Isospin Breaking Effects on the Lattice

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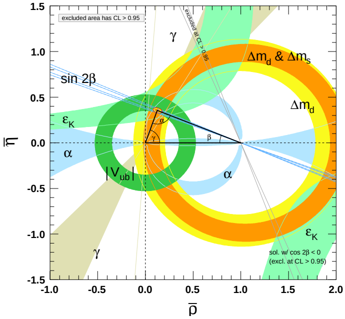
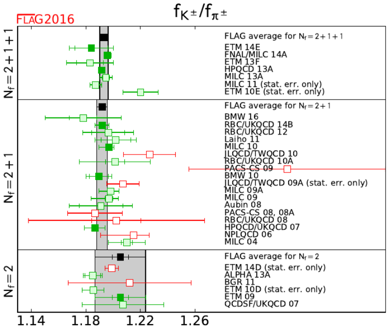
Motivation

- calculation of several quantities in lattice QCD reaching precision of $\lesssim$ **1%**
- e.g. flavor physics



[PDG]



[FLAG]

- e.g. Hadronic Vacuum Polarization contribution to $\mathbf{a}_\mu$ aiming at **1%**

Motivation

- calculations usually done in isospin symmetric limit (treat **u** and **d** as equal)
- two sources of isospin breaking effects
  - different masses for up- and down quark (of $\mathcal{O}((\mathbf{m_d} - \mathbf{m_u})/\Lambda_{\mathrm{QCD}})$)
  - Quarks have electrical charge (of $\mathcal{O}(\alpha)$)
- lattice calculation aiming at **1%** precision requires to include isospin breaking

- Status of calculations including IB corrections on the lattice
  - IB corrections to hadron masses
    [e.g. S. Borsanyi *et al.* Phys. Rev. Lett. **111** (2013) 252001; G. M. de Divitiis *et al.* Phys. Rev. **D87** (2013) 114505; S. Borsanyi *et al.*Science **347** (2015) 1452; R. Horsley *et al.* J. Phys. **G43** (2016) 10LT02; R. Horsley *et al.* JHEP **04** (2016) 093; S. Basek *et al.* PoS LATTICE2015 (2016) 259; Z. Fodor *et al.* Phys. Rev. Lett. **117** (2016) 082001; D. Giusti *et al.* Phys.Rev. **D95** (2017) 114504; V.G. *et al.*,JHEP 09, 153 (2017)]
  - First calculations of IB corrections to hadronic vacuum polarization
    [V.G. *et al.*,JHEP 09, 153 (2017); D. Giusti *et al.*, JHEP **10** 157 (2017); B. Chakraborty et al. Phys. Rev. Lett. **120** 152001 (2018); C. Lehner, V.G. *et al.* arXiv:1801.07224], [see also talk by M. Della Morte]
  - QED corrections to pion/kaon decay rates
    [N. Carrasco *et al.* Phys. Rev. **D91** (2015) 074506; V. Lubicz *et al.* Phys. Rev. **D95** (2017) 034504; D. Guisti *et al.* Phys.Rev.Lett. 120 (2018) no.7, 072001], [see also talk by F. Sanfilippo]

Outline

- Including Isospin Breaking effects on the lattice
  - stochastic method
  - perturbative method
  - direct comparison of results [V.G. et al.,JHEP 09, 153 (2017)]
    - meson masses
    - HVP
  - strong IB
- IB Corrections to HVP at the physical point [C. Lehner, V.G. *et al.* arXiv:1801.07224]
- Finite volume corrections for HVP
- Summary

Outline

QED on the lattice

- Euclidean path integral including QED

$$\langle O\rangle = \frac{1}{Z}\int \mathcal{D}[\Psi,\overline{\Psi}]\mathcal{D}[U]\mathcal{D}[A]\; O\; e^{-S_F[\Psi,\overline{\Psi},U,A]}\, e^{-S_G[U]}\, e^{-S_\gamma[A]}$$

- non-compact photon action

$$S_\gamma[A] = \frac{a^4}{4}\sum_x\sum_{\mu,\nu}\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)^2$$

- two approaches for including QED
  - stochastic QED using **U(1)** gauge configurations
    [A. Duncan, E. Eichten, H. Thacker, Phys.Rev.Lett. **76**, 3894 (1996)]
  - perturbative QED by expanding the path integral in α
    [RM123 Collaboration, Phys.Rev. **D87**, 114505 (2013)]

$$\langle O\rangle = \langle O\rangle_0 + \frac{1}{2}e^2\,\frac{\partial^2}{\partial e^2}\langle O\rangle\Big|_{e=0} + \mathcal{O}(\alpha^2)$$

stochastic method

- Feynman gauge

$$S_\gamma^{\text{Feyn}}[A] = S_\gamma[A] + \frac{1}{2}\sum_x \left(\sum_\mu \partial_\mu A_\mu(x)\right)^2 = -\frac{1}{2}\sum_x \sum_\mu A_\mu(x)\partial^2 A_\mu(x)$$

- in momentum space

$$S_\gamma^{\text{Feyn}}[A] = \frac{1}{2N}\sum_{k,\,\vec{k}\neq 0} \hat{k}^2 \sum_\mu \left|\tilde{A}_\mu(k)\right|^2 \qquad \hat{k}_\mu = 2\sin\left(\frac{k_\mu}{2}\right)$$

with **N** number of lattice sites

- remove photon zero mode
(e.g. spatial zero-modes $\longrightarrow$ QED_L [S. Uno and M. Hayakawa, Prog. Theor. Phys. **120**, 413 (2008)])
- draw $\tilde{A}_\mu(k)$ from Gaussian distribution with variance $2N/\hat{k}^2$
- Fourier Transform to position space

stochastic method

- multiply **SU(3)** gauge links with **U(1)** photon fields

$$\mathbf{U}_\mu(\mathbf{x}) \to \mathbf{e}^{\mathbf{ieA}_\mu(\mathbf{x})}\mathbf{U}_\mu(\mathbf{x})$$

- calculate hadronic observable as without QED
- remove $\mathcal{O}(\mathbf{e})$ noise by averaging over $+\mathbf{e}$ and $-\mathbf{e}$
  [T. Blum et al., Phys. Rev. **D76** (2007) 114508]

- electro quenched approximation
  $\to$ sea quarks electrically neutral
  $\to$ QED configurations generated independently of QCD configurations
- unquenched calculation
  $\to$ generate combined QED+QCD configurations

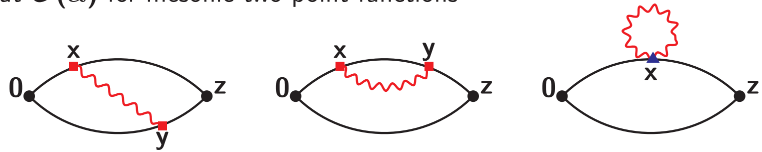
- QED correction to all orders in α

perturbative method

- expand path integral in α [RM123 Collaboration, Phys.Rev. **D87**, 114505 (2013)]

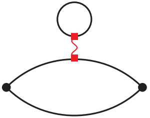
$$\langle O \rangle = \langle O \rangle_0 + \frac{1}{2} e^2 \frac{\partial^2}{\partial e^2} \langle O \rangle \bigg|_{e=0} + \mathcal{O}(\alpha^2)$$

- at $\mathcal{O}(\alpha)$ for mesonic two-point functions



■ conserved vector current, ▲ tadpole operator

- electro-quenched: no disconnected diagrams like



- e.g. photon exchange for a charged Kaon

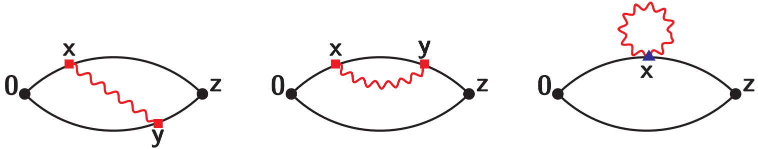
$$C(z_0) = \sum_{\mu,\nu} \sum_{z} \sum_{x,y} \mathrm{Tr}\left[S(z,x)\Gamma^c_\mu S(x,0)\gamma_5 S(0,y)\Gamma^c_\nu S(y,z)\gamma_5 \right] \Delta_{\mu\nu}(x-y)$$

perturbative method

- photon propagator Feynman gauge

$$\Delta_{\mu\nu}(x-y) = \delta_{\mu\nu} \frac{1}{N} \sum_{k,\vec{k}\neq 0} \frac{e^{ik\cdot(x-y)}}{\hat{k}^2}$$

- remove all spatial zero modes → QED_L
  [S. Uno and M. Hayakawa, Prog. Theor. Phys. **120**, 413–441 (2008)]



- calculate diagrams using sequential propagators
- factorize photon propagator $\Delta_{\mu\nu}(x-y) = f(x)g(y)$

perturbative method

- method 1: $\left\langle \eta(\mathbf{u})\eta^\dagger(\mathbf{y})\right\rangle_\eta = \delta_{\mathbf{uy}}$

$$\boldsymbol{\Delta}_{\mu\nu}(\mathbf{x}-\mathbf{y}) = \left\langle \sum_{\mathbf{u}} \boldsymbol{\Delta}_{\mu\nu}(\mathbf{x}-\mathbf{u})\eta(\mathbf{u})\eta^\dagger(\mathbf{y})\right\rangle_\eta = \left\langle \tilde{\boldsymbol{\Delta}}_{\mu\nu}(\mathbf{x})\eta^\dagger(\mathbf{y})\right\rangle_\eta$$

$\rightarrow$ sequential sources for every combination of $\{\mu,\nu\}$, e.g.

$\rightarrow$ **17** inversions in Feynman gauge

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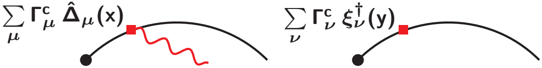
- method 2: $\left\langle \xi_\sigma(\mathbf{u})\xi^\dagger_\nu(\mathbf{y})\right\rangle_\xi = \delta_{\mathbf{uy}}\delta_{\sigma\nu}$ [RM123, Phys.Rev. D87, 114505 (2013)]

$$\boldsymbol{\Delta}_{\mu\nu}(\mathbf{x}-\mathbf{y}) = \left\langle \sum_{\mathbf{u}}\sum_{\sigma} \boldsymbol{\Delta}_{\mu\sigma}(\mathbf{x}-\mathbf{u})\xi_\sigma(\mathbf{u})\xi^\dagger_\nu(\mathbf{y}) \right\rangle_\xi = \left\langle \hat{\boldsymbol{\Delta}}_{\mu}(\mathbf{x})\xi^\dagger_\nu(\mathbf{y}) \right\rangle_\xi$$

$\rightarrow$ sequential sources summed over μ or ν, e.g.



$\rightarrow$ **5** inversions

Stochastic vs Perturbative method

- stochastic method

- perturbative method

Stochastic vs Perturbative method

- stochastic method
  - QED corrections to all orders in α
- perturbative method
  - QED corrections at fixed order in $\mathcal{O}(\alpha)$

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  - QED corrections to all orders in α
  - once the stochastic $\mathbf{U(1)}$ fields are generated, calculation proceeds without QED $\longrightarrow$ computationally cheaper than perturbative method
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  - calculation more involved, requires three- and four-point functions, convolution with photon propagator $\longrightarrow$ more expensive than stochastic method

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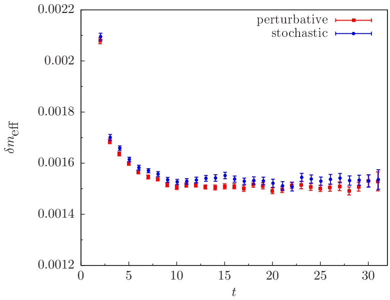
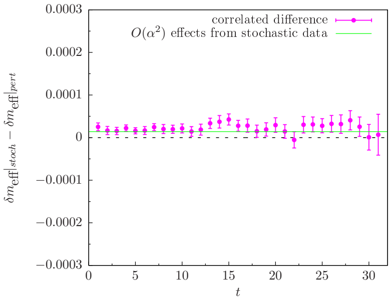
- direct comparison of results and statistical errors for QED corrections to meson masses and hadronic vacuum polarization [V.G. *et al.*, JHEP 1709 (2017) 153]

Direct Comparison of Results - Meson Masses

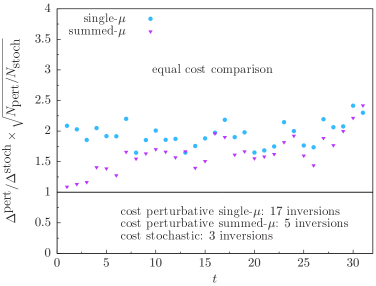
- $\mathbf{N_f = 2 + 1}$ Domain Wall Fermions, $\mathbf{24^3 \times 64}$ lattice, $\mathbf{a^{-1} = 1.78}$ GeV
- isospin symmetric pion mass $\mathbf{m_\pi = 340}$ MeV
- QED correction to effective mass:
  - stochastic:
    $$\delta m_{\text{eff}}^{\text{cosh}}(t) = m_{\text{eff}}(t) - m_{\text{eff}}^{0}(t)$$
  - perturbative:
    $$\delta m_{\text{eff}}^{\text{ratio}}(t) = \frac{\delta C(t)}{C_0(t)} - \frac{\delta C(t+1)}{C_0(t+1)} \qquad (\times \text{corr for periodic boundary})$$





Comparison of statistical errors

- computational cost
  - perturbative: **17** inversions per quark flavor for single-μ insertion
    5 inversions per quark flavor for summed-μ insertion
  - stochastic: **3** inversions per quark flavor
- statistical error $\mathbf{\Delta}$ of QED contribution to effective Kaon mass
- scaled by $\sqrt{\#\,\text{inversions}}$ (equal cost comparison)



stochastic method gives **1.5 − 2** times smaller statistical errors for same cost

Perturbative Expansion HVP

- Vector-Vector correlation function

$$C_{\mu\nu}(x) = \langle V_\mu(x) V_\nu(0) \rangle$$

- HVP tensor

$$\Pi_{\mu\nu}(Q) = \sum_x e^{-iQ\cdot x} C_{\mu\nu}(x)$$

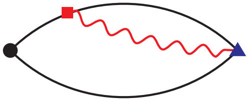
- conserved vector current depends on link variables

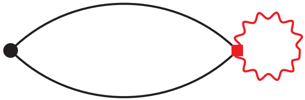
$$U_\mu(x) \to e^{ieA_\mu(x)} U_\mu(x) \quad \text{and thus} \quad V^c_\mu(x) \to V^{c,e}_\mu(x)$$

- perturbative expansion

$$\left\langle V^{c,e}_\mu(x) V^\ell_\nu(0) \right\rangle = \left\langle V^c_\mu(x) V^\ell_\nu(0) \right\rangle_0 + \frac{1}{2} e^2 \frac{\partial^2}{\partial e^2} \left\langle V^{c,e}_\mu(x) V^\ell_\nu(0) \right\rangle \Big|_{e=0}$$

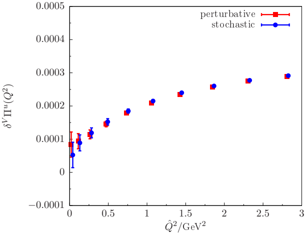
- two additional types of diagrams

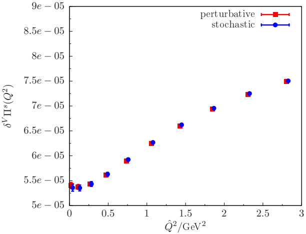




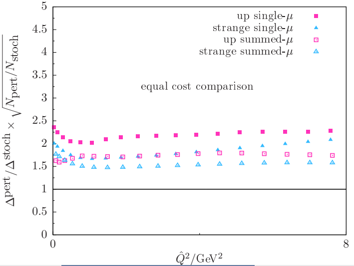
Direct Comparison of Results - HVP

- QED correction to hadronic vacuum polarisation





- $\delta a_\mu < 1\%$ for **u** quarks
- comparision of statistical errors



"Combining" stochastic and perturbative method

- write photon propagator as

$$\Delta_{\mu\nu}(x-y) = \langle A_\mu(x) A_\nu(y) \rangle$$

- use stochastic photon fields $A_\mu(x)$ to estimate $\Delta_{\mu\nu}(x-y)$ [D. Giusti et al. Phys.Rev. D95 (2017) 114504]
- quark - photon vertex insertions of

$$\Gamma^c_\mu A_\mu(x)$$

- path integral

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\Psi, \overline{\Psi}] \mathcal{D}[U] \mathcal{D}[A] \; O \; e^{-S_F[\Psi,\overline{\Psi},U,A]} \, e^{-S_G[U]} \, e^{-S_\gamma[A]}$$

- using the same stochastic photon fields as for stochastic method gives exact $\mathcal{O}(\alpha)$-truncation of results from stochastic method
- 4 inversions

strong Isospin Breaking

- different bare quark masses for up- and down quark

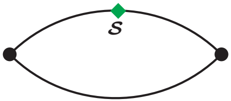
- expansion in $\mathbf{\Delta m = (m_u - m_d)}$ [G.M. de Divitiis *et al*, JHEP 1204 (2012) 124]

$$\langle O \rangle_{m_u \neq m_d} = \langle O \rangle_{m_u = m_d} + \Delta m \, \frac{\partial}{\partial m} \langle O \rangle \Big|_{m_u = m_d} + \mathcal{O}\left(\Delta m^2\right)$$

with

$$\frac{\partial}{\partial m} \langle O \rangle \Big|_{m_u = m_d} = \langle O \, \mathcal{S} \rangle_{m_u = m_d}$$

- scalar current $\mathcal{S} = \sum_x \overline{\psi}(x) \, \psi(x)$



- quark mass tuning at the physical point, e.g. by fixing masses of charged pion, charged and neutral kaon

Outline

Muon a_μ and the hadronic vacuum polarisation (HVP)

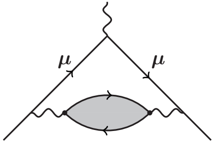
- experiment: polarized muons in a magnetic field [Bennet et al., Phys.Rev. **D73**, 072003 (2006)]

$$a_\mu = 11659208.9(5.4)(3.3) \times 10^{-10}$$

- Standard Model [PDG]

$$a_\mu = 11659180.3(0.1)(4.2)(2.6) \times 10^{-10}$$

- Comparison of theory and experiment: 3.6σ deviation
- largest error on SM estimate from HVP

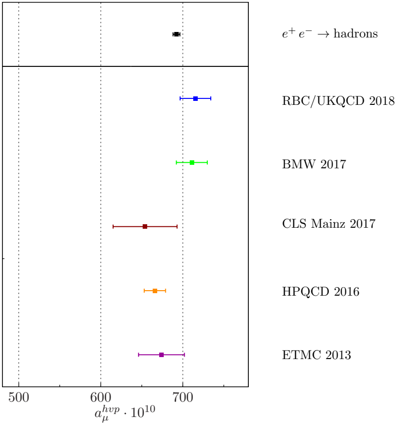


- current best estimate from $e^+e^- \to$ hadrons [Davier et al., Eur.Phys.J. **C71**, 1515 (2011)]

$$(692.3 \pm 4.2 \pm 0.3) \times 10^{-10}$$

HVP from the **R**-ratio ↔ Lattice

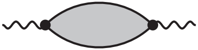
- see also talks on g-2 yesterday



- result using **R**-ratio $\mathbf{a_\mu^{hvp} = (692.3 \pm 4.2 \pm 0.3) \times 10^{-10}}$
- lattice result to be competitive with **R**-ratio requires precision of $\lesssim$ **1%**
  → Isospin Breaking Corrections

Status IB corrections to HVP

- Isospin Breaking corrections to HVP



- QED corrections to HVP:
  - unphysical quark masses [V.G. *et al.*,JHEP **09**, 153 (2017)]
  - strange, charm; extrapolated to physical quark masses [D. Giusti *et al.*, JHEP **10**, 157 (2017)]
  - directly at physical quark masses [C. Lehner, V.G. *et al.* arXiv:1801.07224]

- strong IB corrections to HVP:
  - unphysical quark masses [V.G. *et al.*, JHEP **09**, 153 (2017)]
  - directly at physical quark masses, $\mathbf{N_f = 1 + 1 + 1 + 1}$ [B. Chakraborty *et al.* Phys. Rev. Lett. **120** 152001 (2018)]
  - directly at physical quark masses [C. Lehner, V.G. *et al.* arXiv:1801.07224]

IB corrections to HVP at physical point

- see [C. Lehner, V.G. *et al.* arXiv:1801.07224]

- $\mathbf{N_f = 2 + 1}$ Möbius DWF, $\mathbf{48^3 \times 96}$ lattice, $\mathbf{a^{-1} = 1.730(4)}$ GeV
  [T. Blum *et al.* Phys.Rev. *D93* (2016) no.7, 074505]
- IB corrections from expansion around isospin symmetric calculation

$$\mathbf{C(t) = C^0(t) + \alpha C^{\mathrm{QED}}(t) + \sum_f \Delta m_f C^{\Delta m_f}(t)}$$

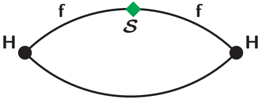
- QED corrections using perturbative method with stochastic photon fields
- isospin symmetric calculation using quark masses determined without QED
  [T. Blum *et al.* Phys.Rev. *D93* (2016) no.7, 074505]
- physical quark masses including QED:
  - tune (**u**, **d**, **s**) quark masses to physical values including QED
    $\rightarrow$ in addition: fix lattice spacing with QED
  - use these tuned masses and perturbative expansion in mass

Tuning the quark masses

- tune $(\mathbf{u},\mathbf{d},\mathbf{s})$ masses to reproduce experimental π^+, $\mathbf{K}^+$ and $\mathbf{K_0}$ mass (and check π^0 mass)

$$\mathbf{a\,m}_{\pi^+}^{\mathrm{exp}} = \left[\hat{\mathbf{m}}_{\pi^+} + \delta^{\mathrm{QED}}\mathbf{m}_{\pi^+} + (\Delta\mathbf{m_d} + \Delta\mathbf{m_u})\ \delta^{\mathrm{sIB},\ell}\mathbf{m}_{\pi^+}\right]$$
$$\mathbf{a\,m}_{\mathbf{K}^+}^{\mathrm{exp}} = \left[\hat{\mathbf{m}}_{\mathbf{K}^+} + \delta^{\mathrm{QED}}\mathbf{m}_{\mathbf{K}^+} + \Delta\mathbf{m_u}\ \delta^{\mathrm{sIB},\ell}\mathbf{m}_{\mathbf{K}^+} + \Delta\mathbf{m_s}\ \delta^{\mathrm{sIB},\mathbf{s}}\mathbf{m}_{\mathbf{K}^+}\right]$$
$$\mathbf{a\,m}_{\mathbf{K}^0}^{\mathrm{exp}} = \left[\hat{\mathbf{m}}_{\mathbf{K}^0} + \delta^{\mathrm{QED}}\mathbf{m}_{\mathbf{K}^0} + \Delta\mathbf{m_d}\ \delta^{\mathrm{sIB},\ell}\mathbf{m}_{\mathbf{K}^0} + \Delta\mathbf{m_s}\ \delta^{\mathrm{sIB},\mathbf{s}}\mathbf{m}_{\mathbf{K}^0}\right]$$

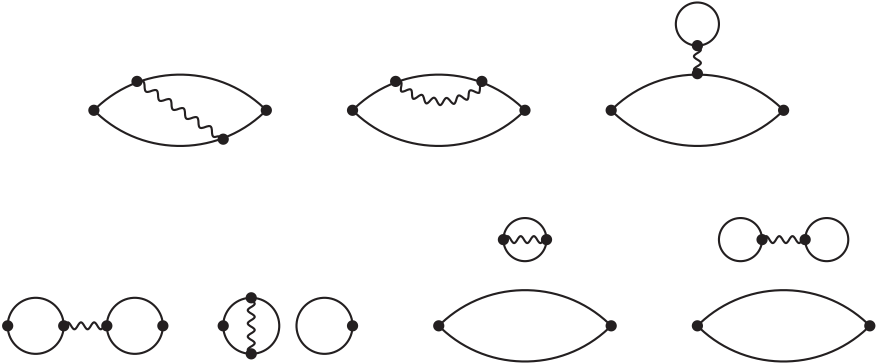
- $\hat{\mathbf{m}}_{\mathbf{H}}$: isospin symmetric mass of $\mathbf{H}$, $\delta^{\mathrm{QED}}\mathbf{m}_{\mathbf{H}}$: QED correction to mass of $\mathbf{H}$
- $\delta^{\mathrm{sIB},\mathbf{f}}\mathbf{m}_{\mathbf{H}}$ from



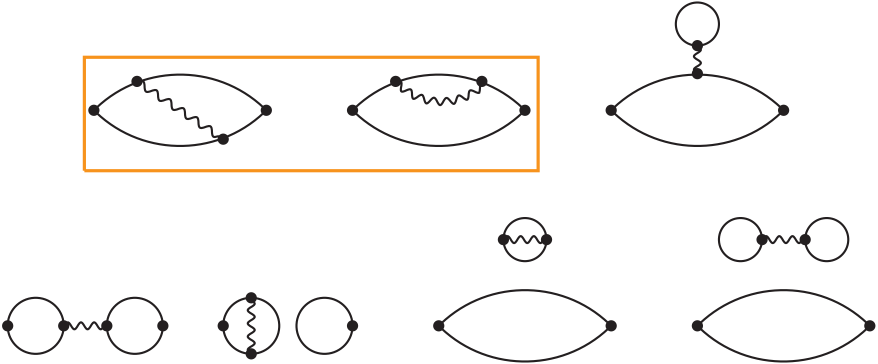
- lattice spacing: fix another mass including QED
  $\rightarrow$ here: Omega-Baryon

$$\mathbf{a} \rightarrow \mathbf{a}(\Delta\mathbf{m_s}) = \left(\hat{\mathbf{m}}_{\Omega} + \delta^{\mathrm{QED}}\mathbf{m}_{\Omega} + 3\,\Delta\mathbf{m_s}\,\delta^{\mathrm{sIB},\mathbf{s}}\mathbf{m}_{\Omega}\right)/\mathbf{m}_{\Omega}^{\mathrm{exp}}$$

QED corrections to the HVP



QED corrections to the HVP



- connected

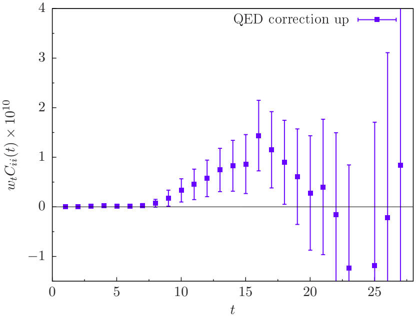
QED corrections to the HVP

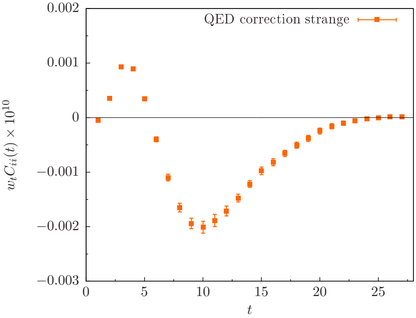
- vector two-point function

$$\mathbf{C}_{\mu\nu}(\mathbf{t}) = \sum_{\vec{\mathbf{x}}} \langle \mathbf{J}_\mu(\mathbf{t}, \vec{\mathbf{x}}) \mathbf{J}_\nu(\mathbf{0}) \rangle$$

- HVP contribution to $\mathbf{a}_\mu$ [Bernecker and Meyer, Eur.Phys.J. A47, 148 (2011); Feng *et al.* Phys.Rev. **D88**, 034505 (2013)]

$$\mathbf{a}_\mu = \sum_{\mathbf{t}} \mathbf{w}_\mathbf{t} \mathbf{C}_{\mathbf{ii}}(\mathbf{t}) \qquad \mathbf{i} = \mathbf{0}, \mathbf{1}, \mathbf{2}$$





QED corrections to the HVP

- Ansatz for $\mathcal{O}(\alpha)$-correction to correlator

$$\delta C(t) = (c_1 + c_0 t) e^{-Et}$$

- lowest lying state w/o QED $\pi\pi$
- lowest lying state with QED $\pi\gamma$
  $\rightarrow$ QED_L: photon has one unit of momentum
- fit data to ansatz with $\mathbf{c_0}$ and $\mathbf{c_1}$ as paramters
- vary **E** between $\pi\gamma$ and $\pi\pi$ $\quad\rightarrow$ systematic error
- result light quarks

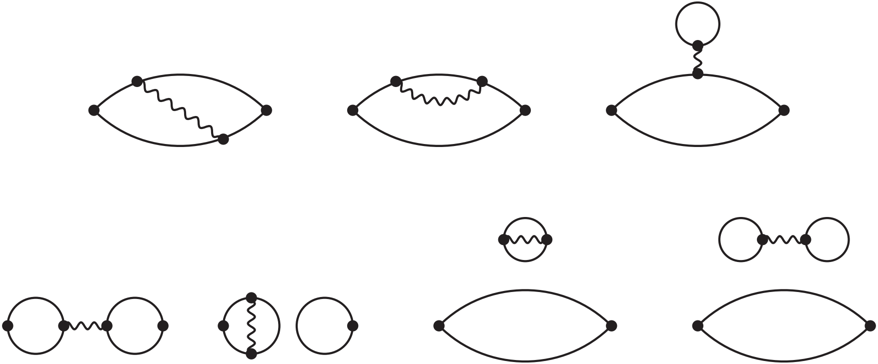
$$a_\mu^{\mathrm{QED},\ell} = 5.9(5.7)(1.7) \times 10^{-10}$$

- results strange quark

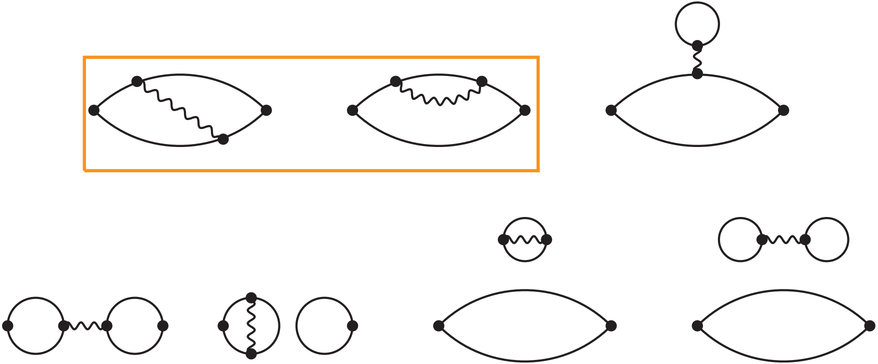
$$a_\mu^{\mathrm{QED,s}} = -0.0149(9)(31) \times 10^{-10}$$

([D. Giusti *et al.*, JHEP **10**, 157 (2017)] $a_\mu^{\mathrm{QED,s}} = -0.018(11) \times 10^{-10}$)

QED corrections to the HVP

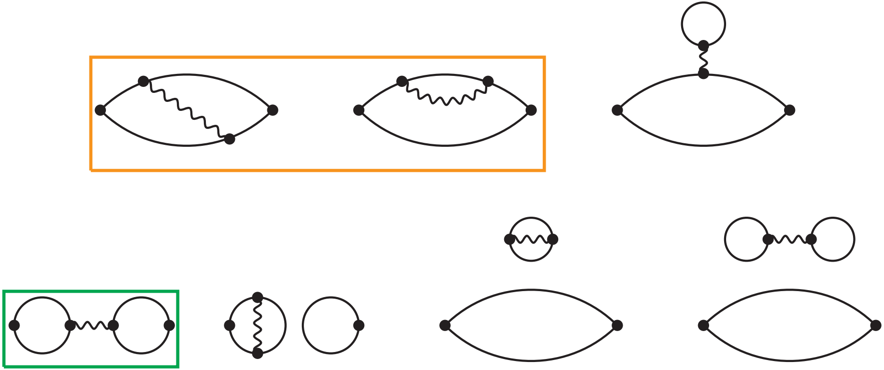


QED corrections to the HVP



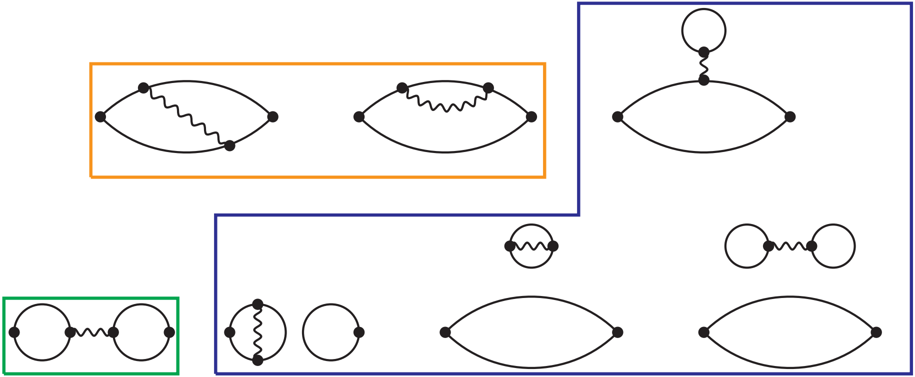
- connected $\mathbf{a}_\mu^{\text{QED, conn}} = \mathbf{5.9(5.7)(1.7)} \times \mathbf{10^{-10}}$

QED corrections to the HVP



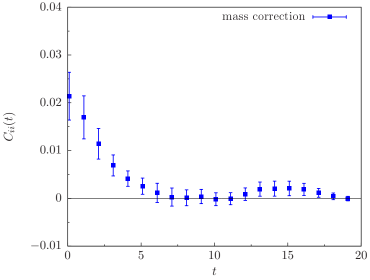
- connected $\mathbf{a}_{\mu}^{\text{QED, conn}} = \mathbf{5.9(5.7)(1.7) \times 10^{-10}}$
- disconnected $\mathbf{a}_{\mu}^{\text{QED, disc}} = \mathbf{-6.9(2.1)(2.7) \times 10^{-10}}$
  using data generated for [T. Blum *et al.* Phys. Rev. Lett. 118, 022005 (2017)]

QED corrections to the HVP



- connected $\mathbf{a}_\mu^{\text{QED, conn}} = \mathbf{5.9(5.7)(1.7)} \times \mathbf{10^{-10}}$
- disconnected $\mathbf{a}_\mu^{\text{QED, disc}} = \mathbf{-6.9(2.1)(2.7)} \times \mathbf{10^{-10}}$
  using data generated for [T. Blum *et al.* Phys. Rev. Lett. 118, 022005 (2017)]
- at least $^1/_{\mathrm{N_c}}$ suppressed

strong Isospin Breaking Corrections to the HVP



- Ansatz

$$\delta C(t) = (c_1 + c_0 t)e^{-Et}$$

- lowest lying state $\pi\pi$
- result

$$a_\mu^{sIB} = 10.6(4.3)(6.8) \times 10^{-10}$$

([B. Chakraborty *et al.* Phys. Rev. Lett. **120** 152001 (2018)] $a_\mu^{sIB} = 9.0(2.3) \times 10^{-10}$)

Outline

QED Finite Volume Corrections

- QED (massless photons) in a finite box with length $\mathbf{L}$
  $\rightarrow$ finite volume (FV) corrections

- QCD: finite volume corrections $\sim \mathbf{e}^{-\mathbf{m}_\pi \mathbf{L}}$
- QED: finite volume corrections $\sim {}^{1}/_{\mathbf{L}^n}$

- can be studied using effective theory, i.e. scalar QED for mesons
- e.g. meson masses with $\text{QED}_\mathbf{L}$ [S. Borsanyi *et al.* Science **347** (2015) 1452]

$$\mathbf{m}^2(\mathbf{L}) \sim \mathbf{m}^2 \left\{ 1 - \mathbf{q}^2 \alpha \left[\frac{\kappa}{\mathbf{mL}} \left(1 + \frac{2}{\mathbf{mL}} \right) \right] \right\} + \mathcal{O}\left(\frac{1}{\mathbf{L}^3} \right)$$

with $\kappa = \mathbf{2.837297}$

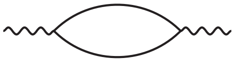
- universal up to $\mathcal{O}\left(\frac{1}{\mathbf{L}^2}\right)$
- [Z. Fodor *et al.* Phys. Rev. Lett. **117** (2016) 082001] $\mathcal{O}\left(\frac{1}{\mathbf{L}^3}\right)$ negligible within errors

- QED corrections to decay amplitudes [V. Lubicz *et al.* Phys. Rev. **D95** (2017) 034504]

QED Finite Volume Corrections for HVP

- analytical calculation for HVP $\longrightarrow$ 2-loop

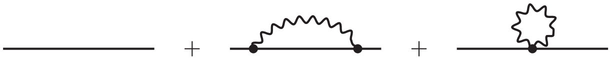
- lattice scalar QED
  $\longrightarrow$ quicker way/cross check for analytical result
- Leading contribution to HVP in effective theory is given by two-pion contribution



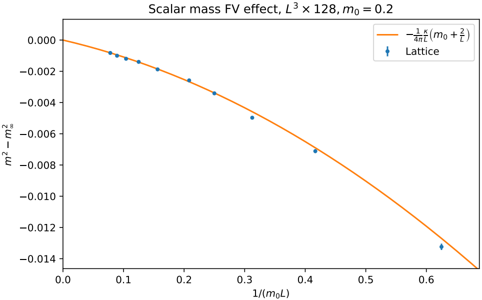
- QED correction $\longrightarrow$ expansion in α
- insertion of stochastic photon fields
- calculate scalar propagators using Fast Fourier Transform (FFT)

Check: FV for hadron masses

- QED correction to meson masses as check
- scalar propagator at $\mathcal{O}(\alpha)$



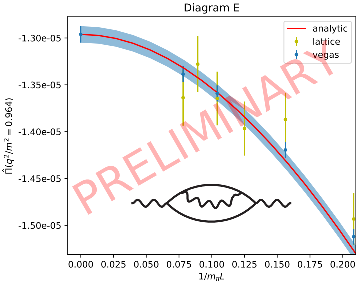
- results [J. Harrison *et al.*, Proceedings Lattice 2017]

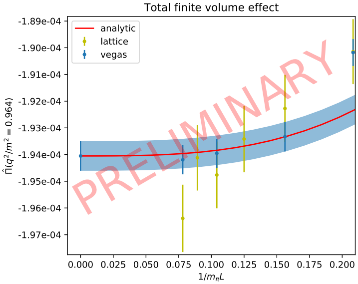


- analytical result from [S. Borsanyi *et al.* Science **347** (2015) 1452], not a fit

Results QED Finite Volume Corrections for HVP

[Plots by T. Janowski]





- numerical results
  - lattice scalar QED calculation [J. Harrison, . . .]
  - lattice PT (vegas) [T. Janowski, . . .]
- analytical results [A. Portelli, J. Bijnens, N. Hermansson Truedsson, T. Janowski, . . .]

Outline

Summary

- Lattice calculations with precision of $\lesssim \mathbf{1\%}$ require inclusion of isospin breaking and QED effects
- challenges for including QED on the lattice
  - photon zero mode [Talk by A. Patella]
  - large finite volume corrections
  - IR divergences for some quantities like kaon/pion decay rate [Talk by F. Sanfilippo]
- comparison stochastic vs perturbative method
- First calculation for IB corrections to HVP at the physical point

Summary

- Lattice calculations with precision of $\lesssim \mathbf{1\%}$ require inclusion of isospin breaking and QED effects
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Thank you!

Outline

Backup

Results HVP window method - total

see [C. Lehner, V.G. et al. arXiv:1801.07224]

$a_\mu^{\text{ud, conn, isospin}}$	$202.9(1.4)_\text{S}(0.2)_\text{C}(0.1)_\text{V}(0.2)_\text{A}(0.2)_\text{Z}$	$649.7(14.2)_\text{S}(2.8)_\text{C}(3.7)_\text{V}(1.5)_\text{A}(0.4)_\text{Z}(0.1)_\text{E48}(0.1)_\text{E64}$
$a_\mu^{\text{s, conn, isospin}}$	$27.0(0.2)_\text{S}(0.0)_\text{C}(0.1)_\text{A}(0.0)_\text{Z}$	$53.2(0.4)_\text{S}(0.0)_\text{C}(0.3)_\text{A}(0.0)_\text{Z}$
$a_\mu^{\text{c, conn, isospin}}$	$3.0(0.0)_\text{S}(0.1)_\text{C}(0.0)_\text{Z}(0.0)_\text{M}$	$14.3(0.0)_\text{S}(0.7)_\text{C}(0.1)_\text{Z}(0.0)_\text{M}$
$a_\mu^{\text{uds, disc, isospin}}$	$-1.0(0.1)_\text{S}(0.0)_\text{C}(0.0)_\text{V}(0.0)_\text{A}(0.0)_\text{Z}$	$-11.2(3.3)_\text{S}(0.4)_\text{V}(2.3)_\text{L}$
$a_\mu^{\text{QED, conn}}$	$0.2(0.2)_\text{S}(0.0)_\text{C}(0.0)_\text{V}(0.0)_\text{A}(0.0)_\text{Z}(0.0)_\text{E}$	$5.9(5.7)_\text{S}(0.3)_\text{C}(1.2)_\text{V}(0.0)_\text{A}(0.0)_\text{Z}(1.1)_\text{E}$
$a_\mu^{\text{QED, disc}}$	$-0.2(0.1)_\text{S}(0.0)_\text{C}(0.0)_\text{V}(0.0)_\text{A}(0.0)_\text{Z}(0.0)_\text{E}$	$-6.9(2.1)_\text{S}(0.4)_\text{C}(1.4)_\text{V}(0.0)_\text{A}(0.0)_\text{Z}(1.3)_\text{E}$
a_μ^{SIB}	$0.1(0.2)_\text{S}(0.0)_\text{C}(0.2)_\text{V}(0.0)_\text{A}(0.0)_\text{Z}(0.0)_\text{E48}$	$10.6(4.3)_\text{S}(0.6)_\text{C}(6.6)_\text{V}(0.1)_\text{A}(0.0)_\text{Z}(1.3)_\text{E48}$
$a_\mu^{\text{udsc, isospin}}$	$231.9(1.4)_\text{S}(0.2)_\text{C}(0.1)_\text{V}(0.3)_\text{A}(0.2)_\text{Z}(0.0)_\text{M}$	$705.9(14.6)_\text{S}(2.9)_\text{C}(3.7)_\text{V}(1.8)_\text{A}(0.4)_\text{Z}(2.3)_\text{L}(0.1)_\text{E48}(0.1)_\text{E64}(0.0)_\text{M}$
$a_\mu^{\text{QED, SIB}}$	$0.1(0.3)_\text{S}(0.0)_\text{C}(0.2)_\text{V}(0.0)_\text{A}(0.0)_\text{Z}(0.0)_\text{E}(0.0)_\text{E48}$	$9.5(7.4)_\text{S}(0.7)_\text{C}(6.9)_\text{V}(0.1)_\text{A}(0.0)_\text{Z}(1.7)_\text{E}(1.3)_\text{E48}$
$a_\mu^{\text{R−ratio}}$	$460.4(0.7)_\text{RST}(2.1)_\text{RSY}$	
a_μ	$692.5(1.4)_\text{S}(0.2)_\text{C}(0.2)_\text{V}(0.3)_\text{A}(0.2)_\text{Z}(0.0)_\text{E}(0.0)_\text{E48}(0.0)_\text{b}(0.1)_\text{c}(0.0)_{\overline{\text{S}}}(0.0)_{\overline{\text{Q}}}(0.0)_\text{M}(0.7)_\text{RST}(2.1)_\text{RSY}$	$715.4(16.3)_\text{S}(3.0)_\text{C}(7.8)_\text{V}(1.9)_\text{A}(0.4)_\text{Z}(1.7)_\text{E}(2.3)_\text{L}(1.5)_\text{E48}(0.1)_\text{E64}(0.3)_\text{b}(0.2)_\text{c}(1.1)_{\overline{\text{S}}}(0.3)_{\overline{\text{Q}}}(0.0)_\text{M}$

TABLE I. Individual and summed contributions to a_μ multiplied by 10^{10}. The left column lists results for the window method with $t_0 = 0.4$ fm and $t_1 = 1$ fm. The right column shows results for the pure first-principles lattice calculation. The respective uncertainties are defined in the main text.

Tuning the quark masses

- isospin symmetric calculation [T. Blum et al. Phys.Rev. D93 (2016) no.7, 074505]

$$\mathbf{m}_\ell = \mathbf{0.0006979(81)}$$
$$\mathbf{m}_s = \mathbf{0.03580(16)}$$

- tune (**u**,**d**,**s**) masses to reproduce experimental π^+, $\mathbf{K}^+$ and $\mathbf{K}_0$ mass (and check π^0 mass), fix lattice spacing using Ω^-

$$\mathbf{\Delta m}_u = \mathbf{0.00050(1)}$$
$$\mathbf{\Delta m}_d = \mathbf{-0.00050(1)}$$
$$\mathbf{\Delta m}_s = \mathbf{-0.0002(2)}$$

- ratio of quark masses

$$\frac{\mathbf{m}_d}{\mathbf{m}_u} = \mathbf{0.449(22)}$$

Coulomb gauge

- projector for photon fields [Borsanyi et al., Science **347** (2015) 1452-1455]

$$(\mathbf{P_C})_{\mu\nu} = \delta_{\mu\nu} - \left|\vec{\hat{\mathbf{k}}}\right|^{-2} \hat{\mathbf{k}}_\mu \left(0, \vec{\hat{\mathbf{k}}}\right)_\nu \qquad \text{with} \;\; \tilde{\mathbf{A}}^{\text{Coul}}_\mu(\mathbf{k}) = (\mathbf{P_C})_{\mu\nu} \tilde{\mathbf{A}}^{\text{Feyn}}_\nu(\mathbf{k})$$

with

$$\hat{\mathbf{k}} = 2\sin\left(\frac{\mathbf{k}_\mu}{2}\right)$$

- Coulomb gauge photon propagator

$$\Delta^{\text{Coul}}_{\mu\nu}(\mathbf{x}-\mathbf{y}) = \begin{cases} \frac{1}{N}\sum_{\mathbf{k}} \frac{1}{\hat{\mathbf{k}}^2}\left[\delta_{ij} - \frac{1}{\vec{\hat{\mathbf{k}}}^{\,2}}\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\right] e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{i\mathbf{k}(\hat{\mu}-\hat{\nu})/2} & \mu = i, \nu = j \\ \frac{1}{N}\sum_{\mathbf{k}} \frac{1}{\vec{\hat{\mathbf{k}}}^{\,2}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{i\mathbf{k}(\hat{\mu}-\hat{\nu})/2} & \mu = 0, \nu = 0 \\ 0 & \text{otherwise} \end{cases}$$

a_μ: Experiment vs. Theory

- $\mathbf{a_\mu = (g_\mu - 2)/2}$
- measured and calculated very precisely $\longrightarrow$ test of the Standard Model
- experiment: polarized muons in a magnetic field [Bennet et al., Phys.Rev. **D73**, 072003 (2006)]

$$\mathbf{a_\mu = 11659208.9(5.4)(3.3) \times 10^{-10}}$$

- Standard Model

em	$\mathbf{(11658471.895 \pm 0.008) \times 10^{-10}}$	[Kinoshita et al., Phys.Rev.Lett. **109**, 111808 (2012)]
weak	$\mathbf{(15.36 \pm 0.10) \times 10^{-10}}$	[Gnendinger et al., Phys.Rev. **D88**, 053005 (2013)]
HVP	$\mathbf{(692.3 \pm 4.2 \pm 0.3) \times 10^{-10}}$	[Davier et al., Eur.Phys.J. **C71**, 1515 (2011)]
HVP(α^3)	$\mathbf{(-9.84 \pm 0.06) \times 10^{-10}}$	[Hagiwara et al., J.Phys. **G38**, 085003 (2011)]
LbL	$\mathbf{(10.5 \pm 2.6) \times 10^{-10}}$	[Prades et al.,Adv.Ser.Direct.High Energy Phys. 20, 303 (2009)]

- Comparison of theory and experiment: $\mathbf{3.6\sigma}$ deviation

$$\mathbf{\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 28.8(6.3)^{\text{Exp}}(4.9)^{\text{SM}} \times 10^{-10}}$$

- new physics?

The zero-mode of the photon field

- shift symmetry of the of the photon action $\mathbf{A}_\mu(\mathbf{x}) \rightarrow \mathbf{A}_\mu(\mathbf{x}) + \mathbf{c}_\mu$
  $\rightarrow$ remove by fixing the zero-mode of the photon field

- different prescriptions of QED:
- $\mathrm{QED_{TL}}$: remove the zero-mode of the photon field, i.e. $\tilde{\mathbf{A}}_\mu(\mathbf{k}=0)=0$
  [A. Duncan, E. Eichten, H. Thacker, Phys.Rev.Lett. **76**, 3894 (1996)]
- $\mathrm{QED_L}$: remove all the spatial zero-modes, i.e. $\tilde{\mathbf{A}}_\mu(\mathbf{k}_0, \vec{\mathbf{k}}=0)=0$
  [S. Uno and M. Hayakawa, Prog. Theor. Phys. **120**, 413 (2008)]
- $\mathrm{QED_m}$: use a massive photon and take $\mathbf{m}_\gamma \rightarrow 0$
  [M. Endres *et al.*,Phys. Rev. Lett. **117** (2016) 072002]
- $\mathrm{QED_C}$: $\mathbf{C}^*$ boundary conditions in spatial direction, i.e. fields are periodic up to charge conjugation [B. Luchini *et al.* JHEP **02** (2016) 076]

- for detailed discussion on different prescriptions of QED see e.g. [A. Patella 1702.03857]

K $\to \ell\nu_\ell$ with QED

- formulated in [N. Carrasco *et al.* Phys.Rev. **D91** (2015) no.7, 074506]
- first results in [D. Guisti *et al.* Phys.Rev.Lett. **120** (2018) no.7, 072001]
- contributions from photon emitted from hadron and absorbed by charged lepton $\to$ hadronic and leptonic part can no longer be factorised
- infrared (IR) divergences
  $\to$ canceled when combining contributions from virtual and real photons
  [F. Bloch and A. Nordsieck, Phys.Rev. **52** (1937) 54]
  $\to$ perturbative method for QED