}^{3\dagger},M}\right]$$

The Rubakov-Callan amplitude

$$\mathcal{A}_{\text{Rubakov-Callan}} \propto \left[u^1\, p^{\flat -}_{u^1,M}\right]\left[u^2\, p^{\flat -}_{u^2,M}\right]\left[\bar{e}^\dagger\, p^{\flat -}_{\bar{e}^\dagger,M}\right]\left[\bar{d}^{3\dagger}\, p^{\flat -}_{\bar{d}^{3\dagger},M}\right]$$

- Violates baryon number

- Saturates J=0 unitarity bound

- Incoming u^1,u^2 part of proton - B violating cross section $\propto$ Λ_{QCD}

- An on-shell derivation of monopole catalysis of proton decay

Callan's unitarity puzzle

- Instead consider the $e^+ + M$ channel

- The only allowed final state by gauge quantum numbers:

$$\bar{u}^{1\dagger} + \bar{u}^{2\dagger} + \bar{d}^{3\dagger}$$

$$\begin{pmatrix} e \\ -\bar{d}^3 \end{pmatrix}, \begin{pmatrix} \bar{u}^1 \\ u^2 \end{pmatrix}, \begin{pmatrix} -\bar{u}^2 \\ u^1 \end{pmatrix}, \begin{pmatrix} d^3 \\ \bar{e} \end{pmatrix} \quad \begin{matrix} e_M \\ \frac{1}{2} \\ -\frac{1}{2} \end{matrix} \qquad \begin{matrix} q = e_M g_m \\ -\frac{1}{2} \\ +\frac{1}{2} \end{matrix}$$

initial state

$$\underbrace{\left[\bar{e}\, p^{\flat -}_{\bar{e},M}\right]}_{J_{\bar{e}} = 0}$$

final state

$$\underbrace{\left[\bar{u}^{\,1\dagger}\, p^{\flat -}_{\bar{u}^{\,1\dagger},M}\right]}_{J_{\bar{u}\dagger 1} = 0} \underbrace{\left[\bar{u}^{\,2\dagger}\, p^{\flat -}_{\bar{u}^{\,2\dagger},M}\right]}_{J_{\bar{u}\dagger 2} = 0} \underbrace{\left[\bar{d}^{\,3\dagger}\, p^{\flat +}_{\bar{d}^{\,3\dagger},M}\right]}_{J_{\bar{d}\dagger 3} = 0}$$

Callan's unitarity puzzle

- Instead consider the $e^+ + M$ channel

- The only allowed final state by gauge quantum numbers:

$$\bar{u}^{1\dagger} + \bar{u}^{2\dagger} + \bar{d}^{3\dagger}$$

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0 by $\left[i\, p^{\flat +}_{ij}\right] = 0$

initial state

final state

$$\underbrace{\left[\bar{e}\, p^{\flat -}_{\bar{e},M}\right]}_{J_{\bar{e}} = 0} \qquad \underbrace{\left[\bar{u}^{1\dagger}\, p^{\flat -}_{\bar{u}^{1\dagger},M}\right]}_{J_{\bar{u}\dagger 1} = 0} \; \underbrace{\left[\bar{u}^{2\dagger}\, p^{\flat -}_{\bar{u}^{2\dagger},M}\right]}_{J_{\bar{u}\dagger 2} = 0} \; \underbrace{\cancel{\left[\bar{d}^{3\dagger}\, p^{\flat +}_{\bar{d}^{3\dagger},M}\right]}}_{J_{\bar{d}\dagger 3} = 0}$$

Callan's unitarity puzzle

- No allowed final states????

- Callan '83: work in truncated 1+1D theory of J=0 states

- Suggests outgoing state $1/2(e^\dagger + \bar{u}^{1\dagger} + \bar{u}^{2\dagger} + d^3)$

- ``Fractional fermions" - semitons. Gauge quantum number only statistically conserved????

- Dawson, Schellekens '84: with bare masses added semitons decay to ordinary fermions

- Polchinsky '84: propagating pulses of vacuum polarization

Callan's unitarity puzzle

- More recent developments:

- Maldacena/Ludwig '97, Boyle Smith/Tong '20: analog 2D theory with SO(8) global symmetry, no gauging - confirmed semiton description.

- Kitano/Matsudo '21: fractional fermions artifacts of 1+1D theory, in 4D they would be domain wall with fermion edge modes ``pancakes”

A possible resolution

- The on-shell formalism suggests another possible simple resolution

- Cannot have $\left[\bar{u}^{1\dagger}\, p^{\flat -}_{\bar{u}^{1\dagger},M}\right]\left[\bar{u}^{2\dagger}\, p^{\flat -}_{\bar{u}^{2\dagger},M}\right]\left[\bar{d}^{3\dagger}\, p^{\flat +}_{\bar{d}^{3\dagger},M}\right]$ since $\left[\bar{d}^{3\dagger}\, p^{\flat -}_{\bar{d}^{3\dagger},M}\right] = 0$

- But CAN have $\left[\bar{u}^{1\dagger}\, p^{\flat -}_{\bar{u}^{1\dagger},M}\right]\left[\bar{u}^{2\dagger}\, p^{\flat +}_{\bar{d}^{3\dagger},M}\right]\left[\bar{d}^{3\dagger}\, p^{\flat -}_{\bar{u}^{2\dagger},M}\right] - (1 \leftrightarrow 2)$

- While individual fermions NOT in J=0 state the total state is J=0 and can penetrate to the core

- Such a state would be missing in the 1+1D effective description since that kept only the individual J=0 states

A possible resolution

- Our proposal:

$$\mathcal{A}_{\text{Puzzle}} \sim \left[\bar{e}\, p^{\flat -}_{\bar{e},M}\right] \left[\bar{u}^{\,1\dagger}\, p^{\flat -}_{\bar{u}^{\,1\dagger},M}\right] \left[\bar{u}^{\,2\dagger}\, p^{\flat +}_{\bar{d}^{\,3\dagger},M}\right] \left[\bar{d}^{\,3\dagger}\, p^{\flat -}_{\bar{u}^{\,2\dagger},M}\right] - (1 \leftrightarrow 2)$$

- Respects all gauge symmetries and SU(4)
- No fractional fermions
- B violating, saturates J=0 unitarity
- Monopole creates entangled fermions

The dynamics of pairwise helicity

- What is the dynamical origin of pairwise helicity?

- Reason for unusual behavior: very soft photons can be exchanged even at large distance, interaction does not die out

- To capture effect of soft photons, can prepare ``dressed states" - Faddeev-Kulish dressing

- Main idea of FK: used to show IR divergences of QED cancel

- Asymptotic interaction $V^{I}_{as\,;\,QED}(t) \equiv \lim_{|t| \to \pm\infty} V^{I}_{QED}(t)$

- Since it doesn't go to zero - modify interaction pic.

The FK dressing

- Include the asymptotic interaction into the states - ``dressed states"

$$|p_1, \ldots, p_f\rangle\!\rangle_{QED} = \mathcal{U}_{QED}\, |p_1, \ldots, p_f\rangle$$

$$\mathcal{U}_{QED} \equiv \mathcal{T}\, \exp\left[-i \int_0^\infty dt\; (V^I_{as;QED})\right]$$

- The S-matrix for these dressed states will be IR finite!

$$S^{finite}_{(1,\ldots,g|1,\ldots,f)} \equiv \langle\!\langle p_1, \ldots, p_g|\, S_{QED}\, |p_1, \ldots, p_f\rangle\!\rangle$$

$$S_{QED} = \mathcal{T}\, \exp\left[-i \int_{-\infty}^\infty dt\; (V_{QED})\right]$$

- $V^I_{QED}(t) = -\int d^3x\, [j^\mu A_\mu]$, need to subtract out V_{as} .

The FK dressing

- We repeated this for QEMD using Zwanziger's Lagrangian - two potentials, but unusual kinetic term making sure only one physical photon

$$\mathcal{L}^I_{int} = -\left[j_e^\mu A_\mu + j_g^\mu B_\mu\right]$$

$$A_\mu(x) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[\varepsilon_\mu^{*\lambda}(\vec{k}) a_\lambda(\vec{k}) e^{ik\cdot x} + \varepsilon_\mu^\lambda(\vec{k}) a_\lambda^\dagger(\vec{k}) e^{-ik\cdot x}\right]$$

$$B_\mu(x) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[\widetilde{\varepsilon}_\mu^{*\lambda}(\vec{k}) a_\lambda(\vec{k}) e^{ik\cdot x} + \widetilde{\varepsilon}_\mu^\lambda(\vec{k}) a_\lambda^\dagger(\vec{k}) e^{-ik\cdot x}\right]$$

- Relation between polarization vectors

$$\widetilde{\varepsilon}_\mu^{\ \lambda} = -A_{\mu\nu}\varepsilon^{\nu\,\lambda}, \qquad A_{\mu\nu} \equiv \frac{\epsilon_{\mu\nu}(n,k)}{n\cdot k + i\epsilon}$$

Dressed states of QEMD

- We calculated the FK dressing factors of QEMD

$$\mathcal{U}_{QEMD} \equiv \mathcal{T} \exp\left[-i \int_{-\infty}^{\infty} dt\, V^{I}_{as\,;\,QEMD}(t)\right] = e^{R_{FK}}\, e^{i\Phi_{FK}}$$

$$R_{FK} = -i \int_{-\infty}^{\infty} dt\; V^{I}_{as\,;\,QEMD}(t)$$

$$\Phi_{FK} = \frac{i}{2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2\, [V^{I}_{as\,;\,QEMD}(t_1), V^{I}_{as\,;\,QEMD}(t_2)]\,.$$

- We found: $U\,[\Lambda]\;|p_1, \ldots, p_f\rangle\!\rangle = e^{i\Phi_{LG}}\,|\Lambda p_1, \ldots, \Lambda p_f\rangle\!\rangle$

- Two steps:

$$\left\{[M^{\mu\nu}, R_{FK}] + \frac{1}{2}\,[[M^{\mu\nu}, R_{FK}]\,, R_{FK}] - \Delta\Phi^{\mu\nu}_{FK}\right\}|p_1, \ldots, p_f\rangle = \Phi^{\mu\nu}_{LG}\;|p_1, \ldots, p_f\rangle$$

Dressed states of QEMD

- Need both phase and real part of FK dressing!

- After heroic efforts: $\Delta\varphi_{FK}(p_1,p_2,n) = 2\,\mathrm{arccos}\left[\hat{\epsilon}(p_1,p_2,\Lambda^{-1}n)\cdot\hat{\epsilon}(p_1,p_2,n)\right]$

$$\Delta\Phi^{\mu\nu}_{FK} = \sum_{l<m} q_{lm}\,\Delta\varphi^{\mu\nu}_{FK;lm} = 2\sum_{l<m} q_{lm}\,\varphi^{\mu\nu}_{LG;lm} = -2\Phi^{\mu\nu}_{LG}$$

- Angular mom. commutator:

$$\left\{[M^{\mu\nu},R_{FK}] + \frac{1}{2}\left[[M^{\mu\nu},R_{FK}],R_{FK}\right]\right\}|p_1,\ldots,p_f\rangle = -\Phi^{\mu\nu}_{LG}\,|p_1,\ldots,p_f\rangle$$

- Sum exactly gives required pairwise LG transformation

$$\left\{[M^{\mu\nu},R_{FK}] + \frac{1}{2}\left[[M^{\mu\nu},R_{FK}],R_{FK}\right] - \Delta\Phi^{\mu\nu}_{FK}\right\}|p_1,\ldots,p_f\rangle = \Phi^{\mu\nu}_{LG}\,|p_1,\ldots,p_f\rangle$$

The calculation of $\Delta\Phi_{FK}$

- $\Phi_{FK} = \frac{i}{4}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, [V^I_{as\,;\,QEMD}(t_{max}), V^I_{as\,;\,QEMD}(t_{min})]$

- Evaluating the commutators:

$$\Phi_{FK} = 4\pi \sum_{l<m} q_{lm} \iint D_l\, p_a\, D_m\, p_b \int_{-\infty}^{\infty} \frac{dt_1}{\omega_a} \int_{-\infty}^{\infty} \frac{dt_2}{\omega_b}\, \mathrm{Im}\,[I(p_a, p_b, n)]$$

- Almost usual Feynman integral but unusual propagator due to magnetic photon

$$I(p_1, p_2, p_3) \equiv -\int \frac{d^4k}{(2\pi)^4} \frac{i\epsilon(p_1, p_2, p_3, k)}{(k^2 + i\epsilon)(p_3 \cdot k + i\epsilon)}\, e^{-ik\cdot\Delta_{12}(p_a,p_b)}$$

$$\Delta^{\mu}_{12}(a, b) = \frac{t_1 a^{\mu}}{\omega_a} - \frac{t_2 b^{\mu}}{\omega_b}$$

Dirac quantization from geometric phase

- Lagrangian depends on Dirac string. Rotate Dirac string adiabatically $n^\mu(\tau) = \exp\left[\tau\omega\right]^\mu_\nu \, n_0^\nu$

- Rotation of dressed states:

$$|p_1, \ldots, p_f\rangle\!\rangle_{n(\tau+\delta\tau)} = e^{-\frac{i\delta\tau}{2}\omega_{\mu\nu}\Phi^{\mu\nu}_{LG}} \, |p_1, \ldots, p_f\rangle\!\rangle_{n(\tau)}$$

- Berry phase: $\gamma_{Berry} = i \int_0^{2\pi} d\tau \, \langle\!\langle p_1, \ldots, p_f | \frac{d}{d\tau} | p_1, \ldots, p_f \rangle\!\rangle = \frac{\omega_{\mu\nu}}{2} \int_0^{2\pi} d\tau \, \Phi^{\mu\nu}_{LG}$

$$= \sum_{l<m} q_{lm} \int_0^{2\pi} d\tau \, \frac{\tau_{lm} \, n_0^\mu \omega_{\mu\nu} \epsilon^\nu \left[p_l(\tau), p_m(\tau), n_0\right]}{\epsilon^2 \left[p_l(\tau), p_m(\tau), n_0\right]} = \pm 2\pi \sum_{l<m} q_{lm}$$

- Demanding overall phase either fermion or boson: Dirac quantization $q_{lm} = n/2$ from purely QFT

Summary

- Pairwise LG provides novel multi-particle states that are not direct products
- Key ingredient to solving magnetic scattering
- Pairwise spinor-helicity new building block
- Can construct all 3pt S-matrix elements, fix angular dependence of $2 \rightarrow 2$ scattering
- Obtain helicity flip, monopole harmonics, Rubakov-Callan, novel resolution to semiton puzzle
- Dynamical origin as dressed states, gives novel QFT derivation of Dirac quantization