QED correction
to Decay Rates

Francesco Sanfilippo, INFN, Roma Tre

"Frontiers in Lattice Quantum Field Theory", Madrid, 22 May 2018

Outline

Introduction

- Motivation to include QED in QCD
- Why lattice QCD+QED
- If Lattice QCD is though, including QED is even harder!
- Include QED: the perturbative approach

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Phenomenology

- Hadron Masses
- Decay rates
  1. In pure QCD (infrared finite)
  2. Ratio of decay rates (infrared finite)
  3. Single decay rate (infrared troubled)
- $g-2$: QED contribution to hadronic vacuum polarization

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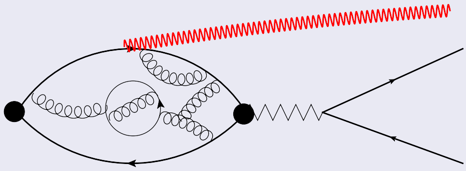
Some final words

- Work in progress
- Future developments

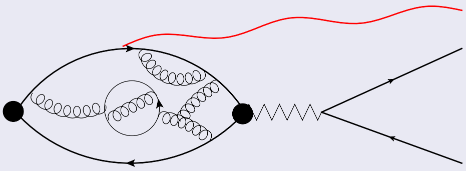
Dealing with photons

Hard photons - $E \sim$ many GeV



Perturbation theory

Ultrasoft photons - $E \sim$ few MeV



Point-like hadrons

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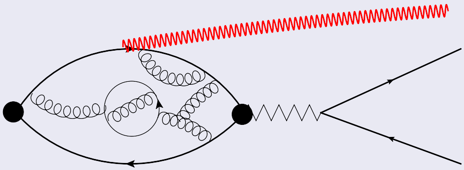

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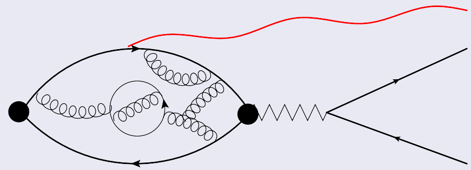
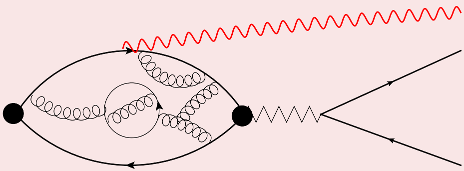
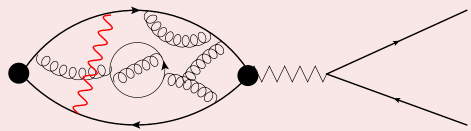


Point-like hadrons

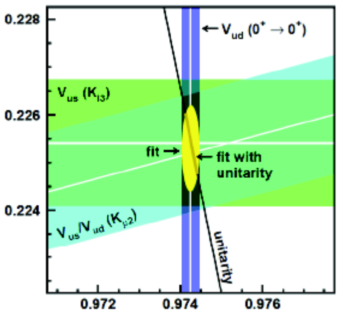
What to do with soft photons?

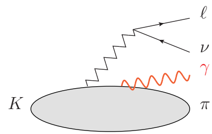




...Here we come to the rescue...

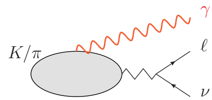
Example: CKM matrix elements from semileptonic and leptonic K and π decays





Semileptonic

$$\underbrace{\Gamma_{K\to\pi l\bar{\nu}(\gamma)}}_{experiments} \propto |V_{us}|^2 \underbrace{\left|f_+^{K\pi}(0)\right|^2}_{QCD}$$



Leptonic

$$\underbrace{\frac{\Gamma_{K\to\ell\bar{\nu}(\gamma)}}{\Gamma_{\pi\to\ell\bar{\nu}(\gamma)}}}_{experiments} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \underbrace{\left(\frac{f_K}{f_\pi}\right)^2}_{QCD}$$

Hadronic matrix elements, lattice results

$$\begin{aligned} f_+^{K\pi}(0) &= 0.956\,(8) \\ f_K/f_\pi &= 1.193\,(5) \end{aligned}$$ in the **isospin symmetric** limit.

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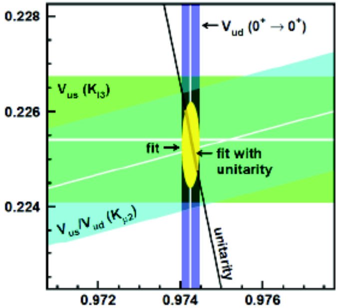

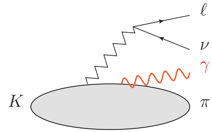

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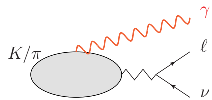

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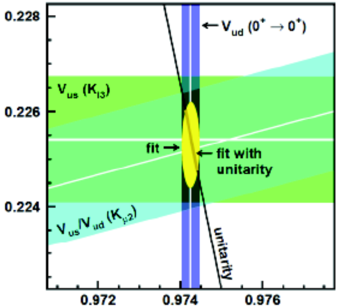

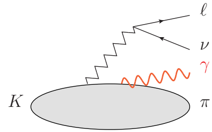

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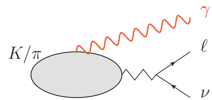

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Indeed ChPT estimates of these effects are:

$\left(f_+^{K^+\pi^0}/f_+^{K^-\pi^+} - 1\right)^{QCD} = 2.9(4)\%$	$\left(\frac{f_{K^+}/f_{\pi^+}}{f_K/f_\pi} - 1\right)^{QCD} = -0.22(6)\%$
A. Kastner, H. Neufeld (EPJ C57, 2008)	V. Cirigliano, H. Neufeld (Phys.Lett.B700, 2011)

More complications from QED

The target: Fully unquenched QCD + QED

$$\mathcal{L} = \sum_i \bar{\psi}_i \left[m_i - i\not{D}_i\right] \psi_i + \mathcal{L}_{gluons} + \mathcal{L}_{photon}, \quad D_{i,\mu} = \partial_\mu + igA^a_\mu T^a + ie_i A_\mu$$

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Introducing photons

Power-like Finite Volume Effects due to long range interaction

Zero mode from photon propagator: $\int \frac{\delta_{\mu\nu}}{k^2} d^4k \;\; \rightarrow \;\; \sum_k \frac{\delta_{\mu\nu}}{k^2}$
massive photons, removal of zero mode, C^* boundary conditions...

Renormalization pattern gets more complicated

Additional divergencies arises!

UV completeness: Nobody knows how to tame QED to all orders!

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Practical problem

- Traditionally, gauge configuration datasets include only gluons
- Dedicated simulations with huge cost
- Even greater cost due to additional zero modes.

The Roman approach - RM123 collaboration

Pioneering papers

- "*Isospin breaking effects due to the up-down mass difference in Lattice QCD*", [JHEP 1204 (2012)]
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3) Roma Tre

D.Giusti,
V.Lubicz,
S.Romiti,
F.S,
S.Simula,
C.Tarantino

1) La Sapienza

M.Di Carlo,
G.Martinelli

2) Tor Vergata

G.deDivitiis,
P.Dimopoulos,
R.Frezzotti,
N.Tantalo

⋆ **Guest Star** from Southampton University: C.T.Sachrajda

The Roman approach - RM123 collaboration

Perturbative expansion

Work on top of the isospin symmetric theory $\boldsymbol{\mathcal{L} = \mathcal{L}_{Iso\,symm} + \mathcal{L}_{Iso\,break}}$

$$\mathcal{L}_{Iso\,break} = e\mathcal{L}_{QED} + \delta m \mathcal{L}_{mass}, \quad e^2 = \frac{4\pi}{137.04}, \quad \delta m = (m_d - m_u)/2$$

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Cleaner: Factorize small parameters e and δm, introduce QED only when needed

Cheaper: No need to generate new QCD gauge field backgrounds (and, newly generated ones are general purpose).

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$\rightarrow$ Only method to include QED in matrix elements (is it? cfr. backup slides)

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Remove "some" zero modes

<u>4D zero mode only</u>: $\sum_k \;\rightarrow\; \sum_{k\neq 0}$

✓ pole removed, irrelevant when $V \rightarrow \infty$

✗ nonlocal constraint, T/L^3 divergence

$\sim$ not tragic when working at fixed T/L.

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Use C^* Boundary conditions

✓ local

✗ needs dedicated simulations

∼ flavor violation across boundaries.

1) The perturbative expansion in δm

Split mass lagrangian in two contributions:

$$\mathcal{L}_{mass} = \underbrace{\left(\frac{m_d + m_u}{2}\right)}_{m_{ud}} (\bar{u}u + \bar{d}d) - \underbrace{\left(\frac{m_d - m_u}{2}\right)}_{\delta m} (\bar{u}u - \bar{d}d)$$

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Split action in **two parts**:

$$S = S_0 - \delta m S_m, \qquad \begin{cases} S_0 & \text{isospin simmetric action} \\ S_m & \text{perturbation} = \sum_x \left(\bar{u}u - \bar{d}d\right) \end{cases}$$

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Expand functional integral:

$$\langle O \rangle = \frac{\int D\psi O e^{-S_0 + \delta m \hat{S}}}{\int D\psi e^{-S_0 + \delta m \hat{S}}} \overset{1st}{\simeq} \frac{\int D\psi O e^{-S_0} (1 + \delta m S_m)}{\int D\psi e^{-S_0} (1 + \delta m S_m)} \simeq \frac{\langle O \rangle_0 + \delta m \langle O S_m \rangle_0}{1 + \delta m \cancel{\langle S_m \rangle_0}}$$

1) The perturbative expansion in δm

Isospin correction determination

Relative correction to an observable O is obtained as:

$$\frac{\delta \langle O \rangle}{\langle O \rangle_0} \equiv \frac{\langle O \rangle - \langle O \rangle_0}{\langle O \rangle_0} \simeq \delta m \frac{\langle \hat{S} O \rangle_0}{\langle O \rangle_0}, \qquad \hat{S} = \sum_x \left(\bar{u}u - \bar{d}d \right)$$

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Predictivity of the method

- A physical observable is needed to fix δm (as for any parameter)
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Diagrammatically

2) The perturbative expansion in e^2

Keep QCD to all orders and QED to $\mathcal{O}\left(e^2\right)$

$$\langle O\rangle = \frac{1}{\mathcal{Z}}\int D\left[A_\mu, U^{QCD}, \psi, \bar{\psi}\right] O\,\left(1 - e^2 S_1 + \mathcal{O}\left(e^4\right)\right)\exp\left[-S_0\right]$$

N.B: $\mathcal{O}\left(e\right)$ vanishes due to charge symmetry.

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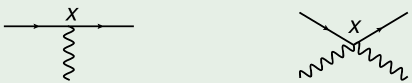
Which on the lattice means...

$$S_1 = \underbrace{\left[\int dx \, V_\mu\left(x\right) A_\mu\left(x\right)\right]^2}_{} + \underbrace{\int dx \, T_\mu\left(x\right) A_\mu^2\left(x\right)}_{}$$



- V^2: Two photon-fermion -fermion vertices (as in the continuum)
- T: One photon-photon-fermion-fermion vertex (tadpole: lattice special).

The case of the pion

Basic correlation function

$$C(t) = \sum_{\vec{x}} \left\langle P(\vec{x},t)\, P^{\dagger}(0) \right\rangle_{QCD+QED}, \qquad P = \bar{\psi}\gamma_5\psi$$

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Functional integral

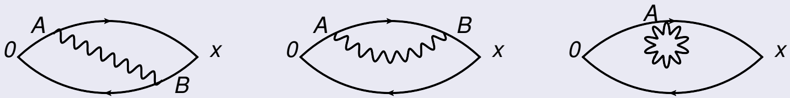
$$C(t) = C_0(t) + C_1(t) =$$

$$\left\langle P(\vec{x},t)\, P^{\dagger}(0) \right\rangle_{QCD} - e^2 \left\langle P(\vec{x},t) \sum_{y} S_1(y) P^{\dagger}(0) \right\rangle_{QCD}$$

Now take all Wick contractions...

Diagrams

Fermionically connected - easy part (so to say)



(gluons not drawn, connecting fermion lines in all possible ways)

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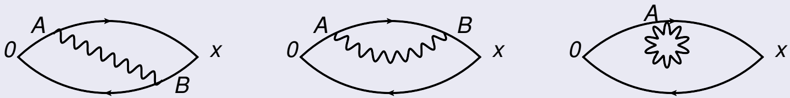

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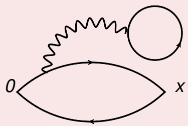
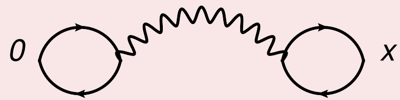
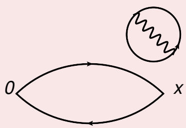
Disconnected - various degree of nastiness - work is in progress to include



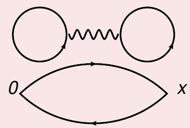
''monocle''



''handcuffs''



"blinking"



"laughing"

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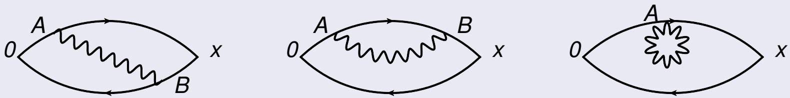
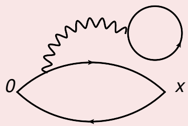
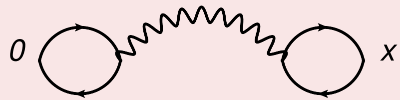


(gluons not drawn, connecting fermion lines in all possible ways)

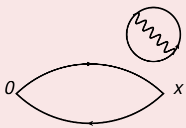
Disconnected - various degree of nastiness - work is in progress to include



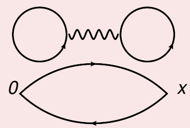
"monocle"



"handcuffs"



"blinking"



"laughing"

"*Disconnected isn't difficult*" [K.Szabo]

Hadron masses

Infrared safe

- Hadron masses are finite quantities (after that the action is properly renormalized)
- True without and with QED.

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- Large euclidean time behaviour (see next slide).

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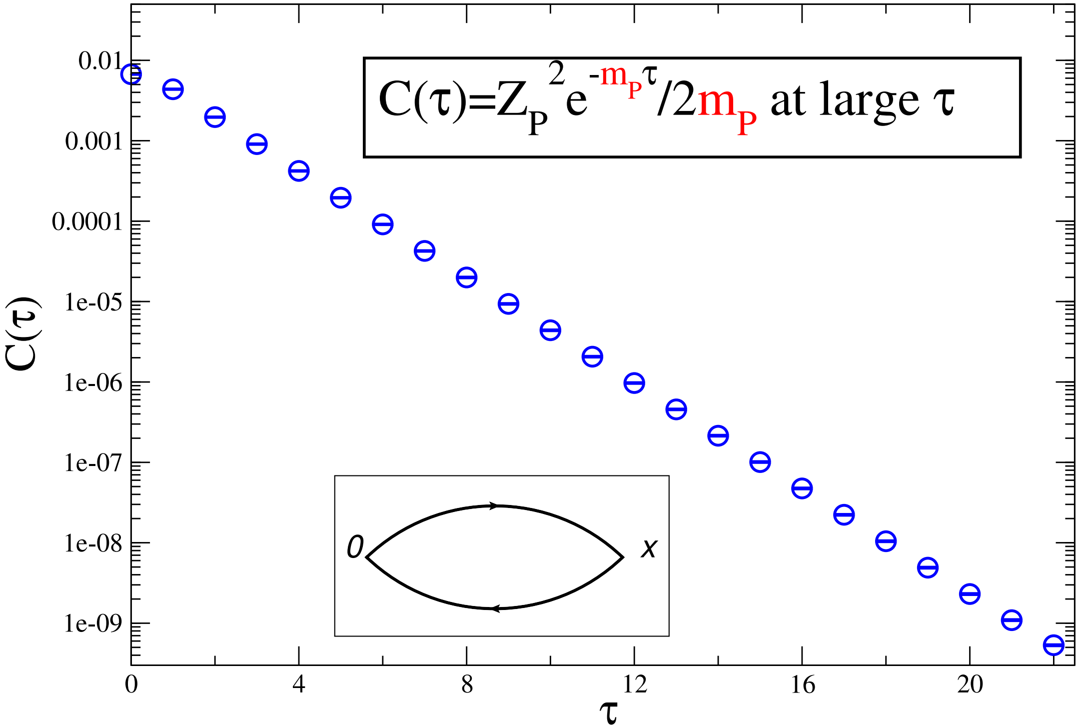
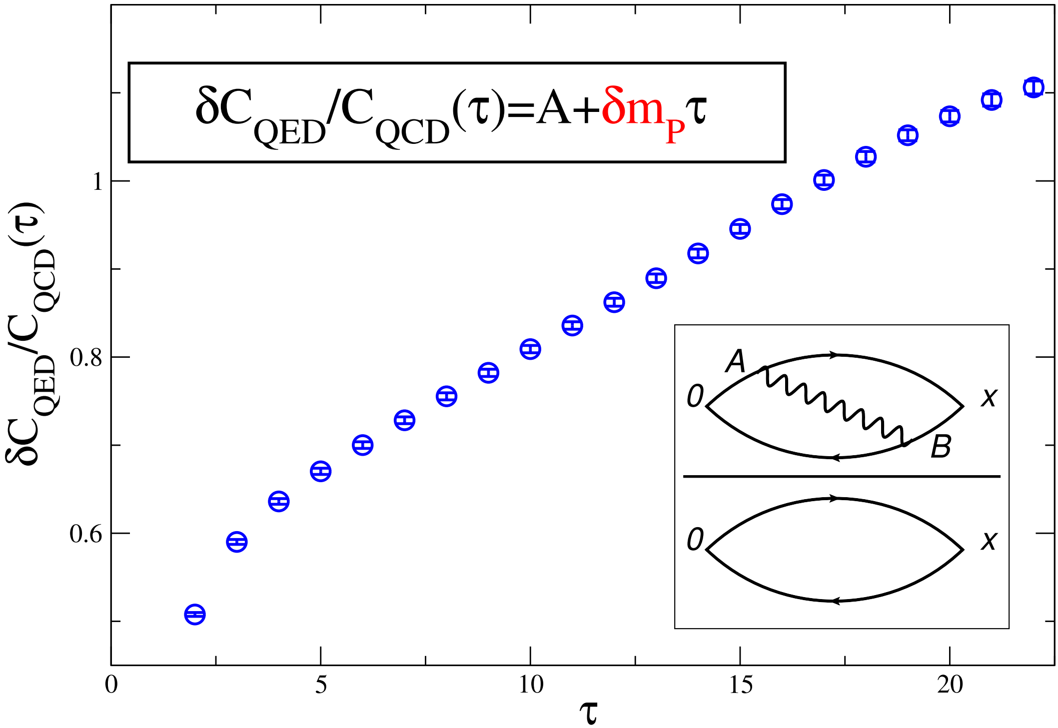
Other collaborations, other approaches

- FNAL/MILC, BMW, QCDSF/UKQCD: fully dynamical simulation of QCD+QED
- RBC/UKQCD: comparison of perturbative and all-order approach.

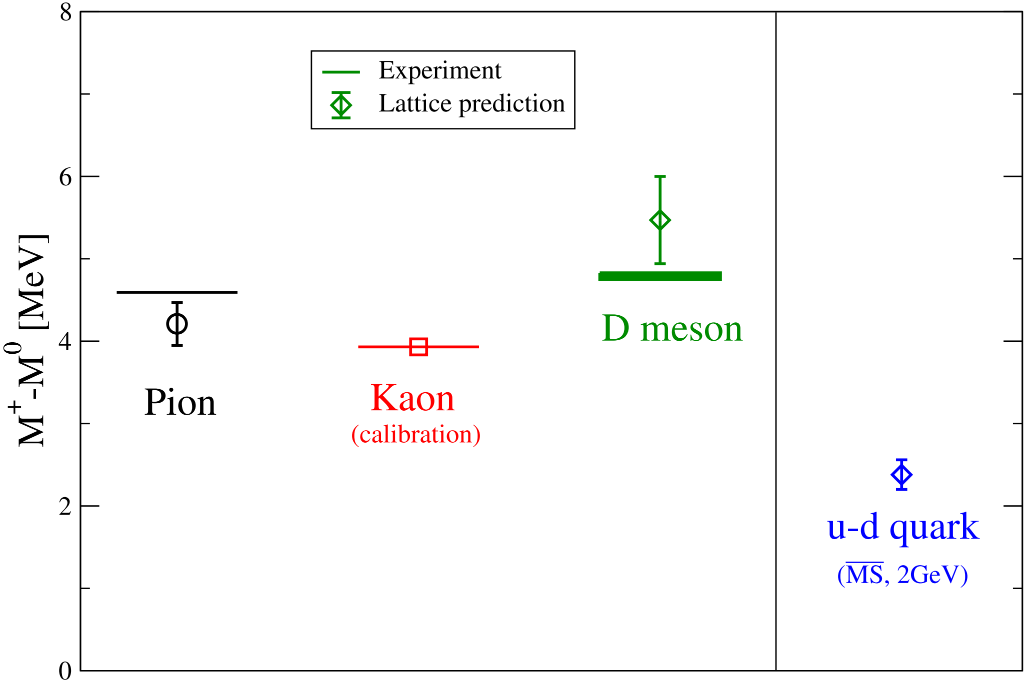
Pseudoscalar meson 2pts. correlation function (no QED)



Hadron masses at $\mathcal{O}\left(e^2\right)$

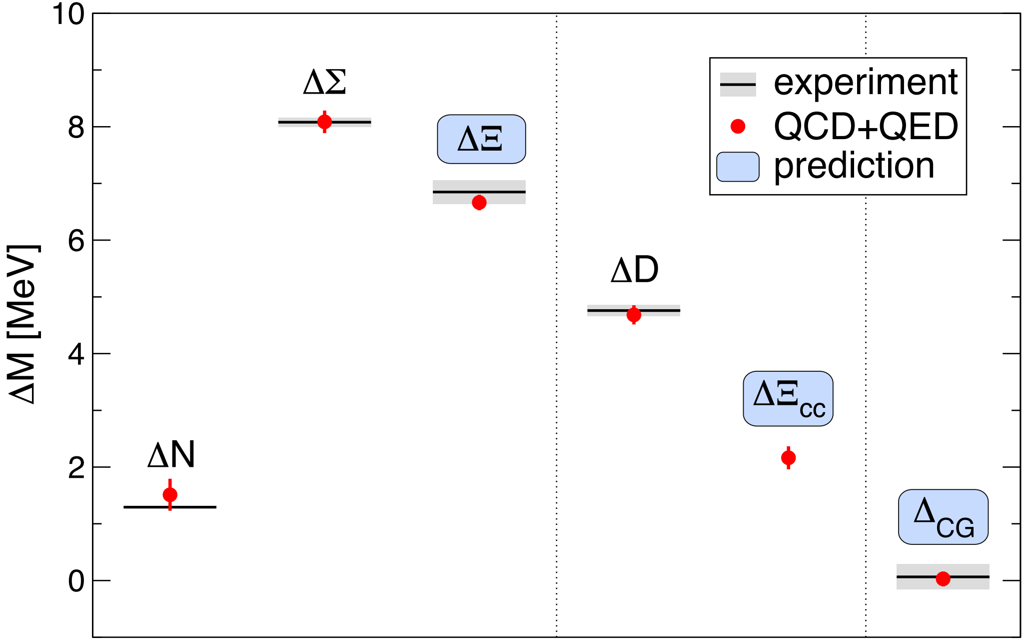


Some results, meson mass (perturbative expansion)



RM123 coll., "*Leading isospin-breaking corrections to pion, K and D mesons*", PRD95 (2017)

Some results, baryons (direct simulation)



BMW coll.: "*Ab initio calculation of the neutron-proton mass difference*", Science 347 (2015)

$$g_{\mu}-2$$

Infrared safe

- Neutral current
- True without and with QED.

$g_\mu - 2$

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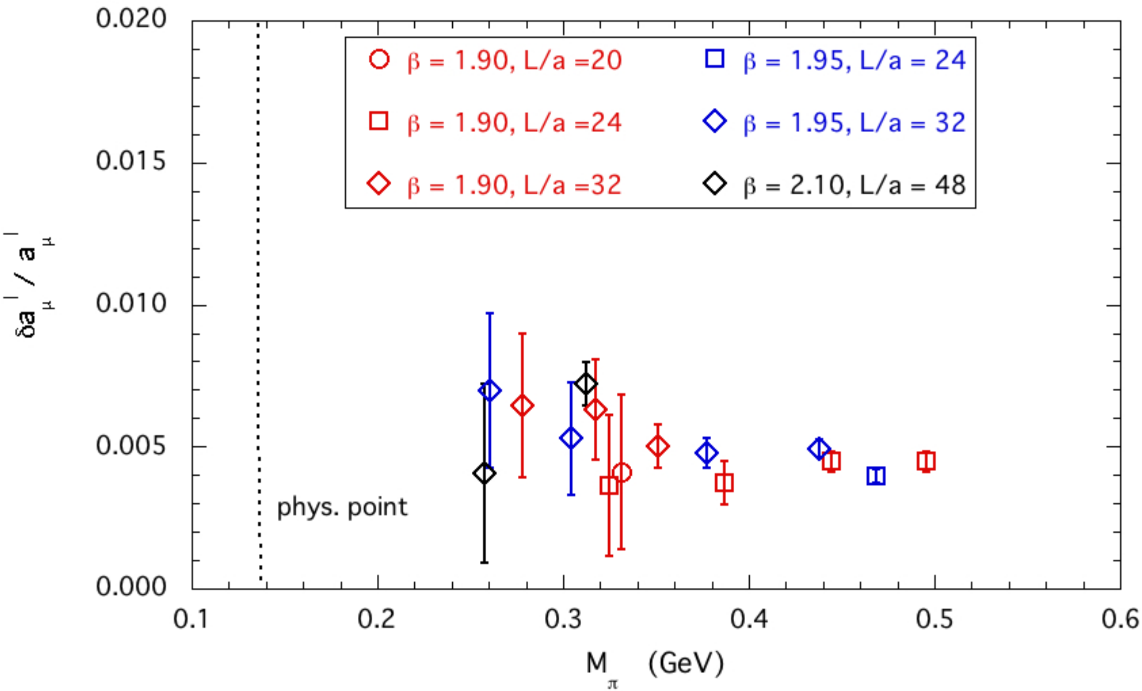
Computation of Strange and Charm contributions [D.Giusti et al, JHEP 1710 (2017) 157]

- Moments method (cfr. yesterday's talk by K.Szabo)
- QED corrections **negligible**
- Now it's the turn of the light channel QED correction...

$g_\mu - 2$ QED corrections to light channel (PRELIMINARY)



In line with dispersive approach estimate (see K.Szabo, yesterday's talk)

Matrix elements

More problems

- In general the amplitudes, are **infrared divergent**
- On the lattice, a natural infrared cutoff is provided by the **finite volume**
- But physically, only combinations of **Real** + **Virtual** contribution is finite.

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- We consider the leptonic decay of a **charged pion**
- The method is general

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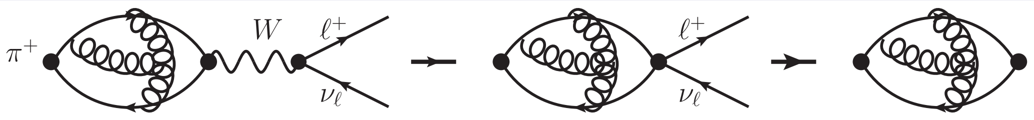
Nobody has gone there before!

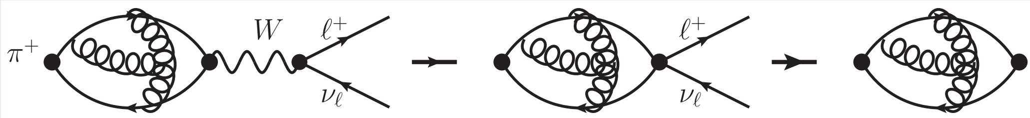
Leptonic decays of mesons (at tree level in QED: $e = 0$)

Full process	Eff. weak hamiltonian	QCD side



Leptonic decays of mesons (at tree level in QED: $e = 0$)

Full process | Eff. weak hamiltonian | QCD side



Two point correlation functions

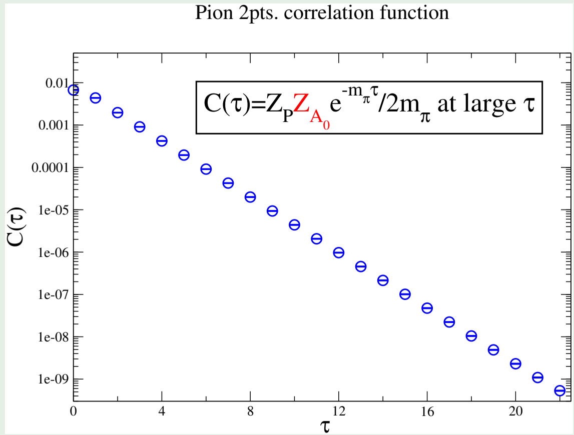
$$\Gamma_{\pi\to\ell\bar{\nu}} = \underbrace{|V_{xy}|^2}_{\text{CKM}} \underbrace{\mathcal{K}\left(m_\ell, m_M\right)}_{\text{kinematics}} \left| \underbrace{\quad f_\pi \quad}_{\text{dec. constant}} \right|^2$$

$$f_\pi = \frac{Z_A}{m_\pi} = \frac{\langle 0|A_0|\pi\rangle}{m_\pi}$$

Z: coupling of current inducing decay

From lattice, 2 point correlation functions:

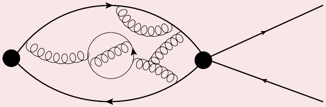
$$C(\tau) = \langle O^\dagger_{A_0}(\tau)\, O_P(0)\rangle,\ O = \bar{\psi}\Gamma\psi$$

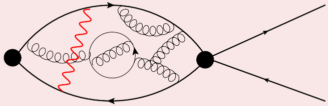
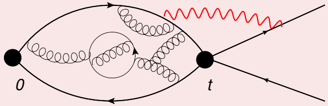


Leptonic decays of mesons (with QED)

Zero photons in the final state, $\mathcal{O}\left(e^2\right)$

$$\Gamma^{0ph}_{\pi^+\to\ell^+\nu} = \left|A^0\right|^2 + 2e^2\left|A^0 A^1\right| + \mathcal{O}\left(e^4\right)$$

$A_0 =$ 

$A_1 =$  $+$ 

$+ \ldots$

IR DIVERGENT

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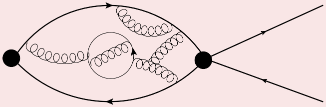
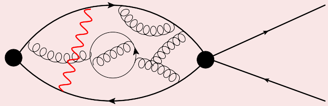
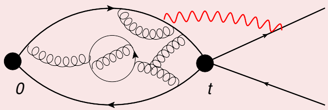
$A_0 =$ 

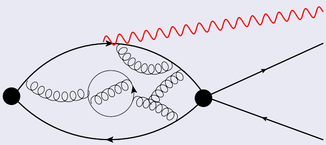
$A_1 =$  $+$ 

$+ \ldots$

IR DIVERGENT

One photon in the final state, $\mathcal{O}\left(e^2\right)$

$$\Gamma^{1ph}_{\pi^+\to\ell^+\nu\gamma} = e^2\left|B^0\right|^2$$

$B_0 =$ 

Again, IR DIVERGENT

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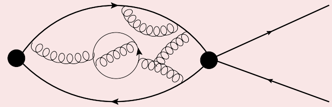
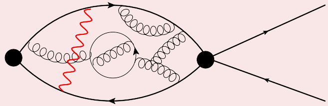
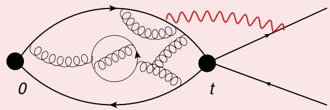
$A_0 =$ 

$A_1 =$  $+$ 

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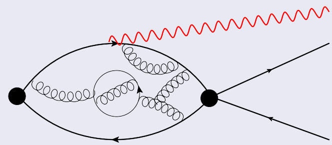
$B_0 =$ 

Again, IR DIVERGENT

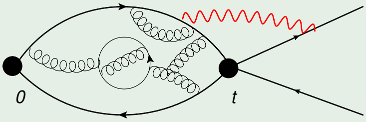
Solution

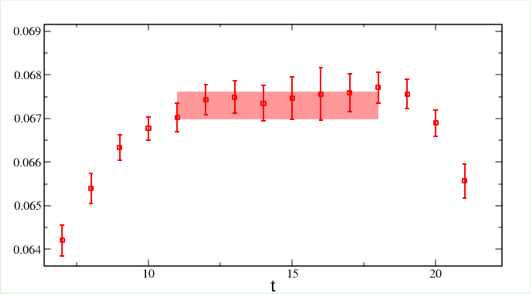
[Bloch and Nordsieck, PR52 (1937)]

$$\Gamma = \Gamma^{0ph} + \Gamma^{1ph} \text{ is } \underline{\text{finite}}$$

Virtual photon

- Needs to implement leptons
- Not too demanding numerically.





The strategy

[N.Carrasco et al., PRD91 (2015)]

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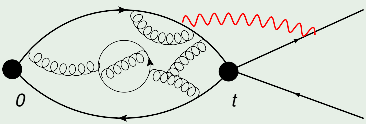
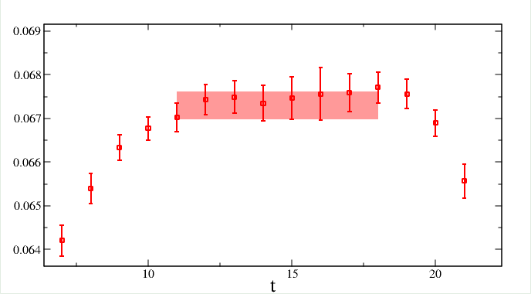
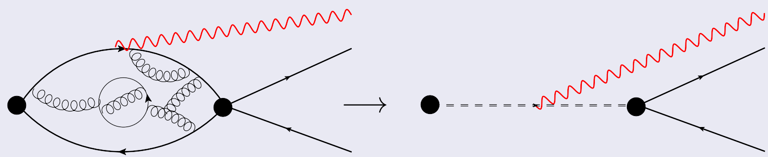




Real photon

- Slightly more numerically demanding/different process
- For the time being, use point-like approximation and consider $E_\gamma < 20$ MeV

$B_0 =$



Cut-off appropriate experimentally (γ detector sensitivity) and theoretically (π inner structure)

(Work is in progress to compute on the Lattice)

Intermediate step

$$\Gamma\left(\Delta E_{\gamma}\right) = \underbrace{\Gamma^{0ph}}_{\text{lattice in a box}} + \underbrace{\Gamma_{pt}^{1ph}\left(\Delta E_{\gamma}\right)}_{\text{perturbation theory, massive photon}}$$

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In our strategy

To ensure proper cancellation of IR divergence with different regulator, add and subtract Γ^0_{pt}

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$\Gamma^0_{pt,V}$: V.Lubicz et al, Phys.Rev. D95 (2017)

- Perturbation theory with pointlike pion in finite volume ($V = L^4$)
- IR divergences $\propto \log L$ cancel in the difference $\Gamma^{0ph}(L) - \Gamma^0_{pt}(L)$ (fit the remainder)
- Also $1/L$ corrections are universal and cancel in the difference.

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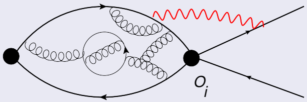
$\lim_{m_\gamma\to 0}\left[\Gamma^0_{pt}(m_\gamma) + \Gamma^{1ph}_{pt}(m_\gamma, \Delta E_\gamma)\right]$: N.Carrasco et al. Phys.Rev. D91 (2015)

- Perturbation theory with pointlike pion with finite photon mass
- Neglected structure dependence: estimated to be small (V.Cirigliano, I.Rosell, JHEP'07)
  This might not be the case for D or B mesons ($m_{B^*} - m_B \sim 45$ MeV)
- Reproduce the the total rate (Berman, PRL 1958, and Kinoshita, PRL 1959).

Matching to the "real world"

Correlation functions computed with bare operators



Needs renormalization: $O_i^{ren} = Z_{ij} O_j^{bare}$

$$O_{1,2} = (V \mp A)_q \otimes (V - A)_\ell$$
$$O_{3,4} = (S \mp P)_q \otimes (S - P)_\ell$$
$$O_5 = \left(T + \tilde{T}\right)_q \otimes \left(T + \tilde{T}\right)_\ell$$

RI-MOM (no QED)

- Compute amputated green functions:

$$\Lambda_O(p) = S^{-1}(p) \left\langle \sum_{x,y} e^{-ip(x-y)} \psi(x)\, O(0)\, \psi(y) \right\rangle S^{-1}(p)$$

- Impose RI-MOM condition at given p^2 (average all equivalent momenta)

$$\boldsymbol{Z_O} = \frac{Z_q}{\mathrm{Tr}\left[\Lambda_O(p)\, \Lambda_O^{tree}(p)^{-1}\right]}$$

- Chiral extrapolate $m \to 0$

Matching to the "real world" (continued)

RI-MOM with QED

- As a first step [D.Giusti et al., PRL '18]: RI-MOM for QCD + perturbation theory for QED
- In the coming-soon paper: RI-(S)MOM for QCD + QED

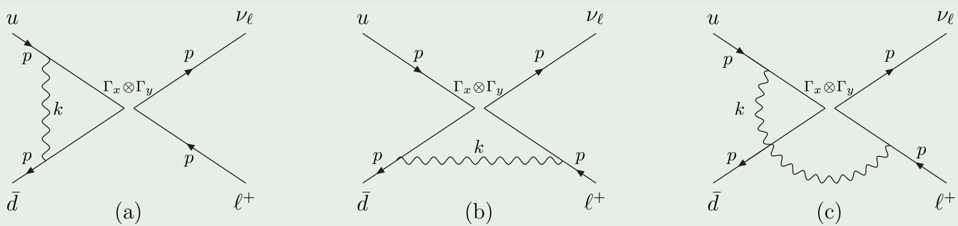
RI-MOM, perturbative expansion: ratio with QCD and QED

$$\frac{\boldsymbol{\delta Z_O^{QED+QCD}}}{\boldsymbol{Z_O^{QCD}\,Z_O^{QED}}} = \frac{\delta Z_q^{QCD+QED}}{Z_q^{QCD}\,Z_q^{QED}} - \frac{\mathrm{Tr}\left[\delta\Lambda_O^{QCD+QED}(p)\,\Lambda_O^{tree}(p)^{-1}\right]}{\mathrm{Tr}\left[\Lambda_O^{QCD}(p)\,\Lambda_O^{tree}(p)^{-1}\right]\mathrm{Tr}\left[\Lambda_O^{QED}(p)\,\Lambda_O^{tree}(p)^{-1}\right]}$$

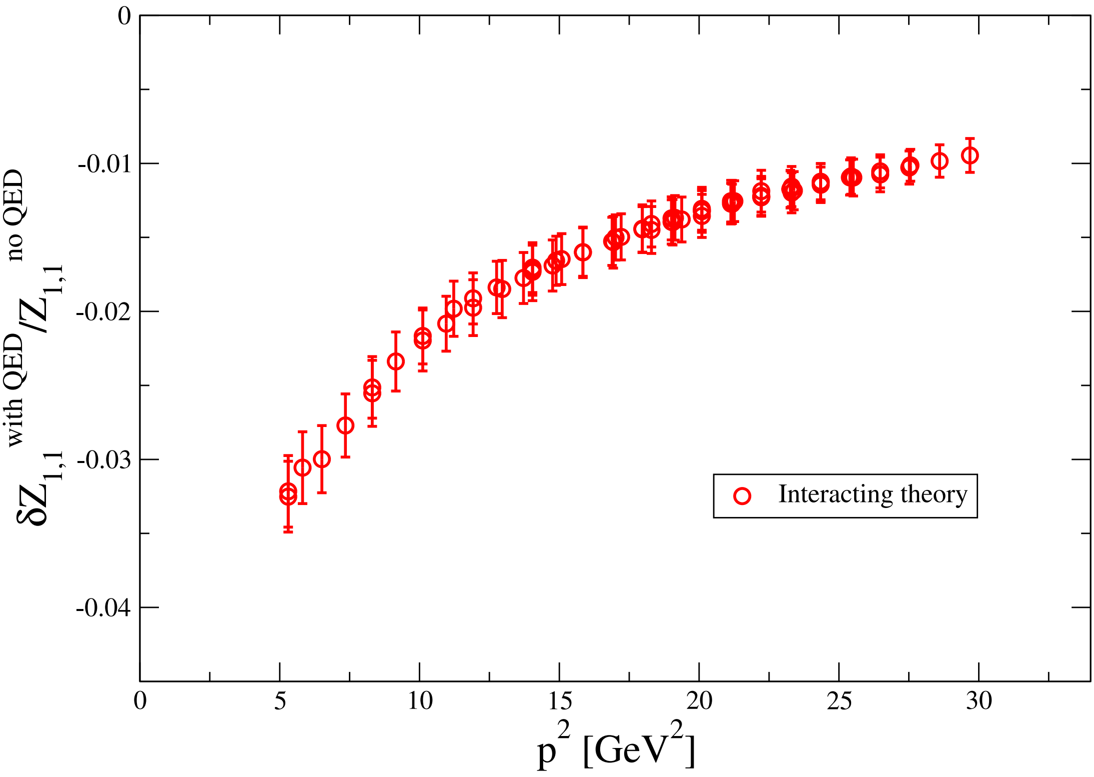
Large cancellation of cut-off effects, anomalous dimensions, noise, etc

Measure of the non-factorizability of the renormalization constants.

Vertices (with or without gluons, not drawn)



An example: QED correction to $Z_{1,1}$



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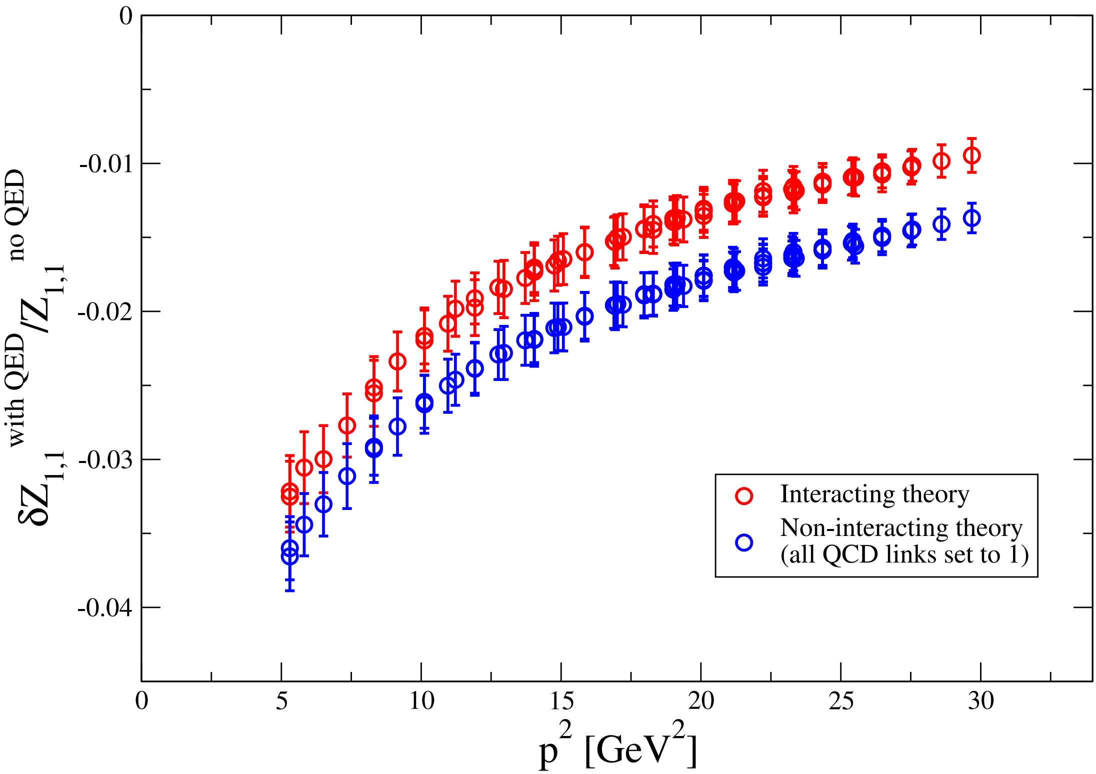

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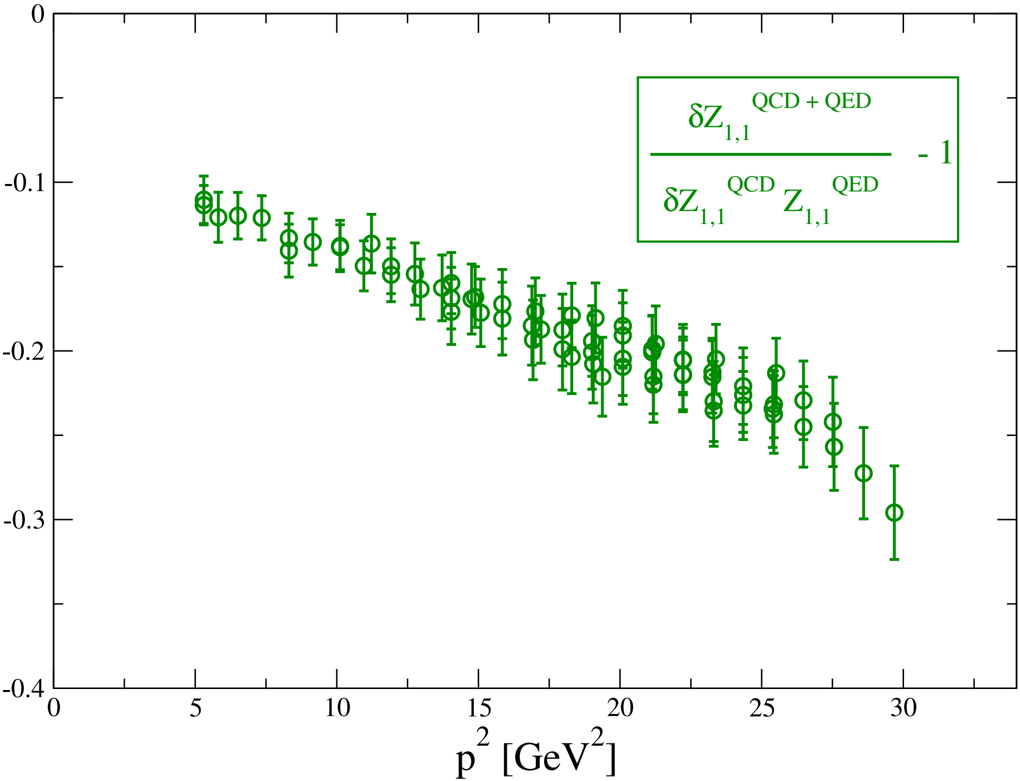

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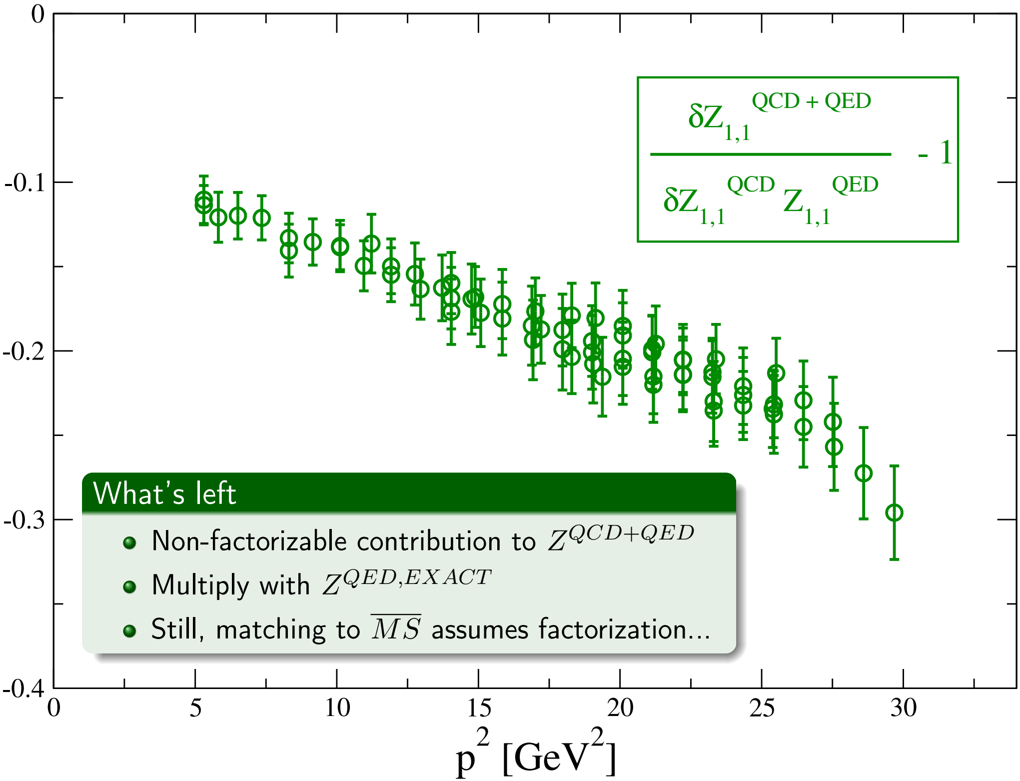


What's left

- Non-factorizable contribution to $Z^{QCD+QED}$
- Multiply with $Z^{QED,EXACT}$
- Still, matching to $\overline{MS}$ assumes factorization...

Matching to W-reg

Effective theory of Weak interaction

Remember that this computation is done in the Weak interaction effective theorys

$$H_{eff} = \frac{1}{\sqrt{2}} \underbrace{G_F}_{\text{muon decay}} V_{ij}^{CKM} \underbrace{\left(1 + \frac{\alpha}{\pi} \log \frac{M_Z}{M_W}\right)}_{\text{SM-Fermi theory matching}} \underbrace{O_1}_{\text{W-reg}}$$

W-reg

Historically, the scheme in which the loop corrections are computed

$$G_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{k^2} \to \frac{M_W^2}{M_W^2 - k^2} \frac{\delta_{\mu\nu}}{k^2}$$

cannot be implemented on the lattice $M_W \gg \frac{1}{a}$

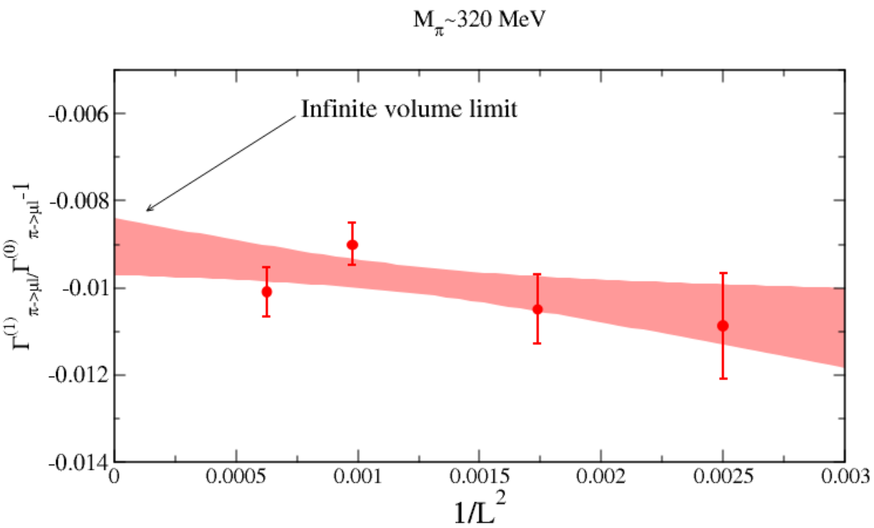
Status

- Matching from RI-MOM to W-reg computed at 1-loop
- Possibly 2-loops in the future?

Infinite volume extrapolation

Volume dependence

- IR divergences $\propto \log L$ cancel in the difference $\Gamma^{0ph}(L) - \Gamma^{0}_{pt}(L)$
- $1/L$ cancel as well (Ward identity)
- Best fit with $1/L^2$ (and $1/L^3$) and extrapolate to $L \to \infty$



Let's start from a slightly simpler quantity

QED contribution to ratio of decay width of Kaon and Pion

$$\frac{\Gamma_{K^+\to\ell^+\nu(\gamma)}}{\Gamma_{\pi^+\to\ell^+\nu(\gamma)}} = \frac{\Gamma_{K^+\to\ell^+\nu}}{\Gamma_{\pi^+\to\ell^+\nu}}\left(1+\delta R_{K\pi}\right), \qquad \delta R_{K\pi} = \delta R_K - \delta R_\pi$$

- Reduction of noise
- Large cancellation of renormalization correction
- Suppression of finite volume dependence.

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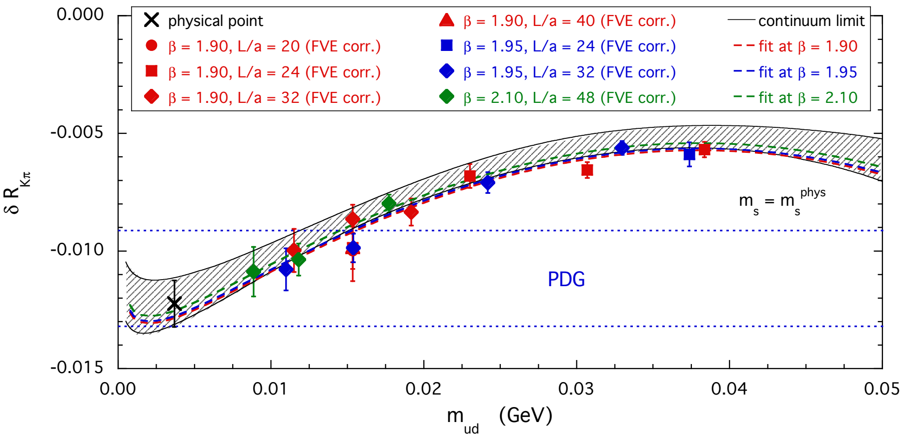
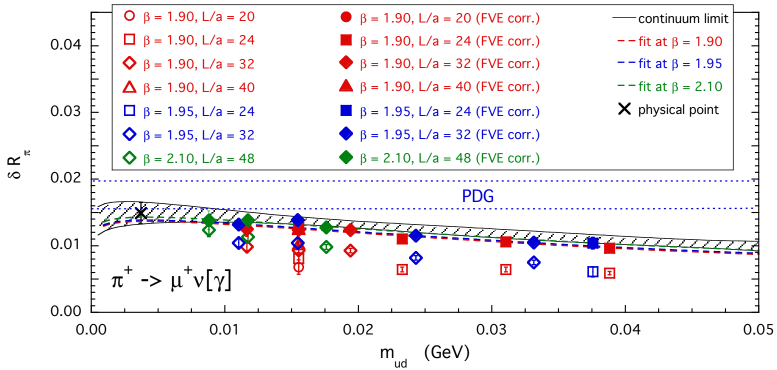
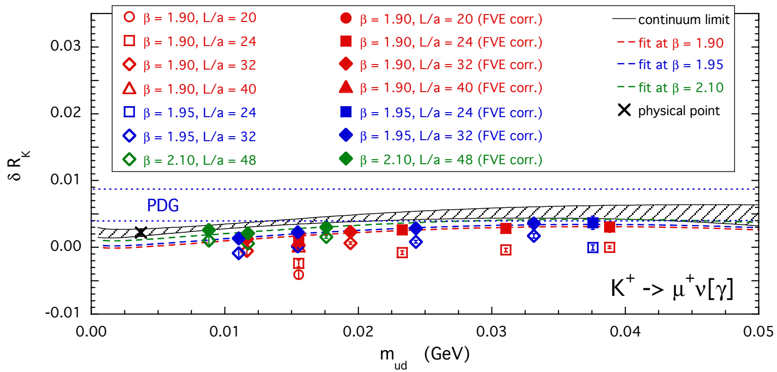


[D.Giusti et al., Phys. Rev. Lett. 120, 072001 (2018)]

Separate Pion and Kaon corrections [PRELIMINARY]





Development

Tomorrow (so to say)

- Finalizing the nonperturbative renormalization with QCD+QED
- Provide final results for δR_K and δR_π

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The day after tomorrow

- Including fermionic-disconnected diagrams
- Lattice calculation of real emission
- Heavy mesons
- $g-2$ hadronic vacuum polarization ($\mathcal{O}\left(1\%\right)$... sleep in peace)

Development

Tomorrow (so to say)

- Finalizing the nonperturbative renormalization with QCD+QED
- Provide final results for δR_K and δR_π

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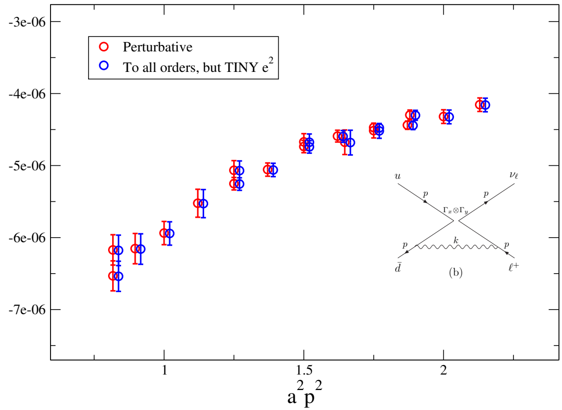
Maybe one day

- Develop a strategy for semileptonic decays (analytic continuation to Minkowsky)
- Corrections to $K \to \pi\pi$
- ...

THANK YOU!

Can't we compute this with the "to all order" approach?

Stochastic = Put the photons in the links $U_{x,\mu}^{QCD} \to U_{x,\mu}^{QCD} \exp(ieA_{x,\mu})$



In the quenched QED approximation

- Can be used to isolate $\propto e^2$ contribution
- $\frac{O(+e)+O(-e)}{2e^2} \xrightarrow{e\to 0} \partial_{e^2} O(e)|_{e=0}$ **"numerical calculation of derivative"**
- strictly the same cost, for 2pts
- easier & cheaper for higher correlations!?
- needs some more investigations

What if you don't take $e \to 0$?

- Higher orders are kept in the calculation
- Can be fine if the observable is not pathological
- Extrapolating has little cost...

Unquenched QED

reweighting: can be used to compute disconnected diagrams

simulations: no easy way to to keep correlation of two independent runs