Prospects for the study of long-distance processes using lattice QCD

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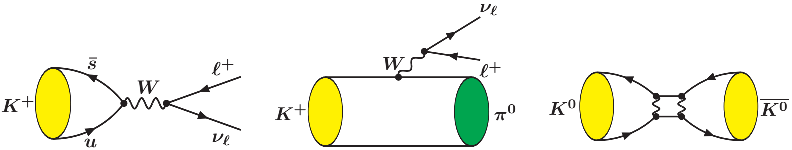
IFT Workshop on "Frontiers in Lattice Quantum Field Theory"
May 29, 2018

A mission of lattice calculations is to evaluate the hadronic effects

- Lattice QCD is powerful for "standard" hadronic matrix elements with



$K_{\ell 2} \to f_K$ $\quad$ $K_{\ell 3} \to f_+(0)$ $\quad$ K^0-$\overline{K^0}$ mixing $\to B_K$

  - single local operator insertion
  - only single stable hadron or vacuum in the initial/final state
  - spatial momenta carried by particles need to be small compared to $1/a$

- FLAG average updated on Nov./Dec. 2016

	N_f	FLAG average	Frac. Err.
f_K/f_π	$2+1+1$	1.1933(29)	0.25%
$f_+(0)$	$2+1+1$	0.9706(27)	0.28%
$\hat{B}_K$	$2+1$	0.7625(97)	1.27%

Frontiers to go beyond "standard"

- QED + isospin breaking
  - QCD+QED and numerical simulations [Agostino Patella]
  - QED corrections to decay rates [Francesco Sanfilippo]
  - Isospin Breaking Effects on the Lattice [Vera Gülpers]
  - Towards isospin breaking corrections to $(g-2)_\mu$ [Michele Della Morte]

Processes involving QED corrections are essentially LD processes

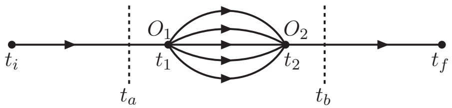
- Multi-hadron interactions
  - Few-body physics: status and outlook [Raúl Briceno]
  - Multi-hadron observables from lattice QCD [Maxwell Hansen]

Studying multi-hadron interactions are important for LD processes

Three generic features of lattice QCD computation of LD processes

- Euclidean time $\Rightarrow$ exponentially growing contamination
- Multiparticle in intermediate state $\Rightarrow$ non-exp finite-volume effects
- Two interpolating operators $\Rightarrow$ new short-distance divergence

LD processes and bilocal matrix elements



Hadronic matrix element for the 2[nd]-order weak interaction

$$\int_{-T}^{T} dt \, \langle f | T \left[O_1(t) O_2(0) \right] | i \rangle$$
$$= \sum_n \left\{ \frac{\langle f|O_1|n\rangle\langle n|O_2|i\rangle}{M_i - E_n} + \frac{\langle f|O_2|n\rangle\langle n|O_1|i\rangle}{M_i - E_n} \right\} \left(1 - e^{(M_i - E_n)T}\right)$$

- For $E_n > M_i$, the exponential terms exponentially vanish at large T
- For $E_n < M_i$, the exponentially growing terms must be removed

Euclidean time $\Rightarrow$ exponentially growing contamination

Feature 1: removal of the exponentially growing terms

- Determine the hadronic matrix element for all low-lying intermediate states

$$\frac{\langle\overline{K^0}|H_W|n\rangle\langle n|H_W|K^0\rangle}{M_K - E_n}\left(1 - e^{(M_K-E_n)T}\right)$$

- Replacing $H_W \to H'_W = H_W + c_s\bar{s}d + c_p\bar{s}\gamma_5 d$ won't affect physical amplitude
  - Apply the chiral Ward identity

$$\begin{aligned}\partial_\mu\bar{s}\gamma_\mu d &= (m_s - m_d)\bar{s}d \\ \partial_\mu\bar{s}\gamma_\mu\gamma_5 d &= (m_s + m_d)\bar{s}\gamma_5 d\end{aligned}$$

  - K^0-$\overline{K^0}$ transition amplitude is given by

$$\int d^4x\,\langle\overline{K^0}|T[H_W(x)H_W(0)]|K^0\rangle$$

$\partial_\mu\bar{s}\gamma_\mu d$ and $\partial_\mu\bar{s}\gamma_\mu\gamma_5 d$ do not contribute to the $\int d^4x$ integral

- Choose appropriate c_s and c_p, e.g.

$$\begin{aligned}\langle 0|H_W + c_p\bar{s}\gamma_5 d|K^0\rangle &= 0 \\ \langle\pi|H_W + c_s\bar{s}d|K^0\rangle &= 0\end{aligned}$$

to remove the contamination from low-lying states such as $|0\rangle$, $|\pi\rangle$ or $|\eta\rangle$

Feature 2: finite-volume effects

- In infinite volume we need to compute the amplitude

$$\mathcal{A} = \int_{-\infty}^{\infty} dt \, \langle \bar{K}^0 | T[H_W(t) H_W(0)] | K^0 \rangle$$

and to determine the K_L-K_S mass difference

$$\Delta M_K = M_{K_L} - M_{K_S} = 2 \mathcal{PV} \sum\!\!\!\!\!\!\!\int_\alpha \frac{\langle \bar{K}^0 | H_W | \alpha \rangle \langle \alpha | H_W | K^0 \rangle}{M_K - E_\alpha}$$

- On the lattice with size L, we can obtain

$$\Delta M_K(L) = 2 \sum_n \frac{\langle \bar{K}^0 | H_W | n \rangle_L \, {}_L\langle n | H_W | K^0 \rangle}{M_K - E_n}$$

  - $|n\rangle$ could be given by multi-hadron state $|\pi\pi\rangle$
  - Significant FV effects, especially when $E_n = E_{\pi\pi} \to M_K$

- Finite volume correction

[Christ, XF, Martinelli, Sachrajda, PRD 91 (2015) 114510]

$$\Delta M_K(L) - \Delta M_K(\infty) = 2 \cot(\phi + \delta) \frac{d(\phi + \delta)}{dE} \bigg|_{E = M_K} \langle \bar{K}^0 | H_W | \pi\pi, M_K \rangle_L \, {}_L\langle \pi\pi, M_K | H_W | K^0 \rangle$$

Finite-volume effects in three related processes

- $\pi\pi$ scattering, Lüscher's quantization condition

$$\phi(q) + \delta(k) = n\pi, \quad n \in \mathbb{Z}, \quad q = \frac{k}{2\pi/L}, \quad E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}$$

- $K \to \pi\pi$ transition, Lellouch-Lüscher formula

$$|\langle 0|\sigma(0)|\pi\pi, E\rangle_\infty|^2 = \frac{2\pi E^2}{k^2}\left(\frac{d(\phi+\delta)}{dk}\right)|\langle 0|\sigma(0)|\pi\pi, E\rangle_L|^2$$

- K_L-K_S mass difference

$$\Delta M_K(L) - \Delta M_K(\infty) = 2\cot(\phi+\delta)\frac{d(\phi+\delta)}{dE}\bigg|_{E=M_K}\langle\bar{K}^0|H_W|\pi\pi, M_K\rangle_L\,{}_L\langle\pi\pi, M_K|H_W|K^0\rangle$$

Similar structure ⇒ Uniform treatment of the FV effects in the three processes

KSS method

Kim, Sachrajda, Sharpe, NPB 727 (2005) 218

- Use "sum=integral + [sum-integral]" where integrand has pole, with [KSS]

$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(P-k)^2-m^2+i\epsilon}g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*)\mathcal{F}\ \ (q^*, q^{*\prime}) g^*(\hat{q}^{*\prime})$$

- Diagrammatically

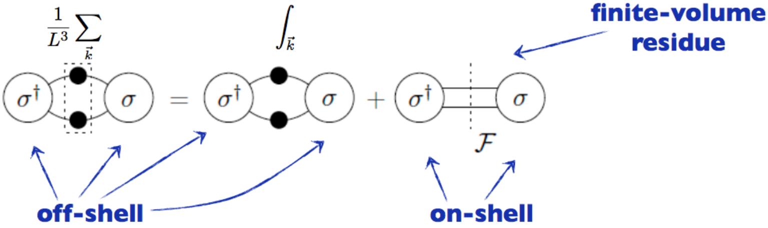
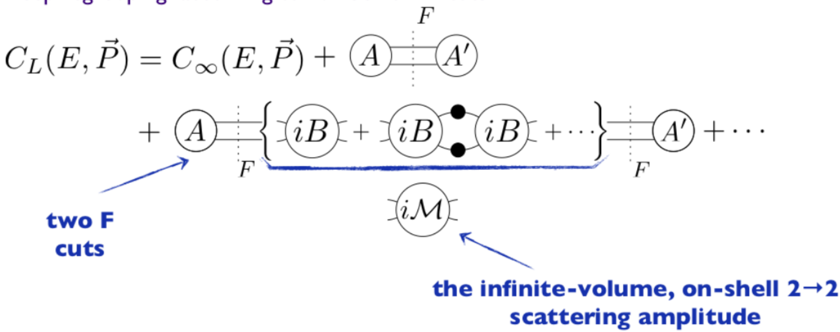


illustration from S. Sharpe's talk at INT workshop INT-18-70W

- And keep regrouping according to number of "F cuts"



Finite volume correction to correlator

$$\Delta C(P) = C_L(P) - C_\infty(P) = A\mathcal{F}\sum_n (i\mathcal{M}\mathcal{F})^n A' = A\frac{1}{\mathcal{F}^{-1} - i\mathcal{M}}A' = \frac{k}{8\pi P_0}\frac{e^{i(\phi+\delta)}}{\sin(\phi+\delta)}|A|^2$$

- $C_L(P)$ has poles at $P_0 = E_n$, which match the poles in $\Delta C(P)$

$$\sin(\phi+\delta)\big|_{P_0=E_n} = 0 \quad \Rightarrow \quad (\phi+\delta)\big|_{P_0=E_n} = n\pi$$

First product from KSS: Lüscher quantization condition

Second product: Lellouch-Lüscher formalism

Agadjanov, Bernard, Meissner, Rusetsky, NPB 886 (2014) 1199
Briceno, Hansen, Walker-Loud, PRD 91 (2015) 034501
Christ, XF, Martinelli, Sachrajda, PRD 91 (2015) 114510

- Let the poles in $\Delta C(P)$ and $C_L(P)$ have the same residues

$$\lim_{P_0\to E_n}(P_0-E_n)\Delta C(P)=\lim_{P_0\to E_n}(P_0-E_n)C_L(P)$$

- Spectral representation for $C_L(P)$ is given by

$$C_L(P)=\sum_n\frac{|\langle 0|\sigma(0)|\pi\pi,E_n\rangle_L|^2}{P_0-E_n}$$

- The expression for ΔC is given by

$$\Delta C(P)=\frac{k}{8\pi P_0}\frac{e^{i(\phi+\delta)}}{\sin(\phi+\delta)}|A|^2$$

- Picking up the residues leads to the Lellouch-Lüscher formula

$$|A|^2=\left.\frac{d(\phi+\delta)}{dk}\right|_{P_0=E_n}\frac{2\pi E_n^2}{k_n^2}|\langle 0|\sigma(0)|E_n\rangle_L|^2$$

Third product: FV correction to long-distance observables

Christ, XF, Martinelli, Sachrajda, PRD 91 (2015) 114510

For the $K_L - K_S$ system, we need to evaluate the principal value of the branch-cut integral, which can be achieved by taking the real part of KSS formula

$$\mathrm{Re}\,\Delta C(P) = C_L(P) - \mathcal{PV}[C_\infty(P)] = \frac{k}{8\pi P_0}\mathrm{ctg}\,(\phi+\delta)|A|^2$$

Using the spectral representation for $C_L(P)$, we can rewrite

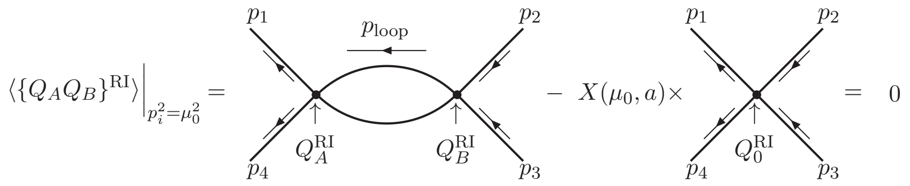
$$\mathcal{PV}[C_\infty(P)] = \sum_n \frac{|\langle 0|\sigma(0)|E_n\rangle_L|^2}{P_0 - E_n} - \frac{k}{8\pi P_0}\mathrm{ctg}\,(\phi+\delta)|A|^2$$

This gives FV correction for ΔM_K (if we replace $|\langle 0|\sigma(0)|E_n\rangle_L|$ by $|\langle K|H_W|E_n\rangle_L|$)

Christ, XF, Portelli, Sachrajda, PRD 93 (2016) 114517

New SD divergence appears in $Q_A(x)Q_B(0)$ when $x \to 0$

- Introduce a counter term $X \cdot Q_0$ to remove the SD divergence



The coefficient X is determined in the RI/(S)MOM scheme

- The bilocal operator in the $\overline{\mathrm{MS}}$ scheme can be written as

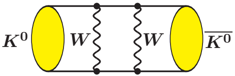
$$\left\{\int d^4x\, T[Q_A^{\overline{\mathrm{MS}}}(x)\, Q_B^{\overline{\mathrm{MS}}}(0)]\right\}^{\overline{\mathrm{MS}}}$$

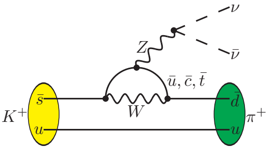
$$= Z_A Z_B \left\{\int d^4x\, T[Q_A^{\mathrm{lat}} Q_B^{\mathrm{lat}}]\right\}^{\mathrm{lat}} + \left(-X^{\mathrm{lat}\to\mathrm{RI}} + Y^{\mathrm{RI}\to\overline{\mathrm{MS}}}\right) Q_0(0)$$

  - $X^{\mathrm{lat}\to\mathrm{RI}}$ is calculated using NPR and $Y^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ calculated using PT

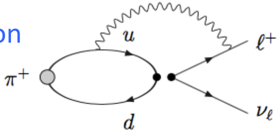
Applications of LD processes

2nd order electroweak interaction

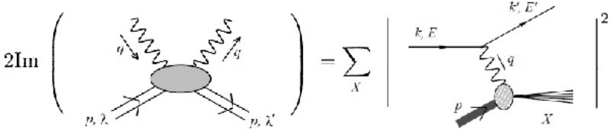




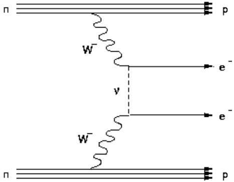
Electromagnetic correction



Inclusive decay and deep inelastic scattering



Neutrinoless double beta decay



Summary of lattice study on rare kaon decays

Proposal to study rare kaon decays using lattice QCD

- Isidori, Martinelli, Turchetti, hep-lat/0506026 / PLB 2006

Method paper and first calculation of $K \to \pi\ell^+\ell^-$

- Christ, XF, Portelli, Sachrajda, 1507.03094/PRD 2015
- Christ, XF, Jüttner, Lawson, Portelli, Sachrajda, 1608.07585/PRD 2016

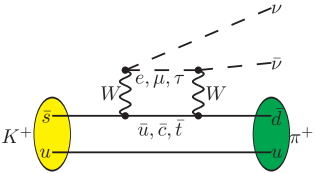
Method paper and first calculation of $K^+ \to \pi^+\nu\bar{\nu}$

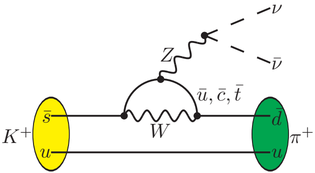
- Christ, XF, Portelli, Sachrajda, 1605.04442/PRD 2016
- Bai, Christ, XF, Lawson, Portelli, Sachrajda, 1701.02858/PRL 2017

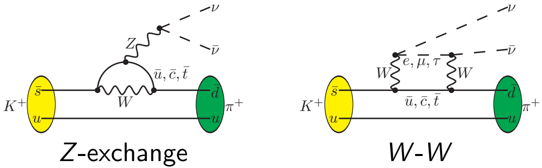
$K^+ \to \pi^+\nu\bar{\nu}$, $32^3 \times 64$, $m_\pi = 170$ MeV, $m_c^{\overline{\mathrm{MS}}}(2\text{ GeV}) = 750$ MeV

$K^+ \to \pi^+\nu\bar{\nu}$, $64^3 \times 128$, $m_\pi = 140$ MeV, $m_c^{\overline{\mathrm{MS}}}(2\text{ GeV}) = 1.2$ GeV

Use $K^+ \to \pi^+ \nu \bar{\nu}$ as an example







$K^+ \to \pi^+ \nu\bar{\nu}$: largest contribution from top quark loop, thus theoretically clean

$$\mathcal{H}_{eff} \sim \frac{G_F}{\sqrt{2}} \cdot \underbrace{\frac{\alpha_{\text{EM}}}{2\pi \sin^2\theta_W} \lambda_t X_t(x_t)}_{\mathcal{N} \sim 2\times 10^{-5}} \cdot (\bar{s}d)_{V-A} (\bar{\nu}\nu)_{V-A}$$

Probe the new physics at scales of $\mathcal{N}^{-\frac{1}{2}} M_W = O(10\text{ TeV})$

Past experimental measurement is 2 times larger than SM prediction

$$\text{Br}(K^+ \to \pi^+ \nu\bar{\nu})_{\text{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10} \quad \text{[BNL E949, '08]}$$

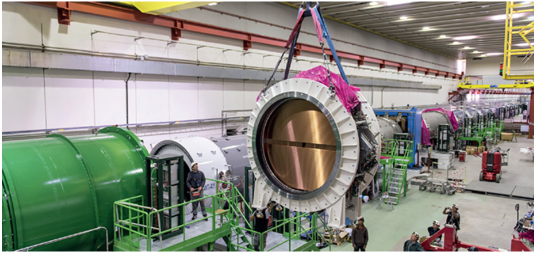
$$\text{Br}(K^+ \to \pi^+ \nu\bar{\nu})_{\text{SM}} = 9.11 \pm 0.72 \times 10^{-11} \quad \text{[Buras et. al., '15]}$$

but still consistent with $> 60\%$ exp. error

New experiments

New generation of experiment: NA62 at CERN

- aims at observation of $O(100)$ events [2016-2018, 2021-2023]
- 10%-precision measurement of $\mathrm{Br}(K^+ \to \pi^+ \nu\bar{\nu})$

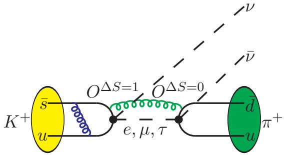


NA62 timeline

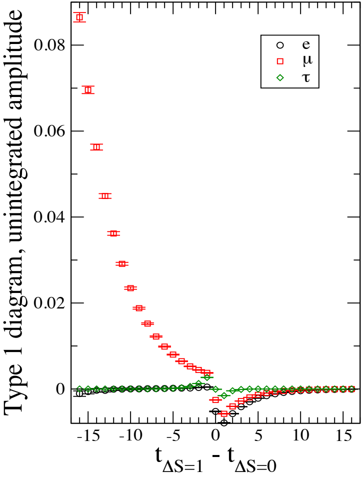
- Detector installation completed in 09.2016
- 2016 run: 2×10^{11} K^+ decays
- 2017 run: 6.7×10^{11} K^+ decays
- 2018 run: starting from April 9, 210 days run

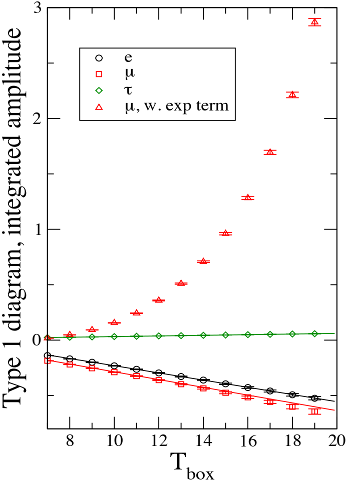
10% of 2016 data are analyzed $\Rightarrow$ expected 0.064 event

W-W diagram, type 1

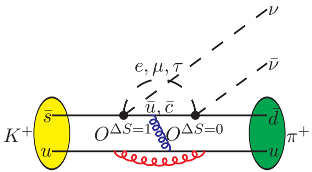


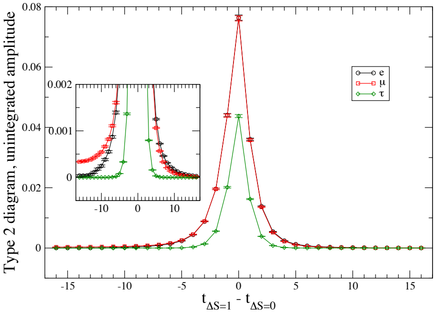
F_{WW}	Type 1
e	$-1.685(47) \times 10^{-2}$
μ	$-1.818(40) \times 10^{-2}$
τ	$1.491(36) \times 10^{-3}$



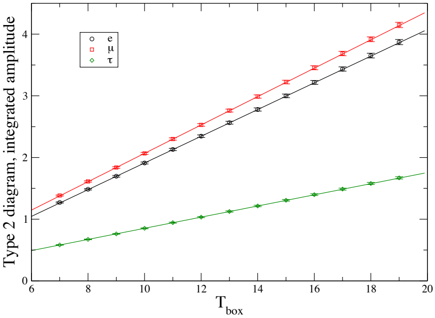


W-W diagram, type 2

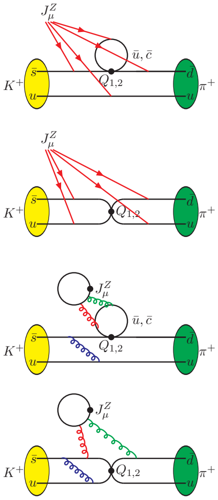


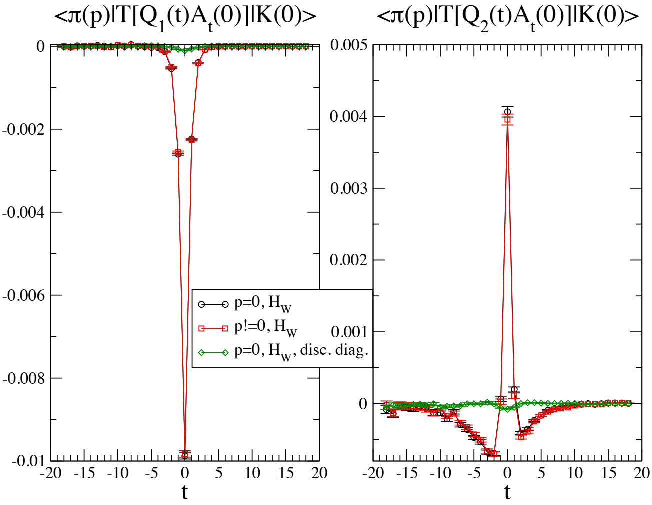


F_{WW}	Type 2
e	$1.123(17) \times 10^{-1}$
μ	$1.194(18) \times 10^{-1}$
τ	$4.690(77) \times 10^{-2}$

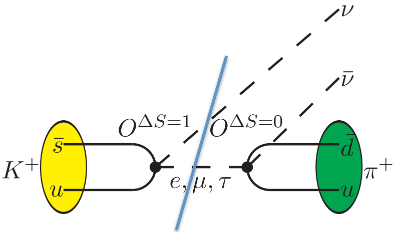


Z-exchange diagram

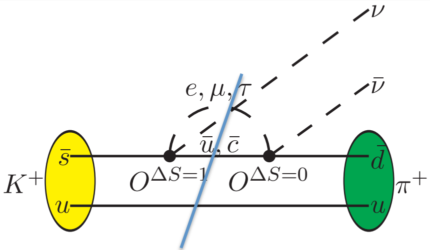




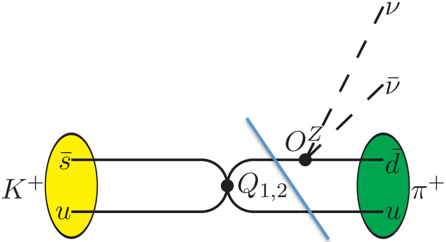
Low lying intermediate states



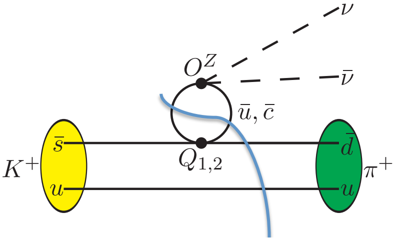
$K^+ \to \ell^+\nu \quad \& \quad \ell^+ \to \pi^+\bar{\nu}$



$K^+ \to \pi^0\ell^+\nu \quad \& \quad \pi^0\ell^+ \to \pi^+\bar{\nu}$



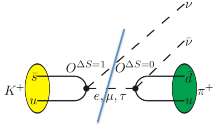
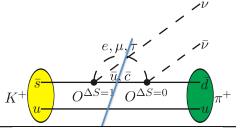
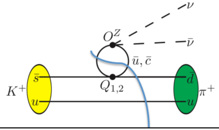
$K^+ \xrightarrow{H_W} \pi^+ \quad \& \quad \pi^+ \xrightarrow{V_\mu} \pi^+$

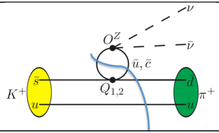


$K^+ \xrightarrow{H_W} \pi^+\pi^0 \quad \& \quad \pi^+\pi^0 \xrightarrow{A_\mu} \pi^+$

Kaon decays serve as an ideal LD process

Branching ratios and decay widths for $K \to \{n\}$ decays

$K \to \{n\}$	Branching ratio	Relevant diagrams
$K^+ \to \mu^+\nu_\mu$	$6.355(11) \times 10^{-1}$	
$K^+ \to 2\pi\mu^+\nu_\mu$	$4.254(32) \times 10^{-5}$	
$K^+ \to \pi^0 e^+\nu_e$	$3.353(34) \times 10^{-2}$	
$K^+ \to 3\pi e^+\nu_e$	$< 3.5 \times 10^{-6}$	
$K^+ \to \pi^+\pi^0$	$2.066(8) \times 10^{-1}$	
$K^+ \to 3\pi$	$7.35(5) \times 10^{-2}$	

$K \to \{n\}$	Decay width [eV]	Relevant diagrams
$K_S \to 2\pi$	$7.343(13) \times 10^{-6}$	
$K_L \to 3\pi$	$4.125(30) \times 10^{-9}$	

FV effects from $\pi\pi$ intermediate state

Calculation has been performed at m_π = 420 MeV and 170 MeV

For m_π = 170 MeV, $K \to \pi\pi$ is possible

- Matrix elements for $K \to \pi\pi$ and $\pi\pi \to \pi$

$a^4 \cdot \langle \pi\pi \vert Q_{1,2} \vert K \rangle$	$a \cdot \langle \pi \vert A_\mu \vert \pi\pi \rangle$	expectation
$-i \cdot 9.653(25) \times 10^{-5}$	$i \cdot 3.1910(60)$	$i \cdot 3.2149(62)$

Expectation value is given by $\langle \pi \vert A_\mu \vert \pi\pi \rangle \approx \langle \pi \vert \pi \rangle \cdot \langle 0 \vert A_\mu \vert \pi \rangle$.

- Parameters relevant for finite-size correction

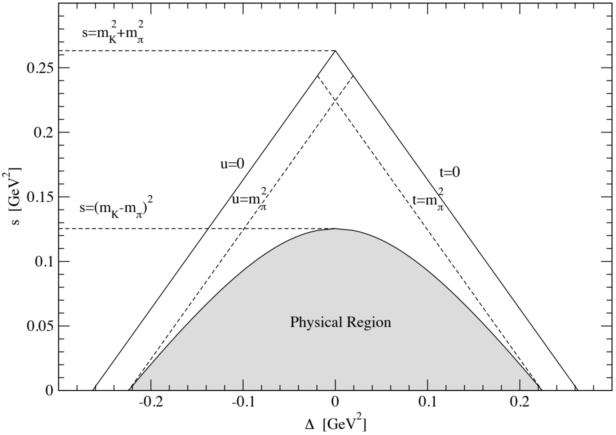
ϕ	$\delta = k \cdot a_{\pi\pi}$	$d\phi/dE$	$d\delta/dE = a_{\pi\pi} \cdot \frac{m_K}{4k}$	$\cot(\phi + \delta)$
1.449(2)	−0.0678(16)	13.858(8)	−0.367(9)	0.192(3)

- Results

$F_0(s)$	$F_0^{(\pi\pi)}(s)$	$\Delta_{FV} F_0(s)$
$2.05(12) \cdot 10^{-2}$	$-1.536(5) \cdot 10^{-3}$	$4.28(7) \cdot 10^{-4}$

$I = 2$ $\pi\pi$-state contributes 7.5%. The finite-size correction is about 2.1%.

Momentum choice



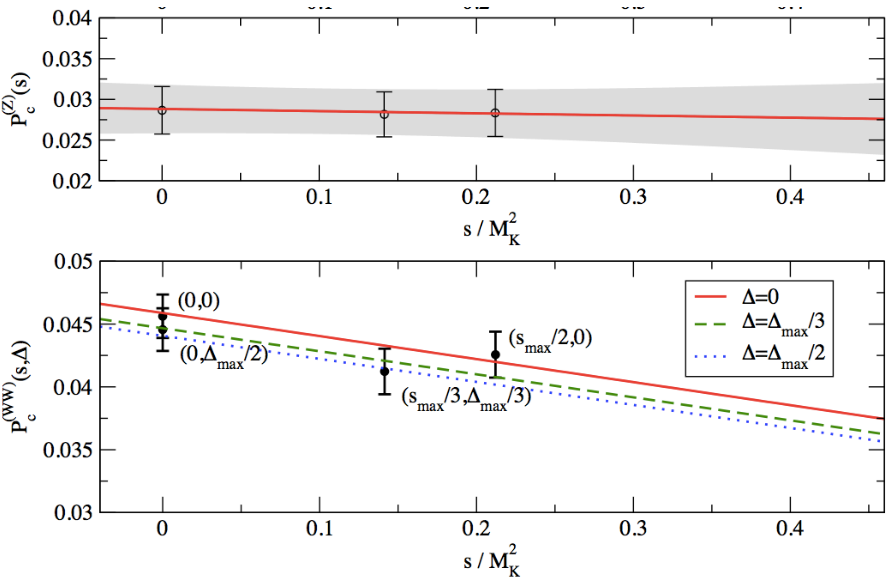
- Two Lorentz invariant variables

$$s = (p_K - p_\pi)^2, \quad \Delta = (p_K - p_\nu)^2 - (p_K - p_{\bar{\nu}})^2$$

- $s_{\max} = (M_K - M_\pi)^2, \quad \Delta_{\max} = M_K^2 - M_\pi^2$
- Momentum choice

$$(s, \Delta) = (0, 0),\ (s_{\max}/2, 0),\ (0, \Delta_{\max}),\ (s_{\max}/3, \Delta_{\max}/3)$$

Momentum dependence



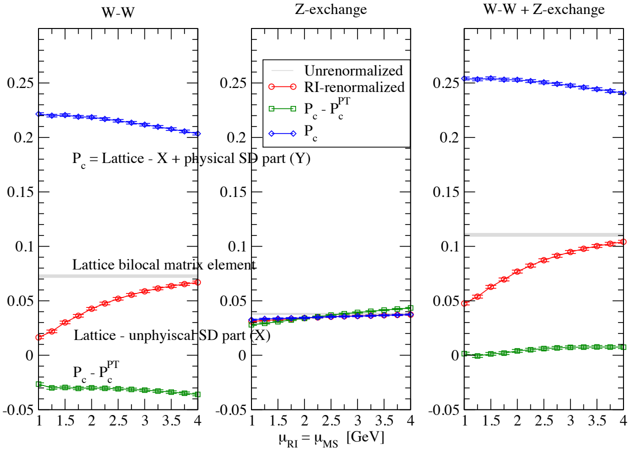
With momentum dependence, the branching ratio can be determined

Lattice results

Published results @ $m_\pi = 420$ MeV, $m_c = 860$ MeV

[Bai, Christ, XF, Lawson, Portelli, Schrajda, PRL 118 (2017) 252001]

$$P_c = 0.2529(\pm 13)_{\rm stat}(\pm 32)_{\rm scale}(-45)_{\rm FV}$$



Lattice QCD is now capable of first-principles calculation of rare kaon decay

- The remaining task is to control various systematic effects

Electromagnetism on $K \to \pi\pi$

[N. Christ & XF, arXiv:1711.09339]

Direct and indirect CP violation

- The experimentally detected states $K_{L/S}$ are not CP eigenstates

$$|K_{L/S}\rangle = \frac{1}{\sqrt{1+\bar{\epsilon}^2}}\left(|K^0_{\mp}\rangle + \bar{\epsilon}|K^0_{\pm}\rangle\right)$$

- $K_L \to 2\pi$ $(CP = +)$
  - $K^0_+ \to 2\pi$ (indirect CP violation, ϵ or ϵ_K)
  - $K^0_- \to 2\pi$ (direct CP violation, ϵ')

- Experimental measurement

$$\frac{A(K_L \to \pi^+\pi^-)}{A(K_S \to \pi^+\pi^-)} \equiv \eta_{+-} \equiv \epsilon + \epsilon'$$
$$\frac{A(K_L \to \pi^0\pi^0)}{A(K_S \to \pi^0\pi^0)} \equiv \eta_{00} \equiv \epsilon - 2\epsilon'$$

- Using $|\eta_{+-}|$ and $|\eta_{00}|$ as input, PDG quotes

$$|\epsilon| \approx \frac{1}{3}\left(2|\eta_{+-}| + |\eta_{00}|\right) = 2.228(11)\times 10^{-3}, \quad \mathrm{Re}[\epsilon'/\epsilon] \approx \frac{1}{3}\left(1 - \frac{|\eta_{00}|}{|\eta_{+-}|}\right) = 1.66(23)\times 10^{-3}$$

ϵ' is 1000 times smaller than the indirect CP violation ϵ

Thus direct CP violation ϵ' is very sensitive to New Physics

$K \to \pi\pi$ decays and CP violation

- Theoretically, Kaon decays into the isospin $I = 2$ and 0 $\pi\pi$ states

$$\Delta I = 3/2 \text{ transition:} \quad \langle \pi\pi(I=2)|H_W|K^0\rangle = A_2 e^{i\delta_2}$$
$$\Delta I = 1/2 \text{ transition:} \quad \langle \pi\pi(I=0)|H_W|K^0\rangle = A_0 e^{i\delta_0}$$

- If CP symmetry were protected $\Rightarrow$ A_2 and A_0 are real amplitudes
- ϵ and ϵ' depend on the $K \to \pi\pi(I)$ amplitudes A_I

$$\epsilon = \bar{\epsilon} + i\left(\frac{\mathrm{Im}[A_0]}{\mathrm{Re}[A_0]}\right)$$
$$\epsilon' = \frac{ie^{i(\delta_2-\delta_0)}}{\sqrt{2}}\frac{\mathrm{Re}[A_2]}{\mathrm{Re}[A_0]}\left(\frac{\mathrm{Im}[A_2]}{\mathrm{Re}[A_2]} - \frac{\mathrm{Im}[A_0]}{\mathrm{Re}[A_0]}\right)$$

The target for lattice QCD is to calculate both amplitude A_2 and A_0

Two remarks:

- $\arg[\epsilon] \approx 43.5°$, $\quad \arg[\epsilon'] = 90° + \delta_2 - \delta_0 \approx 40.4°$
  $\Rightarrow \quad \delta_0$ in $[23.8°, 38.0°]$ only affects $\mathrm{Re}[\epsilon'/\epsilon]$ in $< 2\%$ level
- $\frac{\mathrm{Im}[A_2]}{\mathrm{Re}[A_2]} = -4.7(0.7) \times 10^{-5}$ cancels with $\frac{\mathrm{Im}[A_0]}{\mathrm{Re}[A_0]} = -4.1(3.5) \times 10^{-5}$

How large the EM corrections to $K \to \pi\pi$

Direct CP violation in $K \to \pi\pi$

$$\epsilon' = \frac{1}{3}(\eta_{+-} - \eta_{00}) = \frac{ie^{i(\delta_2-\delta_0)}}{\sqrt{2}} \frac{\mathrm{Re}\,A_2}{\mathrm{Re}\,A_0} \left(\frac{\mathrm{Im}\,A_2}{\mathrm{Re}\,A_2} - \frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0} \right)$$

- Turn on EM interaction, $A_I \to A_I^\gamma$, $\delta_I \to \delta_I^\gamma$, $I = 0, 2$

Though $A_2^\gamma - A_2$ is an $O(\alpha_e)$ effect, its size could be enhanced by a factor of 22 due to the mixing with A_0 and $\Delta I = 1/2$ rule

- ChPT+Large-N_c: Cirigliano et al, hep-ph/0008290, hep-ph/0310351
  –"the isospin violating correction for ϵ' is below 15%"

Technical issues on including electromagnetism

- Lellouch-Lüscher's formalism relies on a short-range interaction
  $\Rightarrow$ long-range EM requires the change in the FV formalism
- EM interaction mixes $I = 0$ and $I = 2$ $\pi\pi$ scattering
  $\Rightarrow$ $K \to \pi\pi$ decay becomes a coupled-channel problem
- Possible photon radiation
  $\Rightarrow$ coupled channels further mixed with 3-particle channel $(\pi\pi\gamma)$

Include EM interaction in the Coulomb gauge

$$\mathcal{L}_{\rm int} = \underbrace{\sum_{q=u,d,s} e_q \vec{A}(x)\cdot \bar{q}\vec{\gamma}q(x)}_{\text{Transverse radiation}} - \underbrace{\sum_{q,q'=u,d,s} \int \frac{d^3\vec{x}'}{4\pi} \frac{\rho_q(\vec{x}',t)\rho_{q'}(\vec{x},t)}{|\vec{x}'-\vec{x}|}}_{\text{Coulomb potential}}$$

- Adding transverse photon to $\pi\pi \quad \Rightarrow \quad$ three-particle problem
- At current stage, focus on Coulomb potential only

Mixing of isospin states

Focus on Coulomb potential, no $\pi\pi\gamma$ state

However, $I = 2$ and $I = 0$ $\pi\pi$ states still mix with each other

- No EM: relation between charged $c = +-, 00$ and isopsin $s = 0, 2$ $\pi\pi$ states

$$|(\pi\pi)_c\rangle^{\rm out} = \sum_{s=0,2} \Omega_{cs} |(\pi\pi)_s\rangle^{\rm out}, \quad \Omega_{cs} = \begin{pmatrix} \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{3} & \sqrt{2}/\sqrt{3} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

- With EM:

$$|(\pi\pi)_c^\gamma\rangle^{\rm out} = \sum_{s=0,2} \Omega_{cs}^\gamma |(\pi\pi)_s^\gamma\rangle^{\rm out}, \quad \Omega_{cs}^\gamma = \begin{pmatrix} \cos\theta^\gamma & \sin\theta^\gamma \\ -\sin\theta^\gamma & \cos\theta^\gamma \end{pmatrix}$$

Define ${}^{\rm out}\langle(\pi\pi)_s^\gamma|H_W|K^0\rangle = e^{i\delta_s^\gamma} A_s^\gamma$

$$\epsilon' = \frac{1}{3}(\eta_{+-} - \eta_{00}) = \frac{\sin 2\theta}{\sin 2\theta^\gamma} \frac{i e^{i(\delta_2^\gamma - \delta_0^\gamma)}}{\sqrt{2}} \frac{\mathrm{Re}\, A_2^\gamma}{\mathrm{Re}\, A_0^\gamma} \left(\frac{\mathrm{Im}\, A_2^\gamma}{\mathrm{Re}\, A_2^\gamma} - \frac{\mathrm{Im}\, A_0^\gamma}{\mathrm{Re}\, A_0^\gamma} \right)$$

$\frac{\sin 2\theta}{\sin 2\theta^\gamma}$ is a small correction $\Rightarrow$ focus on A_s^γ and δ_s^γ

Determination of A_s^γ and δ_s^γ from lattice QCD

Turn off EM and calculate correlators with $I = 0, 2$ operators

$$
\begin{aligned}
C_{II'}(t) &= \langle \phi_{\pi\pi,I}(t)\phi^\dagger_{\pi\pi,I'}(0)\rangle \\
&= \sum_{s=0,2} \langle 0|\phi_{\pi\pi,I}|(\pi\pi)_s\rangle e^{-E_s t}\langle(\pi\pi)_s|\phi^\dagger_{\pi\pi,I'}|0\rangle \delta_{s,I}\delta_{s,I'} \\
&= (UMU^\dagger)_{II'}
\end{aligned}
$$

where

$$
U = \begin{pmatrix} \langle 0|\phi_{\pi\pi,0}|(\pi\pi)_0\rangle & 0 \\ 0 & \langle 0|\phi_{\pi\pi,2}|(\pi\pi)_2\rangle \end{pmatrix}, \quad M = \begin{pmatrix} e^{-E_0 t} & \\ & e^{-E_2 t} \end{pmatrix}
$$

Turn on EM and calculate correlators with the same operators

$$
\begin{aligned}
C^\gamma_{II'}(t) &= \langle \phi_{\pi\pi,I}(t)\phi^\dagger_{\pi\pi,I'}(0)\rangle^\gamma \\
&= \sum_{s=0,2} {}^\gamma\langle 0|\phi_{\pi\pi,I}|(\pi\pi)^\gamma_s\rangle e^{-E^\gamma_s t}\langle(\pi\pi)^\gamma_s|\phi^\dagger_{\pi\pi,I'}|0\rangle^\gamma \\
&= (U^\gamma M^\gamma U^{\gamma\dagger})_{II'}
\end{aligned}
$$

where

$$
U^\gamma = \begin{pmatrix} {}^\gamma\langle 0|\phi_{\pi\pi,0}|(\pi\pi)^\gamma_0\rangle & {}^\gamma\langle 0|\phi_{\pi\pi,0}|(\pi\pi)^\gamma_2\rangle \\ {}^\gamma\langle 0|\phi_{\pi\pi,2}|(\pi\pi)^\gamma_0\rangle & {}^\gamma\langle 0|\phi_{\pi\pi,2}|(\pi\pi)_2\rangle \end{pmatrix}, \quad M^\gamma = \begin{pmatrix} e^{-E^\gamma_0 t} & \\ & e^{-E^\gamma_2 t} \end{pmatrix}
$$

Determination of A_s^γ and δ_s^γ from lattice QCD

- Use the coefficient matrix to construct a ratio $U^{-1}U^\gamma = 1 + \begin{pmatrix} N_{00}^{(1)} & N_{02}^{(1)} \\ N_{20}^{(1)} & N_{22}^{(1)} \end{pmatrix}$
- Build a ratio for the 2×2 correlation matrix: $R(t) = C^{-\frac{1}{2}}(t)C^\gamma(t)C^{-\frac{1}{2}}(t)$
- Time dependence of $R(t)$ yields

$$R(t) = \begin{pmatrix} 1 + 2N_{00}^{(1)} + E_0^{(1)}t & N_{20}^{(1)}e^{(E_2-E_0)t/2} + N_{02}^{(1)}e^{(E_0-E_2)t/2} \\ N_{20}^{(1)}e^{(E_2-E_0)t/2} + N_{02}^{(1)}e^{(E_0-E_2)t/2} & 1 + 2N_{22}^{(1)} + E_2^{(1)}t \end{pmatrix}$$

  - $E_s^{(1)} = E_s^\gamma - E_s$ can be used to determine δ_s^γ, $s = 0, 2$
  - $N_{ll'}^{(1)}$ can be used to construct U^γ and compute $A_s^\gamma = \langle(\pi\pi)_s^\gamma|H_W|K^0\rangle$

Need to modify Lüscher quantization condition and Lellouch-Lüscher relation to include EM effects

Coulomb potential with periodic boundary condition

Encode long-range EM interaction in the finite box – QED_L

- Coulomb potential in periodic box $V_L(\mathbf{r}) = \sum_n V(\mathbf{r} + \mathbf{n}L)$
  - $\forall \mathbf{n}$, $V(\mathbf{r} + \mathbf{n}L)$ have impact on small-$\mathbf{r}$ region and cause divergence
- Modify $V_L(\mathbf{r}) \to \hat{V}_L(\mathbf{r}) = V_L(\mathbf{r}) - \frac{1}{L^3}\int d^3\mathbf{r}\, V(\mathbf{r})$ to remove the divergence
  - This is equivalent to remove zero mode: $\hat{V}_L(\mathbf{r}) = \frac{4\pi\alpha_e}{L^3}\sum_{\mathbf{p}\neq\mathbf{0}} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p^2}$
- However, $\hat{V}_L$ introduces $O(1/L)$ FV effects

$$\delta V(\mathbf{r}) \equiv \hat{V}_L(\mathbf{r}) - V(\mathbf{r}) = \left(\frac{1}{L^3}\sum_{\mathbf{p}\neq\mathbf{0}} - \int \frac{d^3\mathbf{p}}{(2\pi)^3}\right)\frac{4\pi\alpha_e}{p^2}e^{i\mathbf{p}\cdot\mathbf{r}}, \quad \lim_{\mathbf{r}\to 0}\delta V(\mathbf{r}) = -\kappa\frac{\alpha_e}{L} \approx -2.8\frac{\alpha_e}{L}$$

- We get Lüscher quantization $\phi_c(E) + \delta(E) = n\pi$ with $\eta = \frac{\alpha_e\mu}{k} = \frac{\alpha_e}{v}$

$$\cot\phi_c(E) = (1+\pi\eta)\frac{1}{\pi}\frac{1}{kL}\sum_{\mathbf{n}}\frac{1}{-\mathbf{n}^2 + (\frac{kL}{2\pi})^2}$$

$$+\lim_{r\to 0} 8\pi\eta\left\{\sum_{\mathbf{n}\neq\mathbf{m}}\frac{e^{i\mathbf{n}\cdot\mathbf{r}\frac{2\pi}{L}}}{\pi(2\pi)^4}\frac{1}{\mathbf{n}^2-(\frac{kL}{2\pi})^2}\frac{1}{(\mathbf{n}-\mathbf{m})^2}\frac{1}{\mathbf{m}^2-(\frac{kL}{2\pi})^2} - \frac{1}{4\pi}\ln(1/kr) + \frac{1}{4\pi}\right\}$$

(See also formula for scattering length [Bean & Savage, 1407.4846])

Coulomb potential with truncated range $R_T \leq L/2$

Not unique way to build periodic potential

$$V_L^{(T)}(\mathbf{r}) = \sum_{\mathbf{n}} V^{(T)}(\mathbf{r} + \mathbf{n}L), \quad V^{(T)}(\mathbf{r}) = \begin{cases} \alpha_e/r, & \text{for } r < R_T \\ 0, & \text{for } r > R_T \end{cases}$$

Lüscher's quantization condition holds for $V_s(\mathbf{r}) + V^{(T)}(\mathbf{r})$

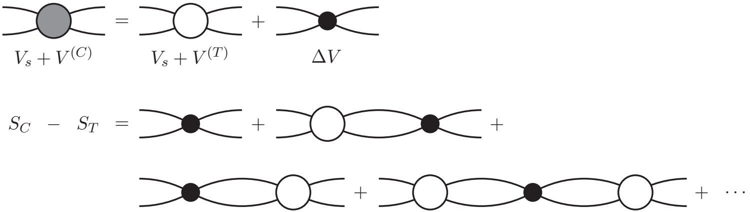
$$\phi(q) + \delta_T(k) = n\pi, \quad q = \frac{kL}{2\pi}$$

- Unfortunately $\delta_T(k)$ is unphysical

Relate the truncated scattering phase to the physical one

$$S_C = \langle E', -, C | E, +, C \rangle = 2\pi\delta(E - E') e^{2i\delta_C}$$
$$S_T = \langle E', -, T | E, +, T \rangle = 2\pi\delta(E - E') e^{2i\delta_T}$$

Skeleton expansion in Potential Theory



Truncation effects in scattering phase

The relation for scattering amplitude

$$S_C = S_T - i\,2\pi\delta(E - E')\,\langle E, -, T|\Delta V|E, +, T\rangle$$

The relation for scattering phase

$$\boxed{\delta_C = \delta_T - \frac{1}{2}\langle E, +, T|\Delta V|E, +, T\rangle}$$

- $\Delta V(r)$ is non-zero only for $r > R_T$
- For $\psi(r) = \langle r|E, +, T\rangle$, the functional form is known for $r > R_T$

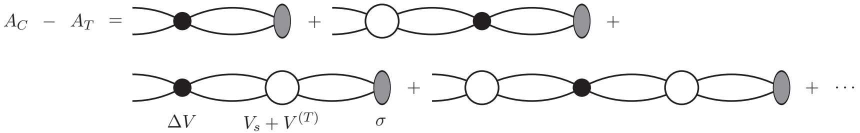
$$\psi(r) = \sqrt{\frac{\mu}{\pi k}}\frac{\sin(kr + \delta_T)}{r}, \quad \text{for S-wave}$$

- Correction to scattering phase can be evaluated

$$\langle E, +, T|\Delta V|E, +, T\rangle = \int_{R_T}^{R_\infty} d^3\mathbf{r}\left[\sqrt{\frac{\mu}{\pi k}}\frac{\sin(kr + \delta_T)}{r}\right]^2 \frac{\alpha_e}{r}$$

$\sigma \to \pi\pi$ decay amplitude

$$A_C = \langle E, -, C|\sigma\rangle, \quad A_T = \langle E, -, T|\sigma\rangle$$



The relation for decay amplitude

$$A_C - A_T = \langle E, -, T|\Delta V\, G_{TS}^{(+)}|\sigma\rangle = \langle E, -, T|\Delta V\, G_0^{(+)}\left(1 + V_{TS}G_{TS}^{(+)}\right)|\sigma\rangle$$

- ΔV is non-zero at $r > R_T$; $V_{TS} = V_s + V^{(T)}$ is non-zero at $r < R_T$
- The free Green function $\langle \mathbf{r}|G_0^{(+)}|\mathbf{r}'\rangle$ for $r > R_T$ and $r' < R_T$ is given by

$$\langle \mathbf{r}|G_0^{(+)}|\mathbf{r}'\rangle = \int \frac{dE'}{2\pi}\langle \mathbf{r}|E'\rangle \frac{1}{E - E' + i\varepsilon}\langle E'|\mathbf{r}'\rangle \quad \xrightarrow{r>r'} \quad -\frac{1}{2}\sqrt{\frac{\mu}{\pi k}}\frac{e^{ikr}}{r}\langle E|\mathbf{r}'\rangle$$

Truncation effects can be determined

$$A_C - A_T = \int d^3\mathbf{r}\, \psi(r)\frac{\alpha}{r}\left(-\frac{1}{2}\sqrt{\frac{\mu}{\pi k}}\frac{e^{ikr}}{r}\right) A_T$$

Outlook

- As the precision increases, our research frontiers expand
  $\Rightarrow$ LD processes, multihadron interactions, E&M corrections, $\cdots$
- Calculation of the non-local matrix element is highly non-trivial
  $\Rightarrow$ but we manage to calculate them now
- Other interesting bilocal system
  - rare B decays: $B \to K^* \ell^+ \ell^-$ and DIS processes
  - electromagnetic corrections to (semi-)leptonic decay and $K \to \pi\pi$
  - double beta decay: $0\nu\beta\beta$
  - ...

LD processes: an exciting and new area for lattice QCD

Backup slides

Results for $\mathrm{Re}[A_0]$, $\mathrm{Im}[A_0]$ and $\mathrm{Re}[\epsilon'/\epsilon]$

- Lattice results for A_2 @ $m_\pi = 140$ MeV [RBC-UKQCD, PRD91 (2015)]

$$\mathrm{Re}[A_2] = 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8} \text{ GeV}$$
$$\mathrm{Im}[A_2] = -6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13} \text{ GeV}$$

- Lattice results for A_0 @ $m_\pi = 140$ MeV [RBC-UKQCD, PRL115 (2015)]

$$\mathrm{Re}[A_0] = 4.66(1.00)_{\mathrm{stat}}(1.26)_{\mathrm{syst}} \times 10^{-7} \text{ GeV}$$
$$\mathrm{Im}[A_0] = -1.90(1.23)_{\mathrm{stat}}(1.08)_{\mathrm{syst}} \times 10^{-11} \text{ GeV}$$

- Experimental measurement

$$\mathrm{Re}[A_2] = 1.479(3) \times 10^{-8} \text{ GeV}$$
$$\mathrm{Re}[A_0] = 3.3201(18) \times 10^{-7} \text{ GeV}$$
$$\mathrm{Im}[A_2] \,\&\, \mathrm{Im}[A_0] \text{ are unknown}$$

- Determine the direct CP violation $\mathrm{Re}[\epsilon'/\epsilon]$

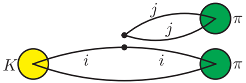
$$\mathrm{Re}[\epsilon'/\epsilon] = 0.14(52)_{\mathrm{stat}}(46)_{\mathrm{syst}} \times 10^{-3} \quad \text{Lattice}$$
$$\mathrm{Re}[\epsilon'/\epsilon] = 1.66(23) \times 10^{-3} \quad \text{Experiment}$$

2.1 σ deviation $\Rightarrow$ require more accurate lattice results

Resolve the puzzle of $\Delta I = 1/2$ rule

$\Delta I = 1/2$ rule: $A_0 = 22.5 \times A_2$ $\Rightarrow$ a > 50 year puzzle

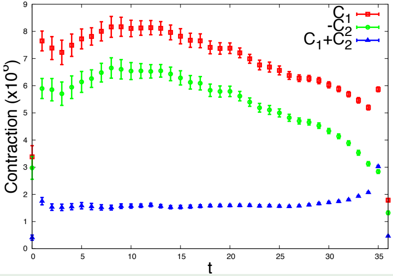
- Wilson coefficient only contributes a factor of ~ 2
- $\mathrm{Re}[A_2]$ and $\mathrm{Re}[A_0]$ are dominated by diagrams C_1 and C_2



C_1: color diagonal C_2: color mixed

Color counting in LO PT $\Rightarrow$ $C_2 = C_1/3$; Non-PT effects $\Rightarrow$ $C_2 \approx -0.7 C_1$

- $\mathrm{Re}[A_2] \propto C_1 + C_2$, while $\mathrm{Re}[A_0] \propto 2C_1 - C_2$ $\Rightarrow$ another factor of ~ 10
  - Such cancellation is first observed in an earlier calculation [RBC-UKQCD, PRL110 (2013) 152001]
  - It is further confirmed in the latest calculation of A_2 [RBC-UKQCD, PRD91 (2015) 074502]



Puzzle of $\Delta I = 1/2$ rule is resolved from first principles

Solve the coupled-channel problem

Generalize Lüscher quantization condition to coupled-channel scattering

[He, XF, Liu, hep-lat/0504019; Hansen & Sharpe, 1204.0826]

$$\sin(\delta_0 + \phi(q_{+-}))\sin(\delta_2 + \phi(q_{00})) + \sin^2\theta^\gamma \sin(\delta_0 - \delta_2)\sin(\phi(q_{+-}) - \phi(q_{00})) = 0$$

where $q_c = \frac{k_c L}{2\pi}$, $c = +-, 00$, $k_{+-} = \sqrt{E^2/4 - m_{\pi^+}^2}$, $k_{00} = \sqrt{E^2/4 - m_{\pi^0}^2}$, $E \approx m_K$

Quantization condition involves three unknown quantities: δ_0, δ_2, θ^γ

- Solution 1: tune three volumes to make $E_0(L_1) = E_1(L_2) = E_2(L_3) = E$?
- Solution 2: Introduce functional form for E-dependence of δ_0, δ_2, θ'

If mixing is caused by EM effect, the situation is simpler

- Note that $\sin(\phi(q_{+-}) - \phi(q_{00}))$ is an $O(\alpha_e)$ quantity

$$\sin(\phi(q_{+-}) - \phi(q_{00})) = \frac{d\phi}{dq}(q_{+-}^{(1)} - q_{00}^{(1)}) + O(\alpha_e^2)$$

where $q_{+-}^{(1)}$ and $q_{00}^{(1)}$ originate from $O(\alpha_e)$ correction to m_{π^+} and m_{π^0}

- Tune the volume to have $E_{I=0}^{(0)} = M_K$

$$\sin(\delta_0 + \phi(q_{+-})) = \delta_0^{(1)} + \frac{d\phi}{dq}q_{+-}^{(1)} + \frac{d(\delta_0^{(0)} + \phi)}{dE}E_{I=0}^{(1)} + O(\alpha_e^2)$$

$$\sin(\delta_2 + \phi(q_{00})) = \sin(\delta_2^{(0)} - \delta_0^{(0)}) + O(\alpha_e)$$

Coupled-channel Lüscher quantization condition is simplified

- Tune the volume to have $E^{(0)}_{I=0} = M_K$

$$\delta_0^{(1)} + \left(\cos^2\theta\, q_{+-}^{(1)} + \sin^2\theta\, q_{00}^{(1)}\right)\frac{\partial\phi}{\partial q} + E^{(1)}_{I=0}\frac{\partial}{\partial E}\left(\delta_0^{(0)} + \phi\right) = 0$$

- Tune the volume to have $E^{(0)}_{I=2} = M_K$

$$\delta_2^{(1)} + \left(\cos^2\theta\, q_{00}^{(1)} + \sin^2\theta\, q_{+-}^{(1)}\right)\frac{\partial\phi}{\partial q} + E^{(1)}_{I=2}\frac{\partial}{\partial E}\left(\delta_2^{(0)} + \phi\right) = 0$$

The relations for $E^{(1)}_{I=0} \to \delta_0^{(1)}$ and $E^{(1)}_{I=2} \to \delta_2^{(1)}$ are established

Extract infinite-volume amplitude

Coupled-channel Lellouch-Lüscher relation [Hansen & Sharpe, 1204.0826]

- Tune the volume to have $E_{I=2} = M_K \quad \Rightarrow \quad \sin(\delta_2 + \phi) = O(\alpha_e)$

$$\left(2\frac{d\Delta}{dE}\right)^{\frac{1}{2}} |A^{\gamma}_{2,L}| = \underbrace{\sqrt{\frac{\sin(\delta_0+\phi_{00})\sin(\delta_0+\phi_{+-})}{\sin(\delta_0-\delta_2)}}}_{=\sqrt{\sin(\delta_0-\delta_2)}+O(\alpha_e)} |A^{\gamma}_2| - s\underbrace{\frac{\sin 2\theta\sin(\phi_{00}-\phi_{+-})}{2\sqrt{\sin(\delta_0-\delta_2)}}}_{=O(\alpha_e)} |A^{\gamma}_0|$$

where $\phi_c = \phi(q_c)$, $c = \pm, 00$ and $s = \mathrm{sgn}(A^{\gamma}_0/A^{\gamma}_2)$

- On the LHS, $|A^{\gamma}_{2,L}|$ is the amplitude calculated in the finite volume

$$\Delta = \sin(\delta_0+\phi_{+-})\sin(\delta_2+\phi_{00}) + \sin^2\theta'\sin(\delta_0-\delta_2)\sin(\phi_{+-}-\phi_{00})$$

$$\Rightarrow \quad \Delta = 0 \text{ is the coupled-channel quantization condition}$$

$$\Rightarrow \quad \frac{d\Delta}{dE} = \frac{d(\delta_2+\phi)}{dE}\sin(\delta_0-\delta_2) + O(\alpha_e)$$

  - If turn off EM $\quad \Rightarrow \quad$ single-channel Lellouch-Lüscher relation

$$\left(2\frac{d(\delta_2+\phi)}{dE}\right)^{\frac{1}{2}} |A^{\gamma}_{2,L}| = |A^{\gamma}_2|$$