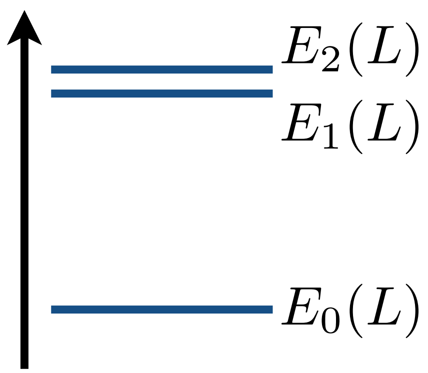
Multi-hadron observables from lattice QCD

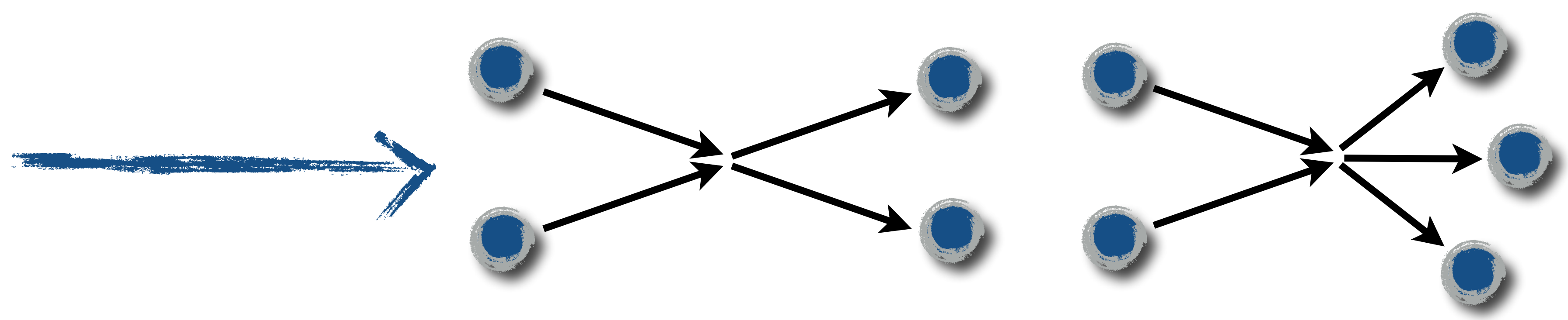
Maxwell T. Hansen

May 25th, 2018

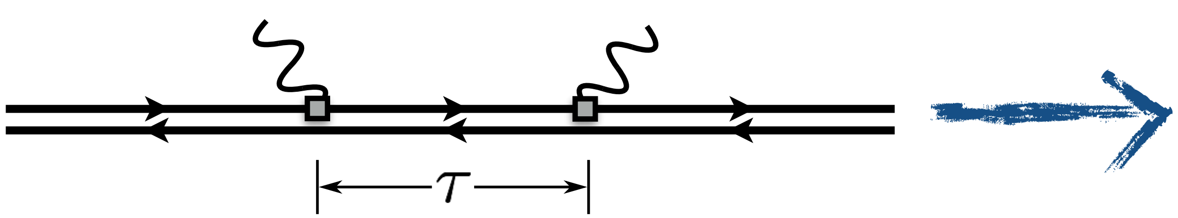
Outline

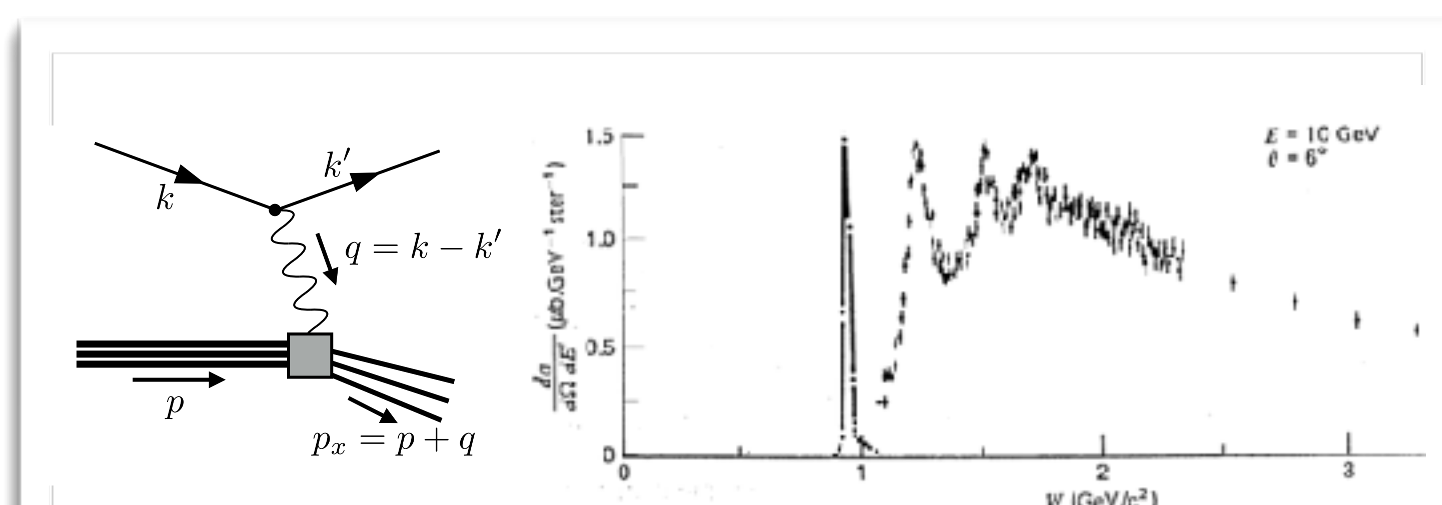
Three-particle scattering





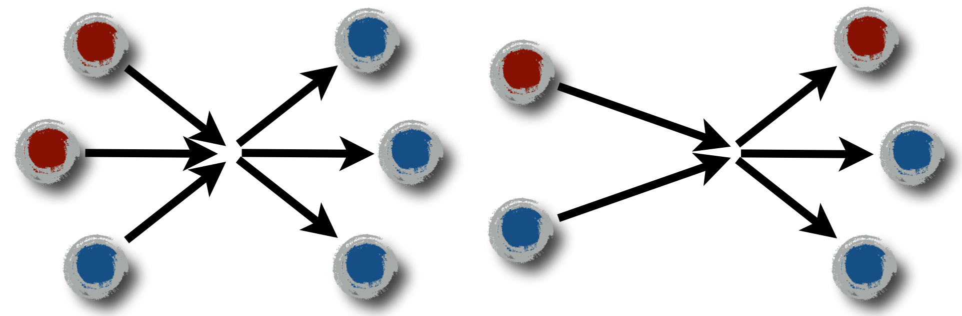
n-particle inclusive rates





taken from Halzen and Martin

The aim is to derive a formalism for studying relativistic two- and three-particle systems from lattice QCD



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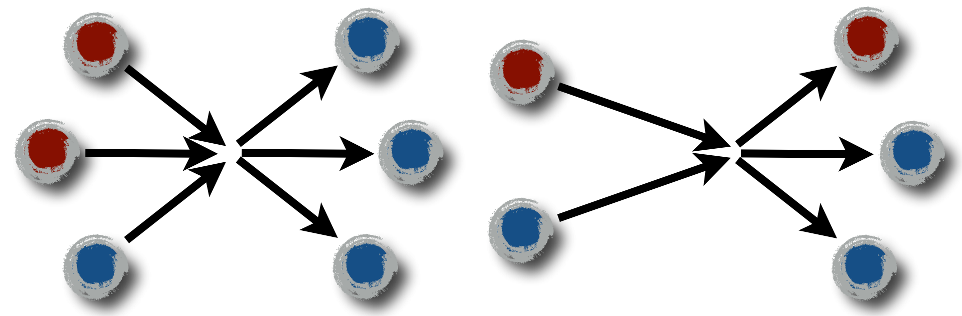
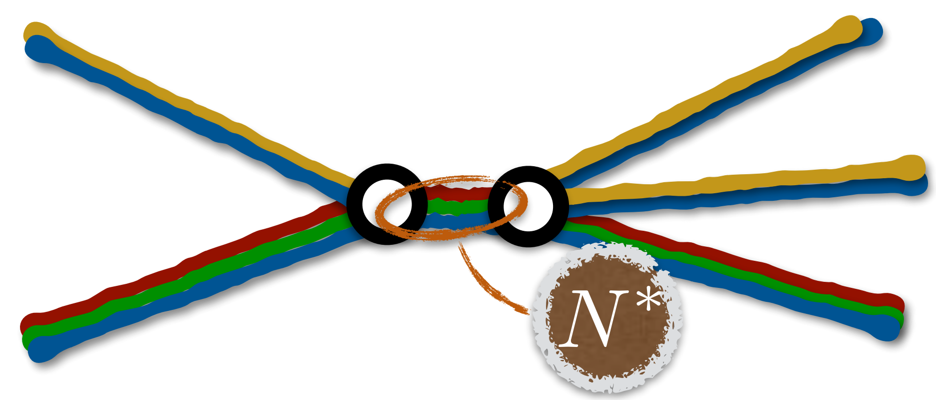


Potential applications…

Studying three-particle resonances

$$\omega(782),\ a_1(1420) \to \pi\pi\pi$$

$$N(1440) \to N\pi, N\pi\pi$$

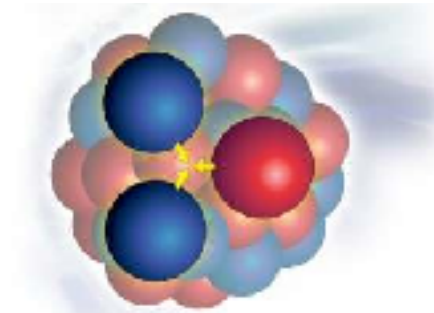


Calculating weak decays, form factors and transitions

$$K \to \pi\pi\pi \qquad N\gamma^* \to N\pi\pi$$

Determining three-body interactions

NNN three-body forces needed as EFT input for studying larger nuclei and nuclear matter



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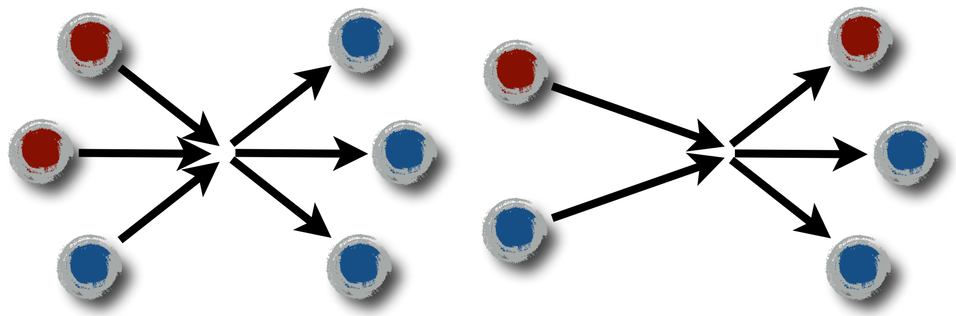

Other motivations…

The Lüscher and Lellouch-Lüscher formalism only applies for

$\sqrt{s} = E_{\rm cm} <$ **multi-particle threshold**

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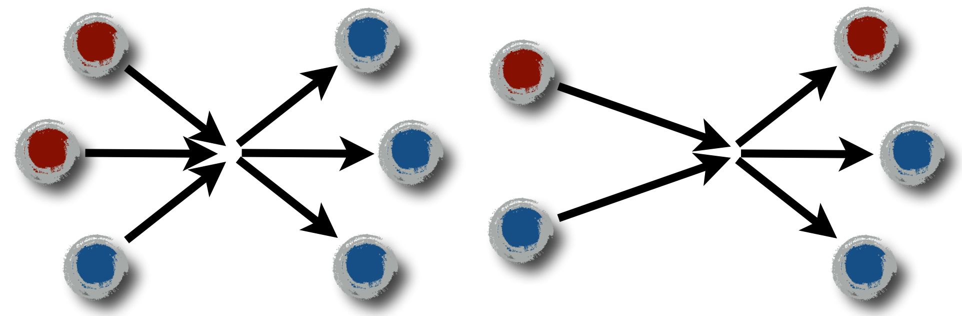

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$\pi K \to \pi K$		$\sqrt{s} = 2m_\pi + m_K$
$\pi\pi \to \pi\pi$	only up to	$\sqrt{s} = 4m_\pi$
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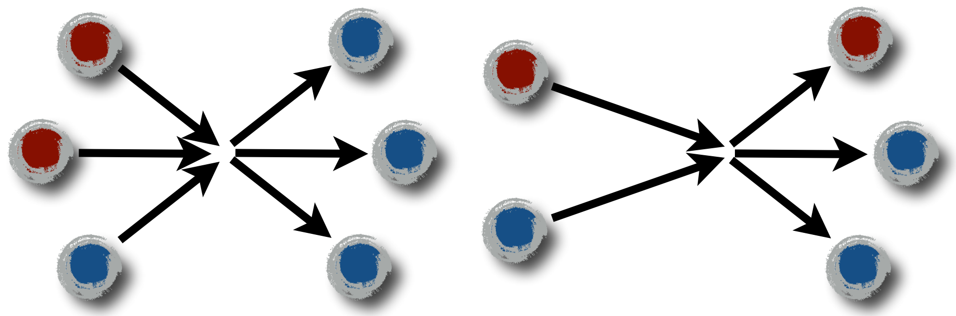

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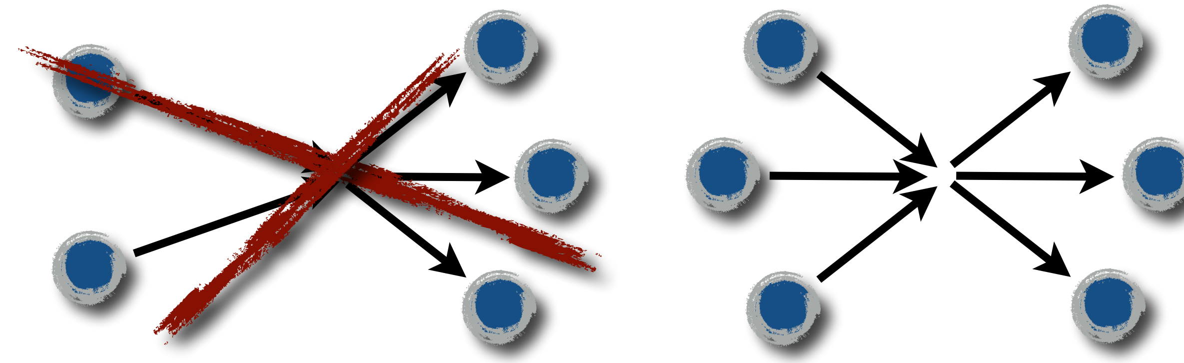
3-particle formalism is a step on the way to *n*-particle formalism

We focus here on identical scalar particles



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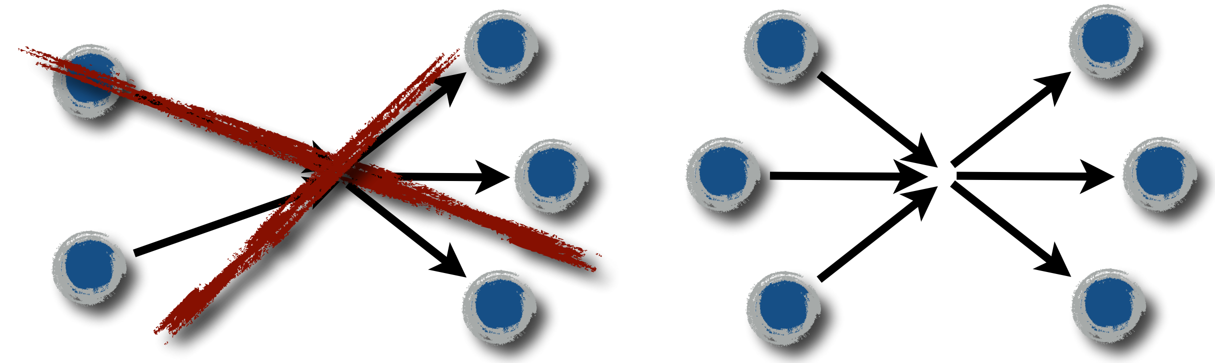

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Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\to 3} \equiv$ fully connected correlator with six external legs amputated and projected on shell

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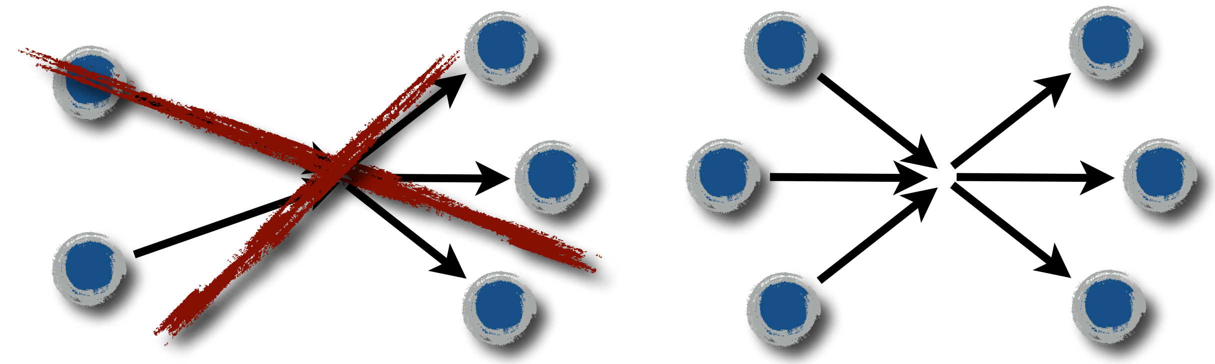

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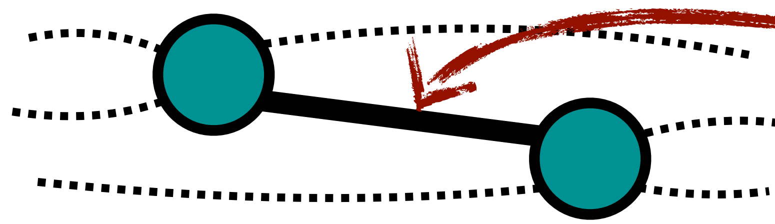
$=$  $+ \cdots$

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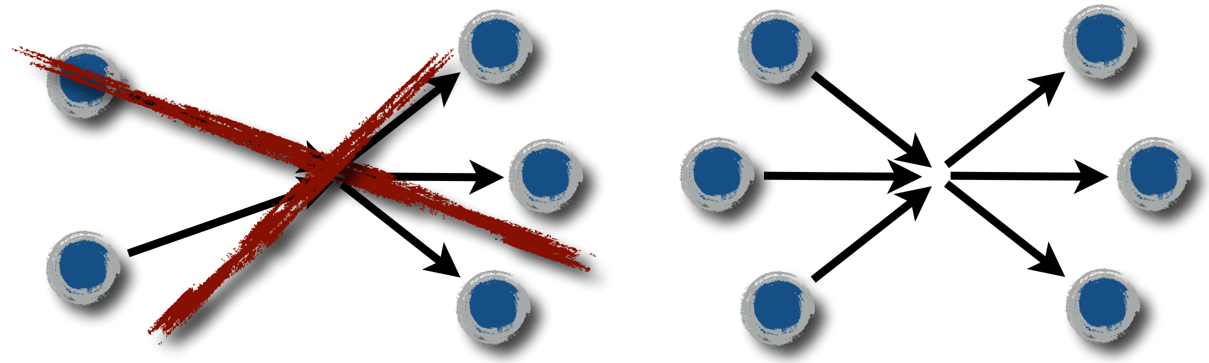

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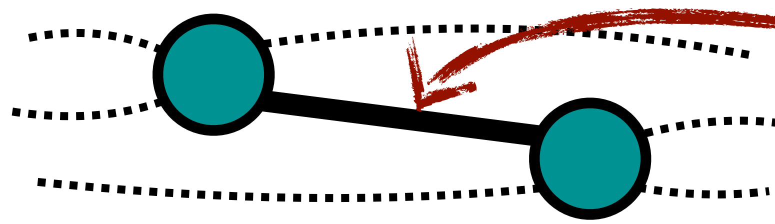
$=$  $+\cdots$

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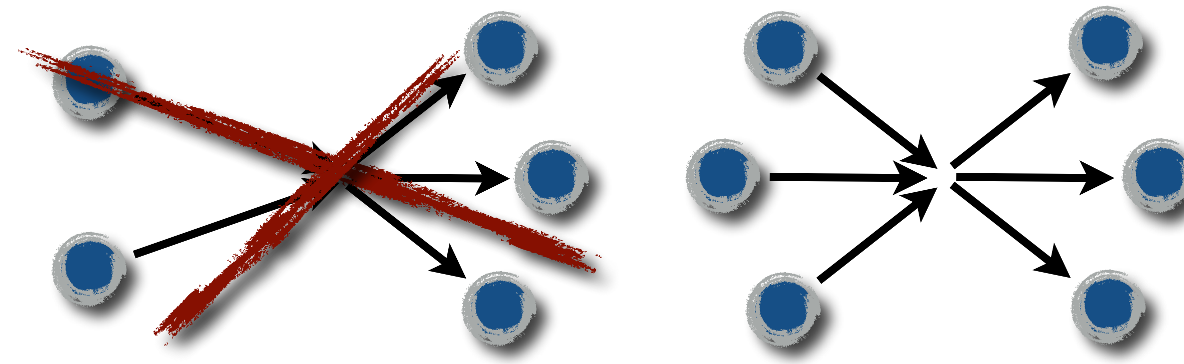

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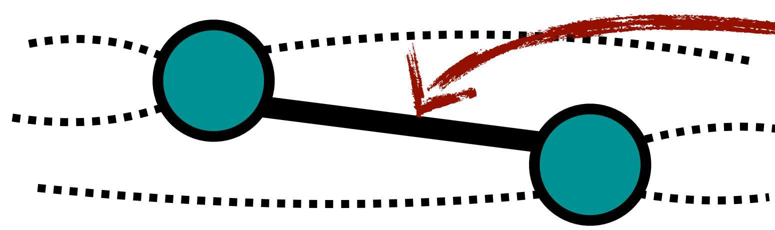
$=$  $+\cdots$

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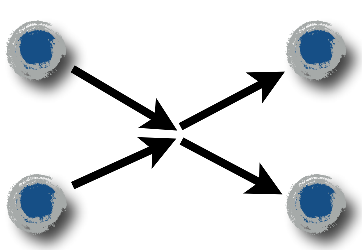

12 momentum components

-10 Poincaré generators

2 degrees of freedom

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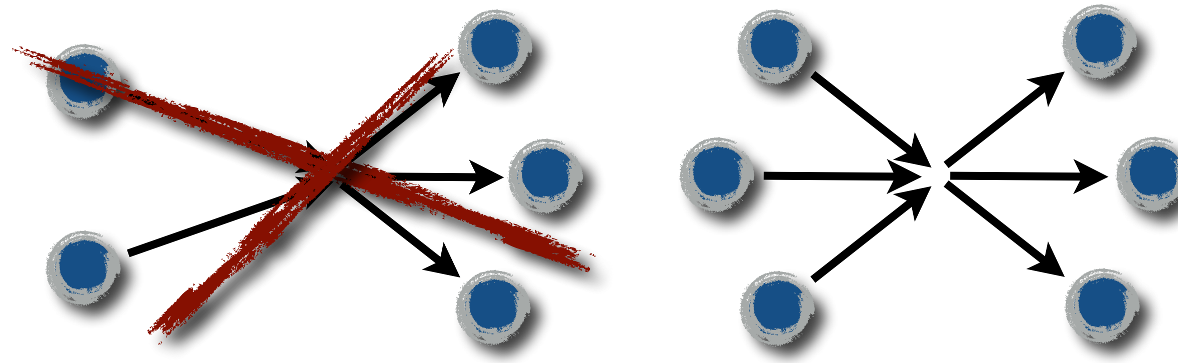

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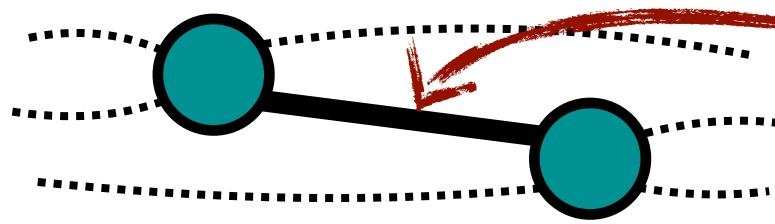

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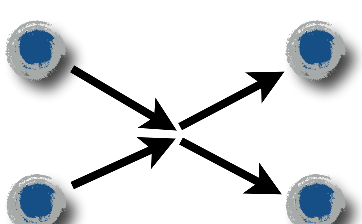

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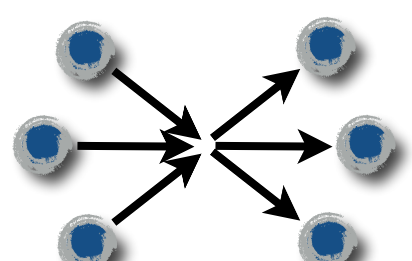


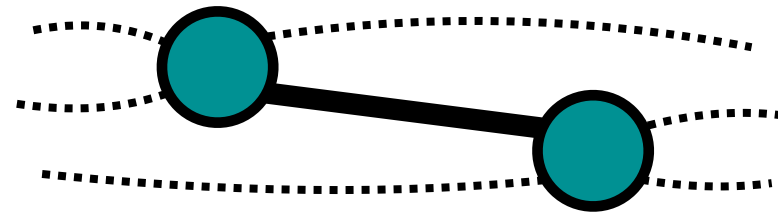
18 momentum components

-10 Poincaré generators

8 degrees of freedom

How can we extract a singular, eight-coordinate function using finite-volume energies?

Spectrum depends on a modified quantity with singularities removed

$$\mathcal{K}_{\mathrm{df},3} \not\ni$$ 

PV pole prescription

Same degrees of freedom as $\mathcal{M}_3$

df stands for "divergence free"

Smooth, real function (easier to extract)

Relation to $\mathcal{M}_3$ is known (depends only on on-shell $\mathcal{M}_2$)

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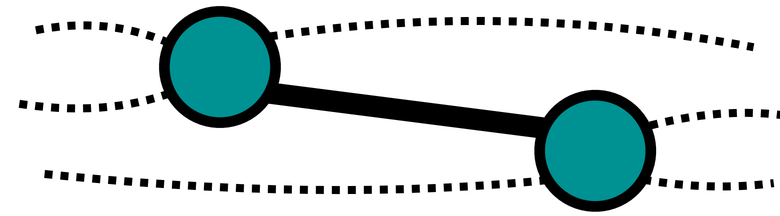

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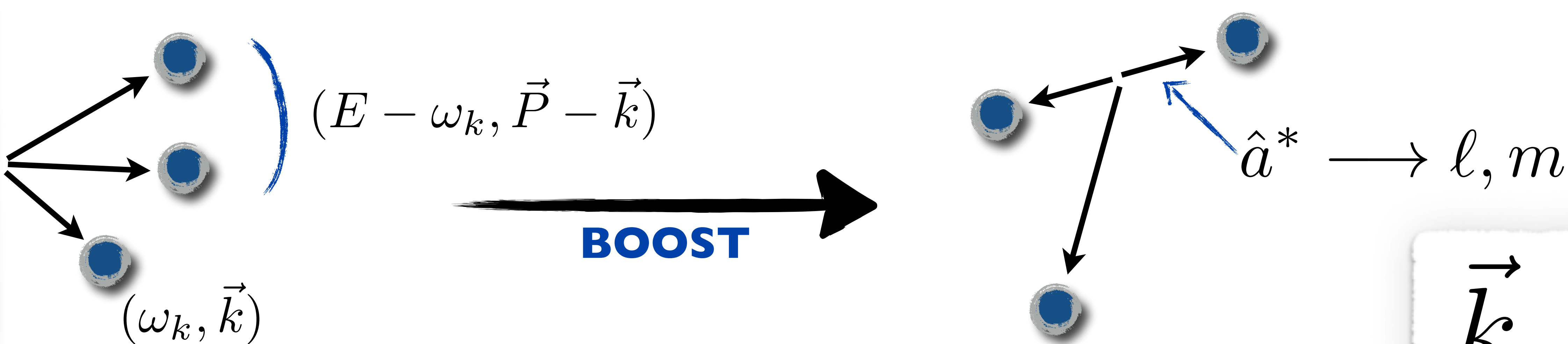
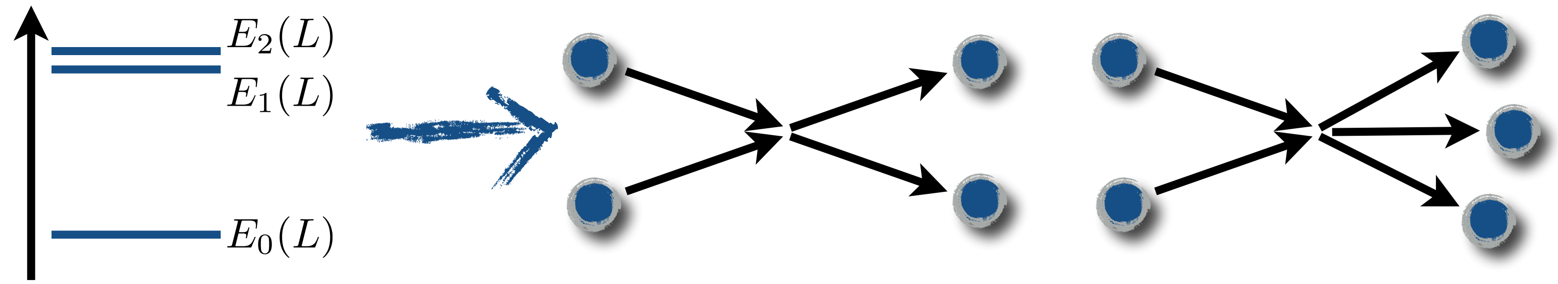
Degrees of freedom encoded in an extended matrix space



($\vec{k}$ is restricted to finite-volume momenta)

$$\boxed{\vec{k}, \ell, m}$$

Finite-volume correlators



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

Total 4-momentum

$$P = (E, \mathbf{P}) = (E, 2\pi\mathbf{n}_P/L)$$

c.m. frame energy: $E^{*2} = E^2 - \mathbf{P}^2$

n-particle interpolator

e.g. $\pi(\mathbf{p})\pi(-\mathbf{p})$ **or** $\bar{q}\Gamma q$

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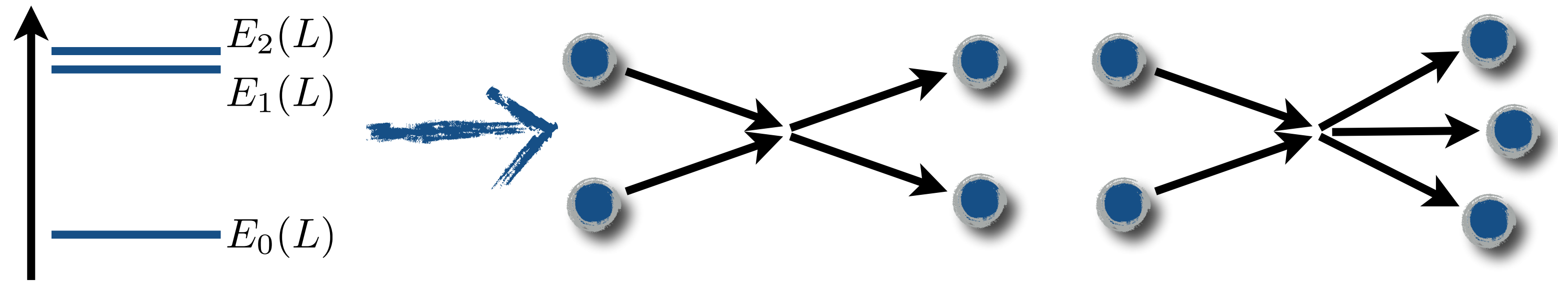

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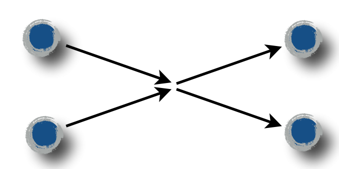
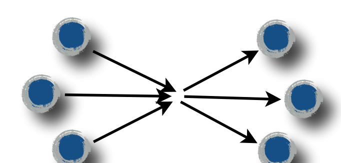
Focus on a widow of energies to isolate particular on-shell states

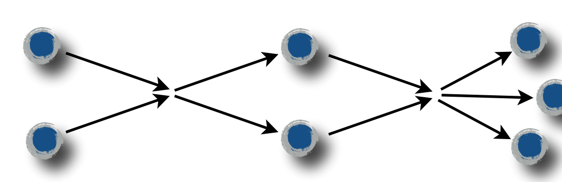
- **Two particles:** $\boxed{\mathbb{Z}_2 \checkmark}$ $0 < E^* < 4m$ $\boxed{\cancel{\mathbb{Z}_2}}$ $m < E^* < 3m$



- **Three particles** $\boxed{\mathbb{Z}_2 \checkmark}$ $m < E^* < 5m$



- **Two and three particles** $\boxed{\cancel{\mathbb{Z}_2}}$ $m < E^* < 4m$

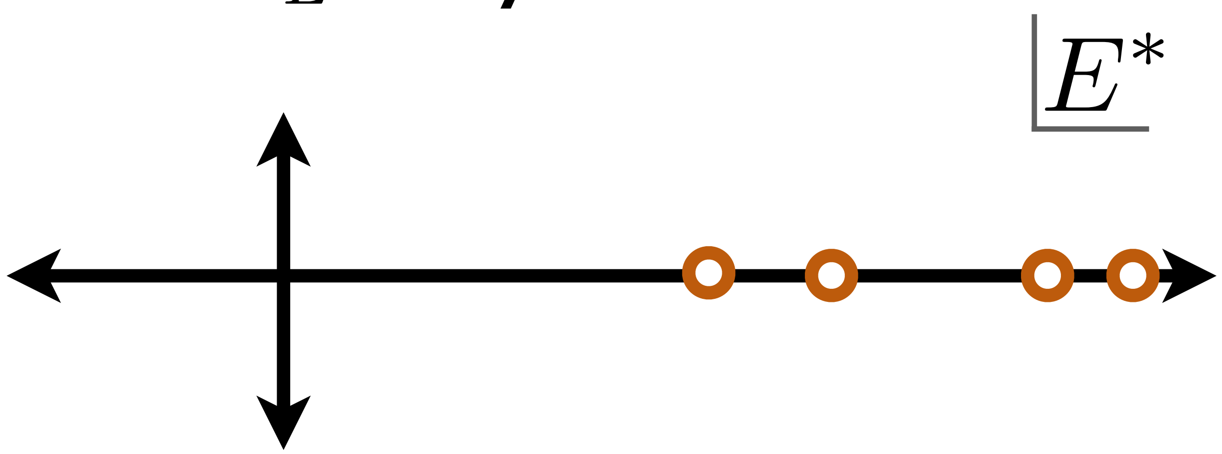


Of poles and branch cuts

$$C_L(P) \equiv \int_L d^4x\ e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

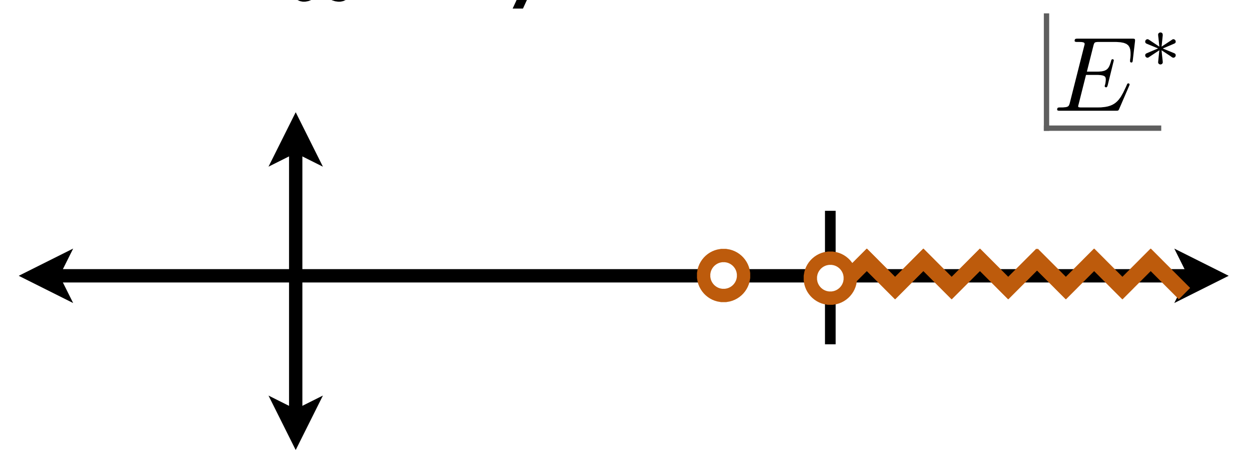
At fixed $L, \mathbf{P}$ poles in C_L give the finite-volume spectrum

C_L analytic structure



- Real function (unitarity hidden)
- No cuts → only one sheet
- No resonance poles

C_∞ analytic structure



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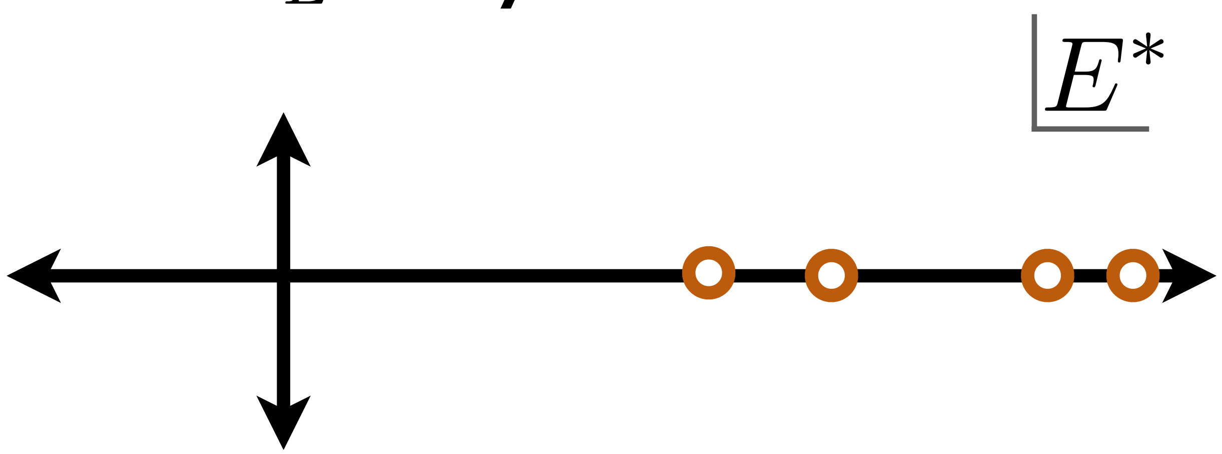

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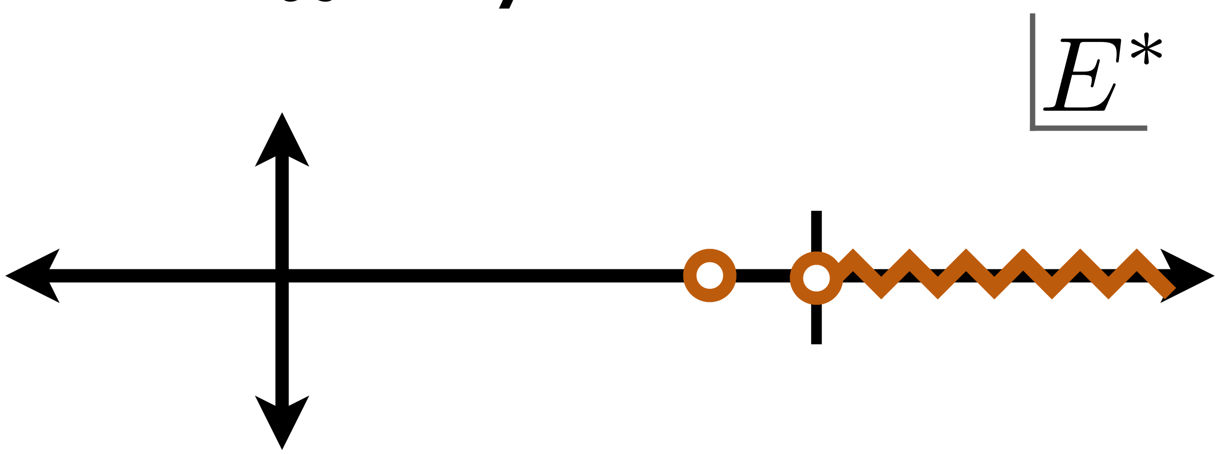

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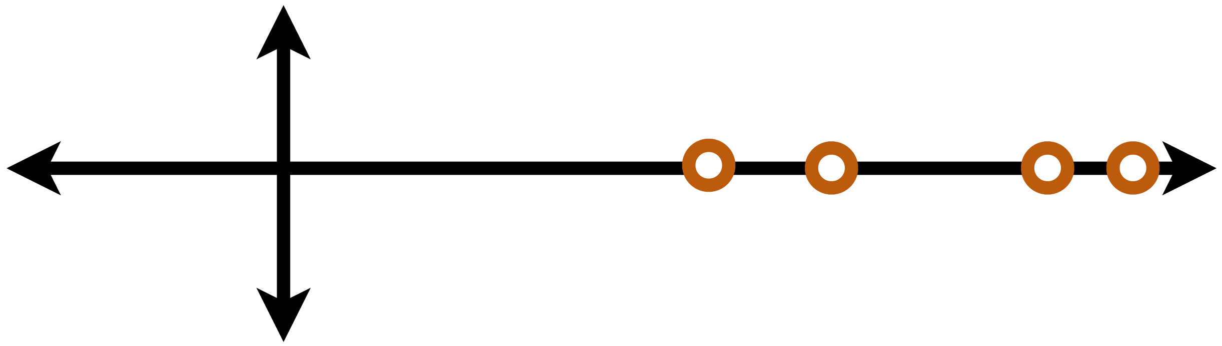
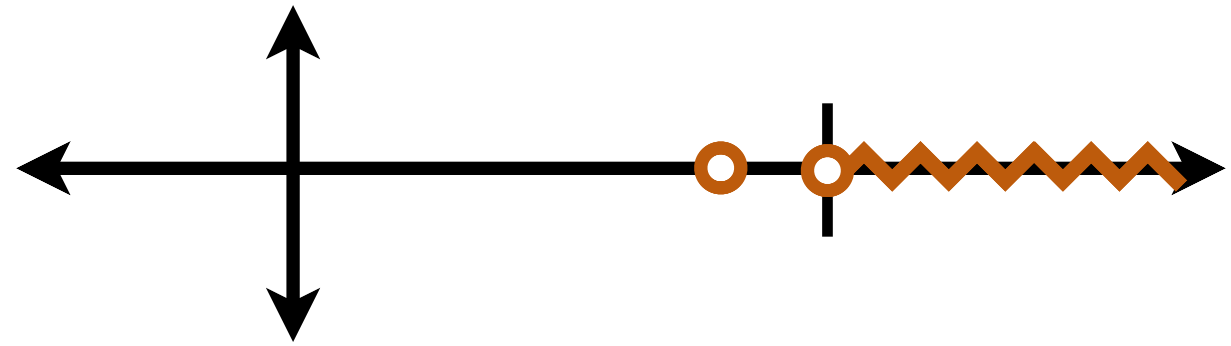
Want to relate $C_L \longleftrightarrow C_\infty$ ($\mathcal{M}_{n\to m}$ is just a specific choice of C_∞)

The idea is to "reach in" and correct the singularity structure

From poles to cuts: Toy example





$$f(x,y,\Delta n) \equiv \sum_{n=-10,-10+\Delta n,\cdots}^{10} \Delta n \,\mathrm{Im}\frac{1}{n-x-iy}$$

$$f_\infty(x,y) \equiv \mathrm{Im}\log\left[\frac{10-x-iy}{-10-x-iy}\right]$$

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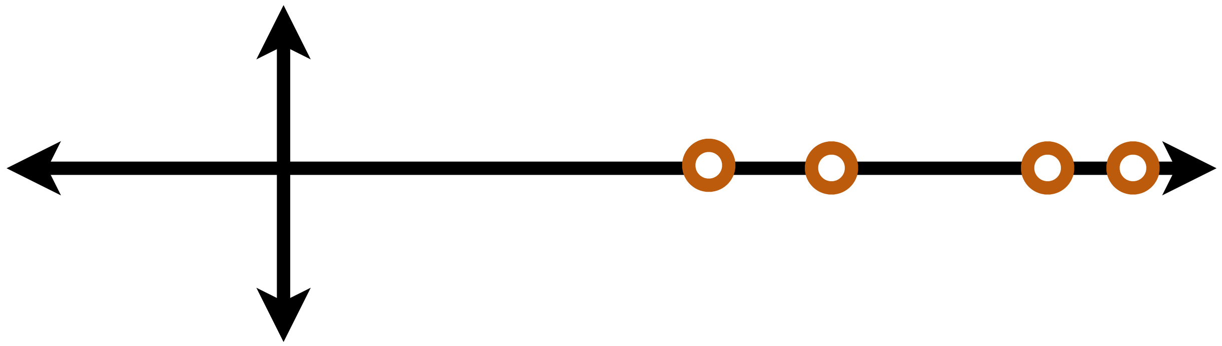

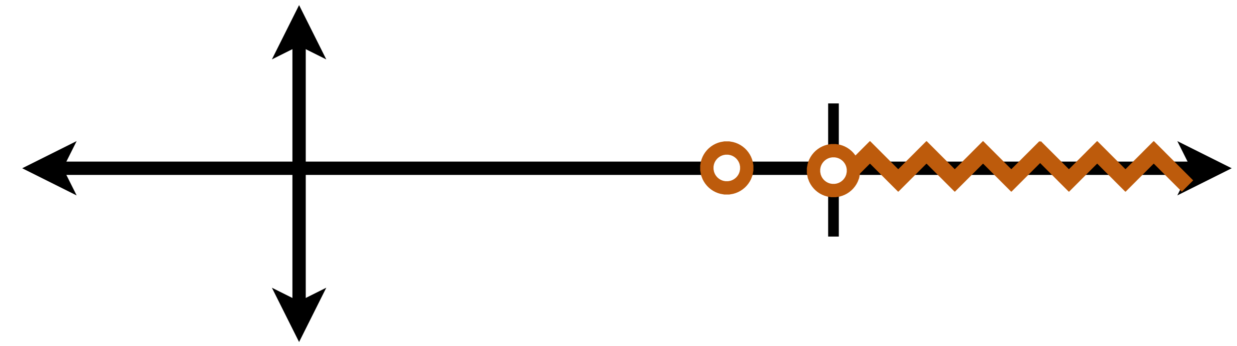

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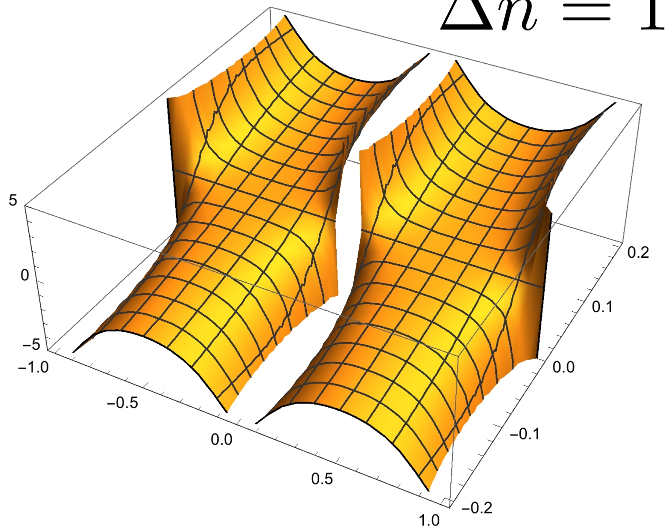
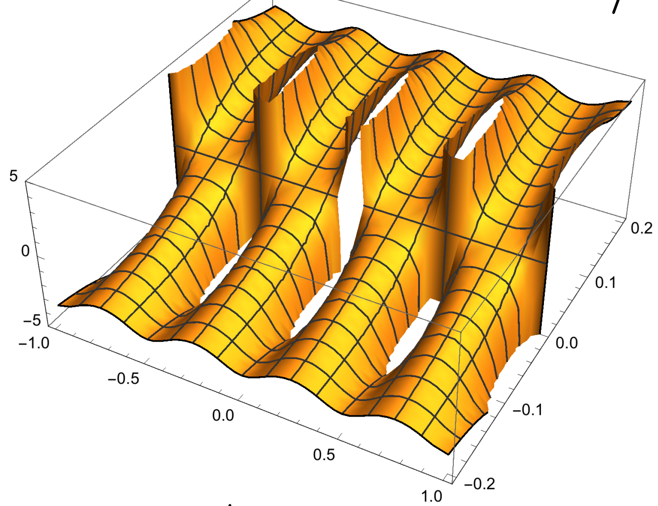


$\Delta n = 1/2$



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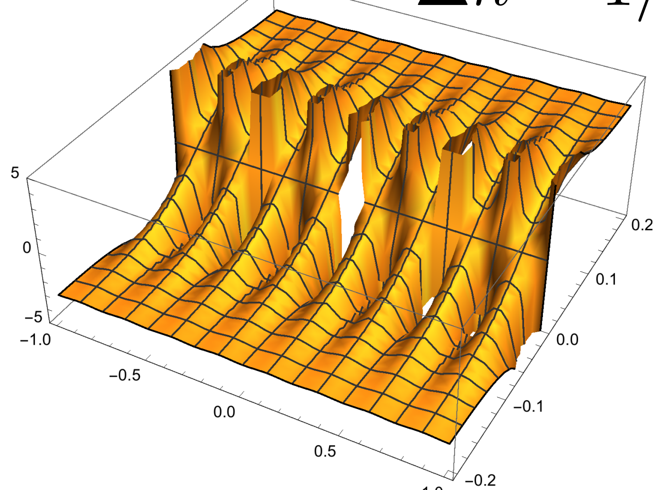

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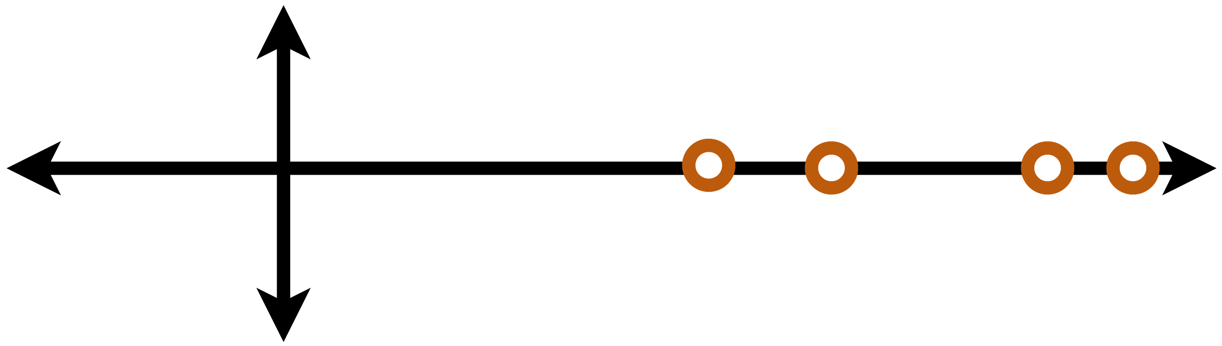

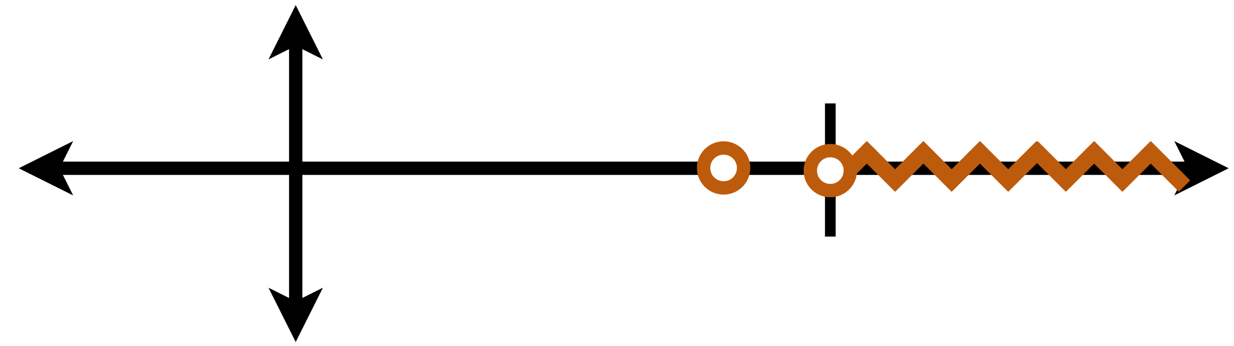

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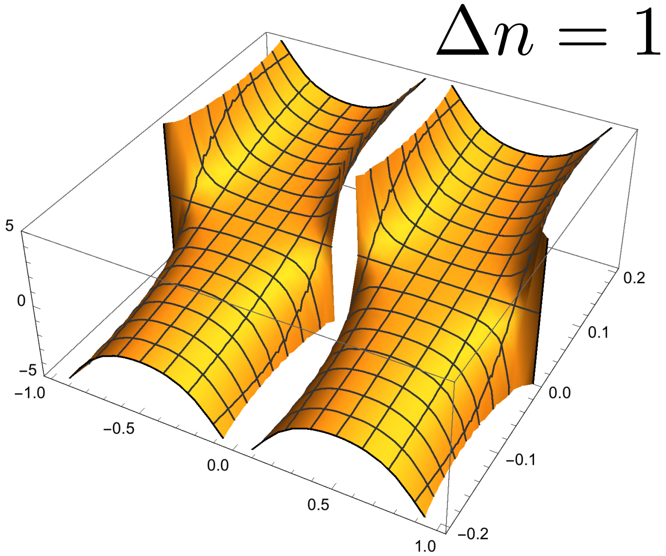

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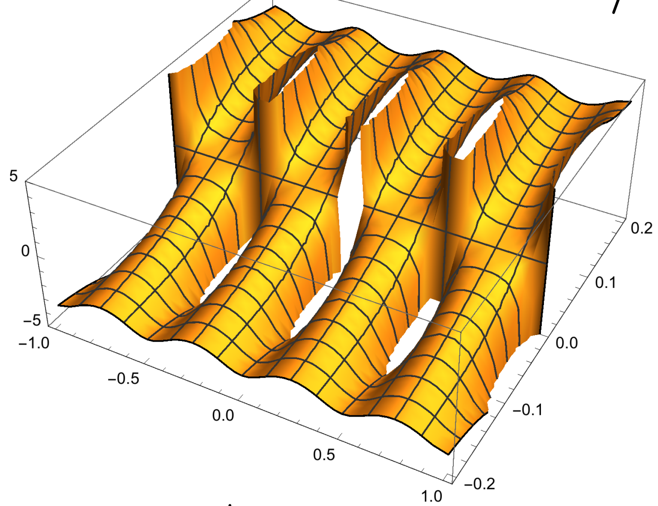

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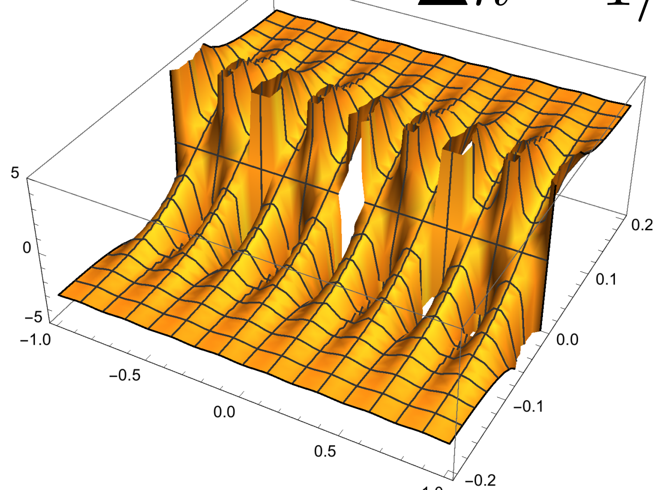
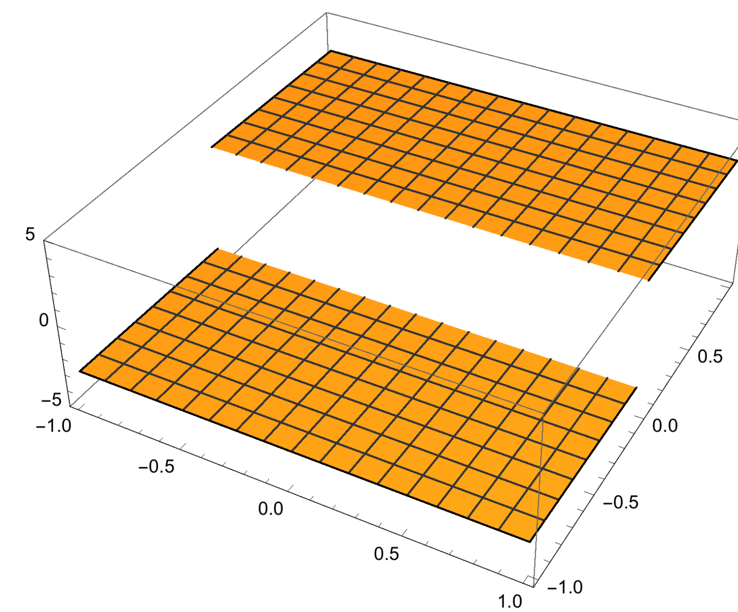
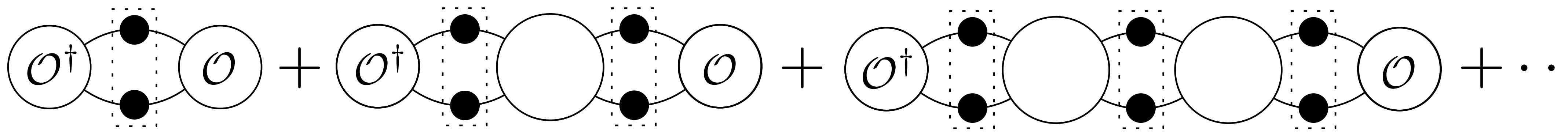




Only in the infinite-volume limit can we generate the singularity that leads to a second sheet

New skeleton expansion

Recall for two particles we started with a "skeleton expansion"

$C_L(P) =$ 

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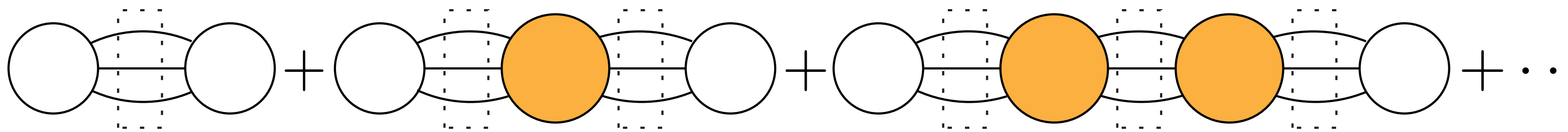
$C_L(E, \vec{P}) \stackrel{?}{=}$ 

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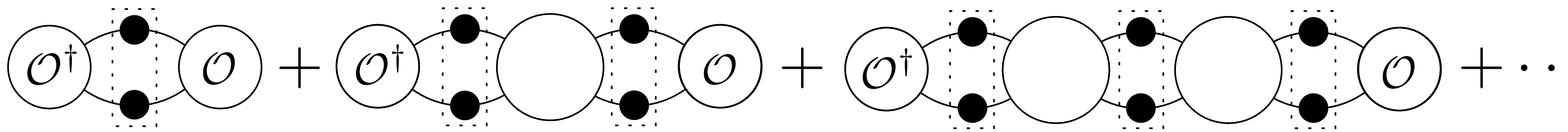
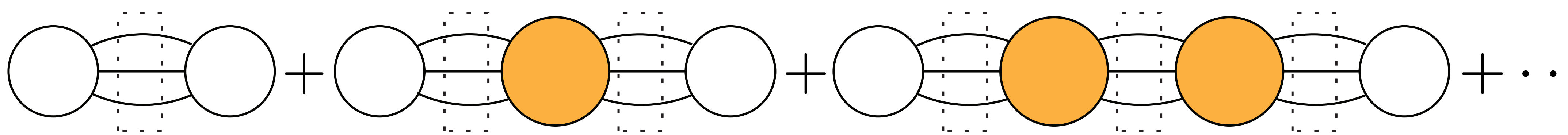
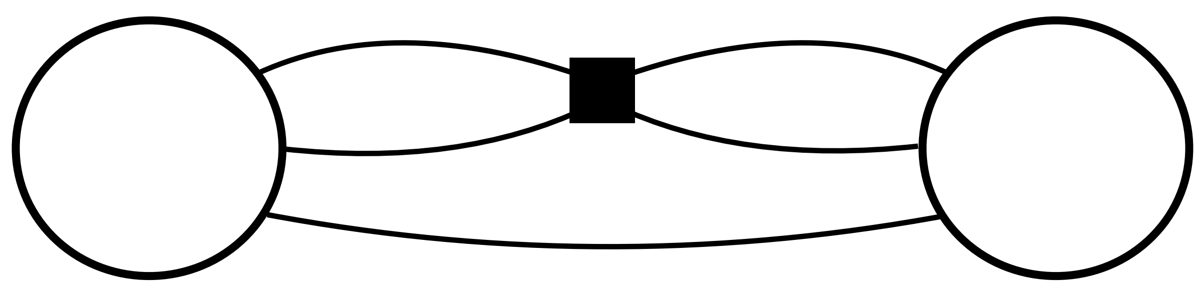
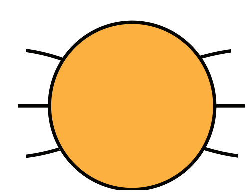
$C_L(P) =$ 

So now we need the same for three…

$C_L(E, \vec{P}) \overset{?}{=}$ 

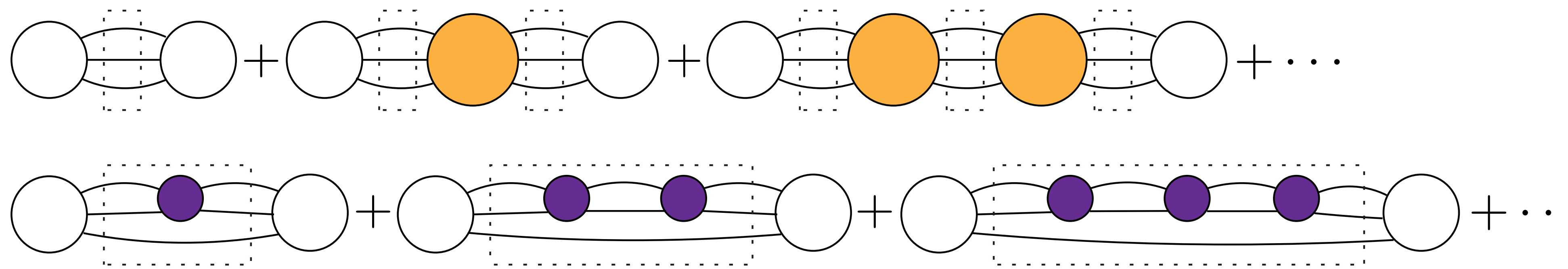
No!… We must also accommodate diagrams like



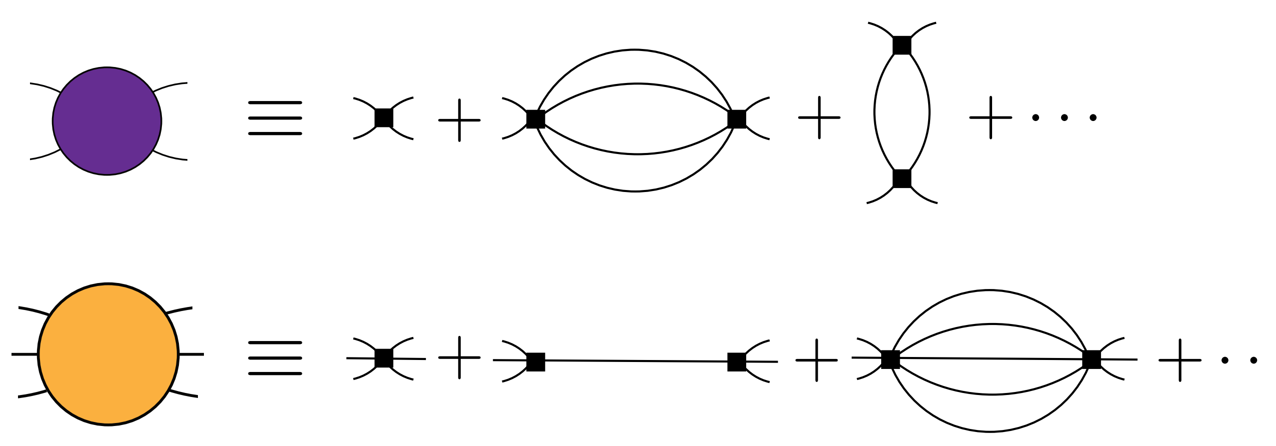
Disconnected diagrams in  **lead to singularities that invalidate the derivation**

Back to three: new skeleton expansion

$C_L(E,\vec{P}) =$



Kernel definitions:



New skeleton expansion

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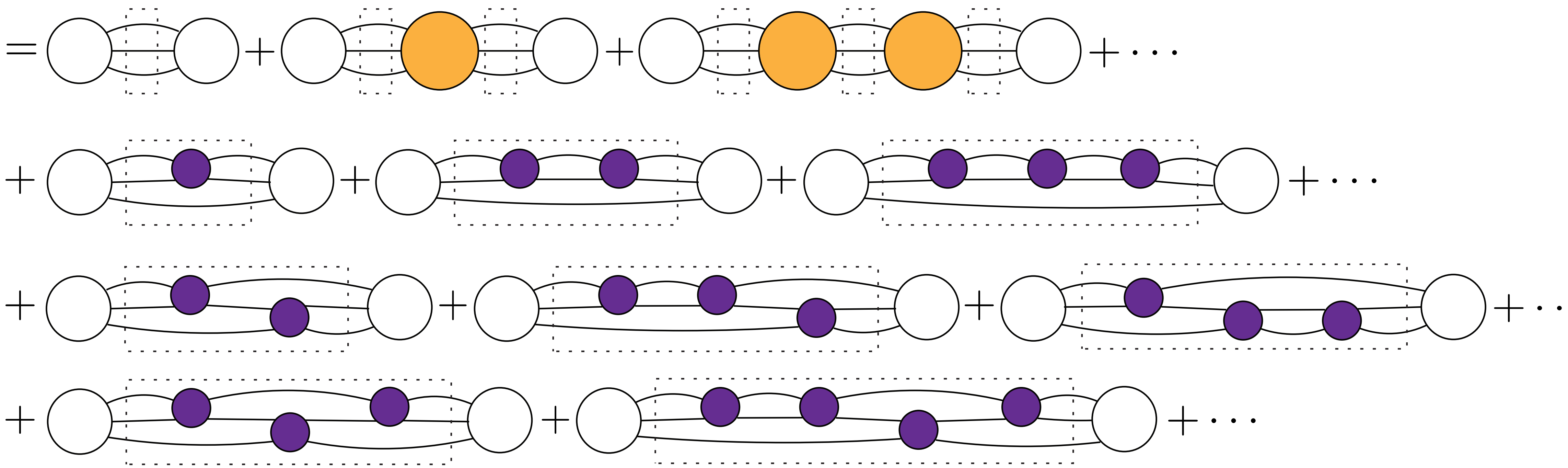

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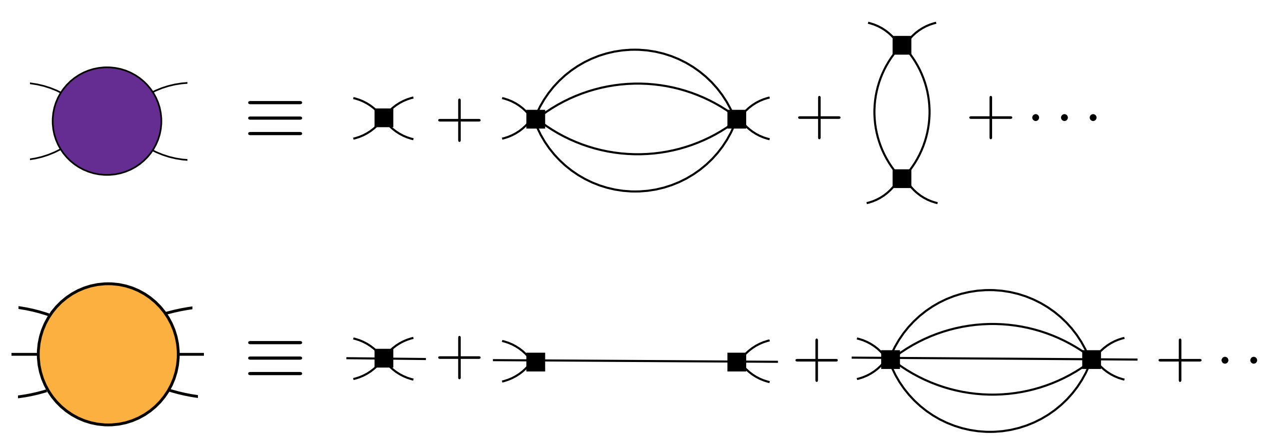

New skeleton expansion

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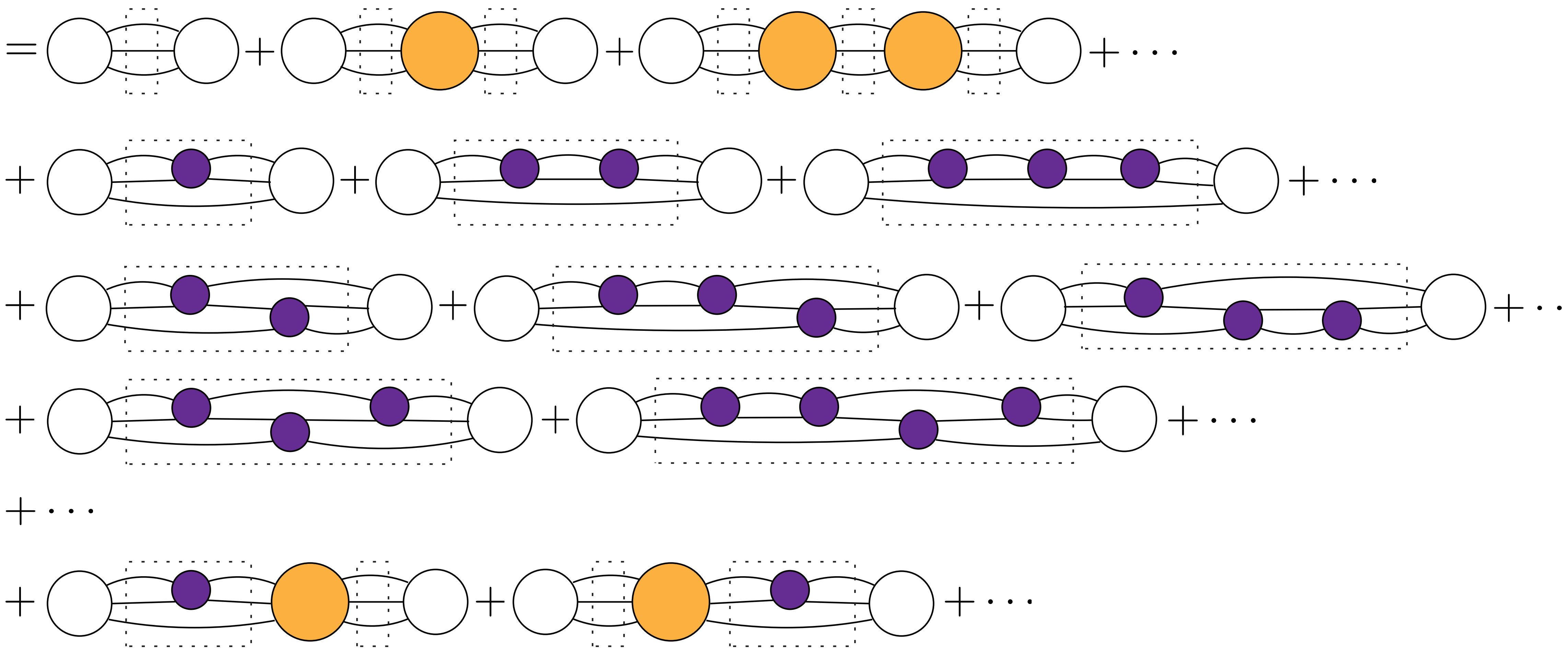

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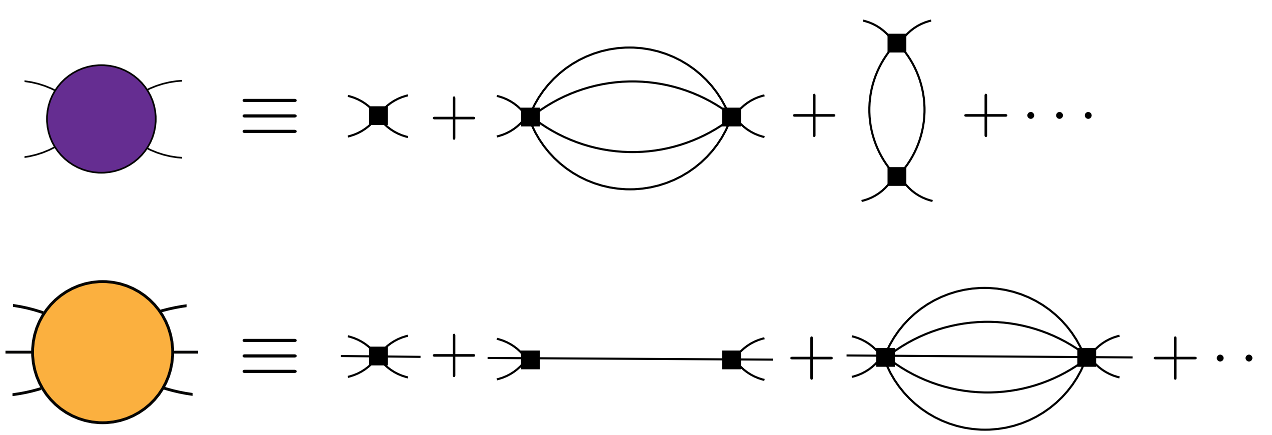
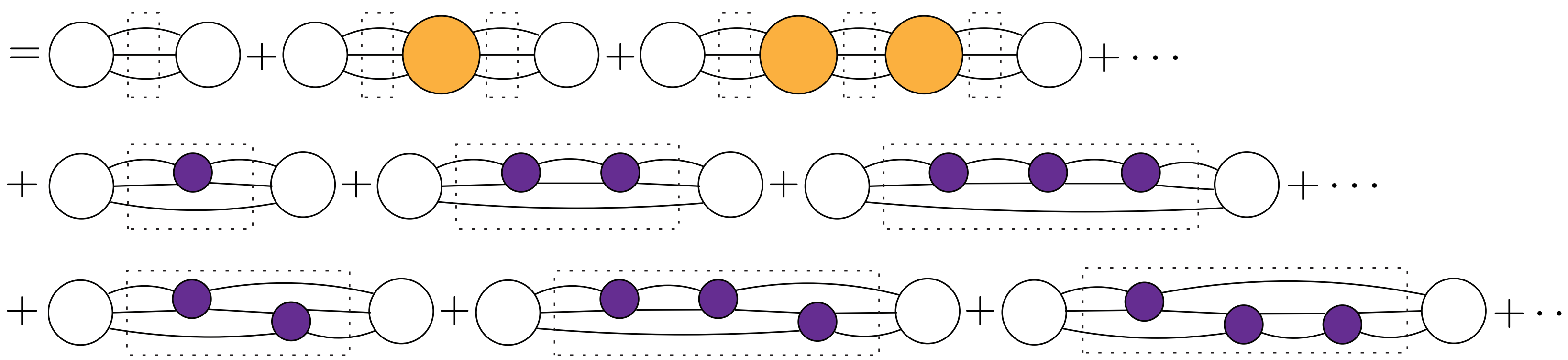


- All lines are fully dressed propagators
- Boxes represent sums over finite-volume momenta
- Kernels may contain fixed poles

Basic approach

1. Work out the three particle skeleton expansion

$C_L(E,\vec{P}) =$



2. Break diagrams into finite- and infinite-volume parts

3. Organize and sum terms to identify
infinite-volume observables

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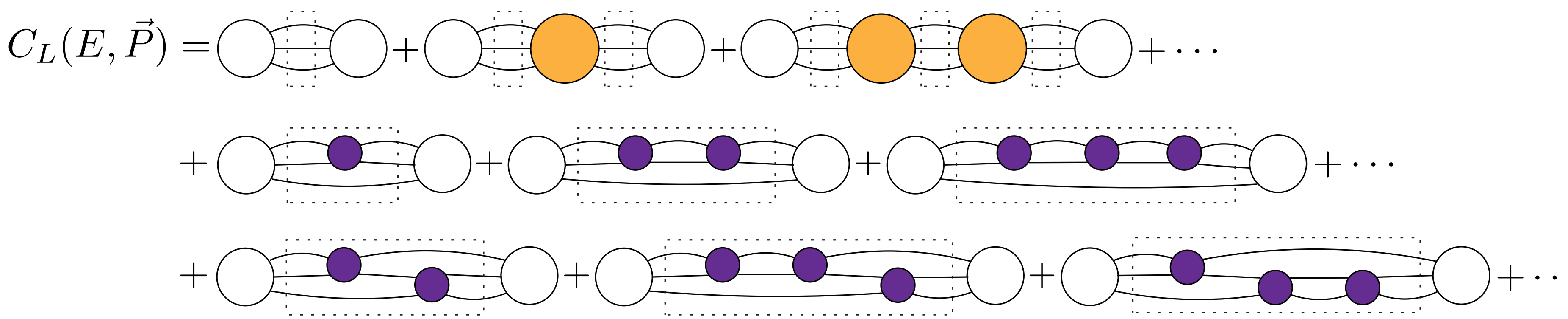

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Result

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) - A' F_3 \frac{1}{1+\mathcal{K}_{\mathrm{df},3} F_3} A$$

- Looks similar to the two-particle case
- All quantities defined with PV-pole prescription
- F_3 depends on finite-volume and two-to-two scattering

Quantization condition

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

$$\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$$

$F_3 \equiv$ **matrix that depends on geometric functions and** $\mathcal{M}_{2\to2}$.

MTH and Sharpe (2014)

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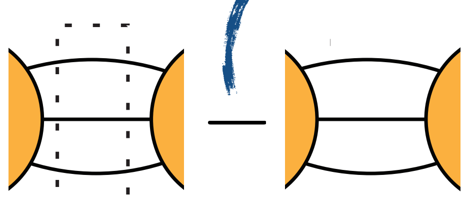
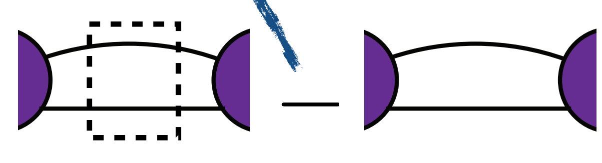
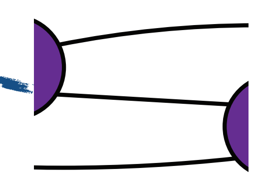
MTH and Sharpe (2014)

All of the complication is buried inside F_3

$$F_3 = \frac{F}{6\omega L^3} - \frac{F}{2\omega L^3}\frac{1}{1+\mathcal{M}_{2,L}G}\mathcal{M}_{2,L}F$$







These are all matrices with indices

momentum of one particle

$$\vec{k} = \frac{2\pi\vec{n}}{L}$$

$\otimes$

angular momentum of the other two

$$\ell, m$$

F and G are geometric functions

$\mathcal{M}_{2,L}$ **depends on F and** $\mathcal{M}_2$

Determining $\mathcal{K}_{\mathrm{df},3}$

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MTH and Sharpe (2014)

(1). Use two-particle q.c. to constrain $\mathcal{M}_2$ and determine $F_3(E, \vec{P}, L)$.

$$\det[\mathcal{M}_2^{-1} + F_2] = 0 \longrightarrow \mathcal{M}_2 \longrightarrow F_3(E, \vec{P}, L)$$

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$$\mathcal{K}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) \approx \mathcal{K}_{\mathrm{df},3}[\alpha_1, \cdots, \alpha_N] \longleftarrow$$ **Recall, this is a real, smooth function**

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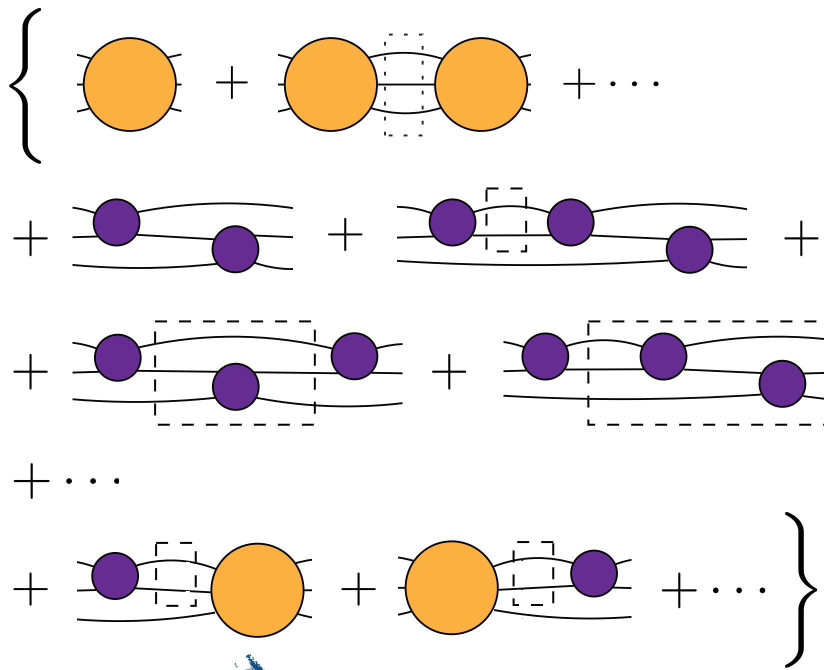
(3). Use three-particle q.c. with finite-volume energies to determine $\mathcal{K}_{\mathrm{df},3}(E^*)$.

$$\det[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3] = 0 \longrightarrow \boxed{\mathcal{K}_{\mathrm{df},3}(E^*) \checkmark}$$

Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we relate $\mathcal{K}_{\mathrm{df},3}$ to a special choice of finite-volume correlator

$$i\mathcal{M}_{L,3\to 3} \equiv \mathcal{S}$$



$$\mathcal{M}_{L,3} = \mathcal{S}\left[\mathcal{D}_L + \mathcal{L}_L \frac{1}{\mathcal{K}_{\mathrm{df},3}^{-1} + F_3}\mathcal{R}_L\right]$$

MTH and Sharpe (2015)

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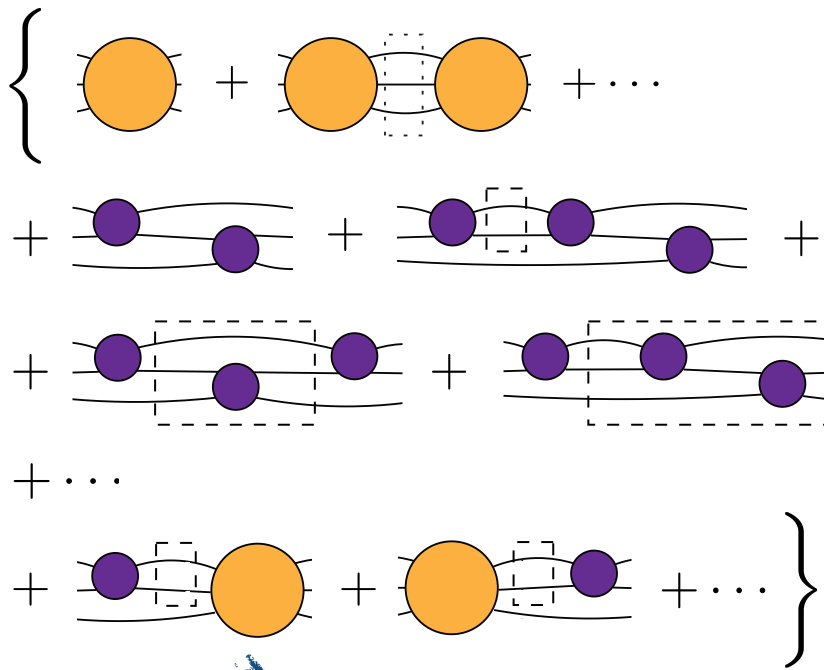

$$\mathcal{M}_{L,3} = \mathcal{S}\left[\mathcal{D}_L + \mathcal{L}_L \frac{1}{\mathcal{K}_{\mathrm{df},3}^{-1} + F_3} \mathcal{R}_L\right]$$

With this analytic relation in hand we can…

(a) Set $E \to E + i\epsilon$, **(b) Send** $L \to \infty$, **(c) Send** $\epsilon \to 0^+$.

Leads to an integral equation for the scattering amplitude

$$\mathcal{M}_3(E^*) = \mathcal{I}\left[\mathcal{K}_{\mathrm{df},3}(E^*), \mathcal{M}_2, \mathcal{M}_3(E^*)\right]$$

MTH and Sharpe (2015)

Current status

Model- & EFT-independent relation between
finite-volume energies and relativistic two-and-three particle scattering

- ☑ **Requires energy is below four- or five-particle production threshold**
- ☑ **Derivation ignores (drops) suppressed volume effects ($e^{-M_\pi L}$)**
- ☑ **Only useful if one truncates angular momentum space**

Briceño, MTH, Sharpe (2017) see also ***Pang, Hammer, Rusetsky*** and ***Döring, Mai***

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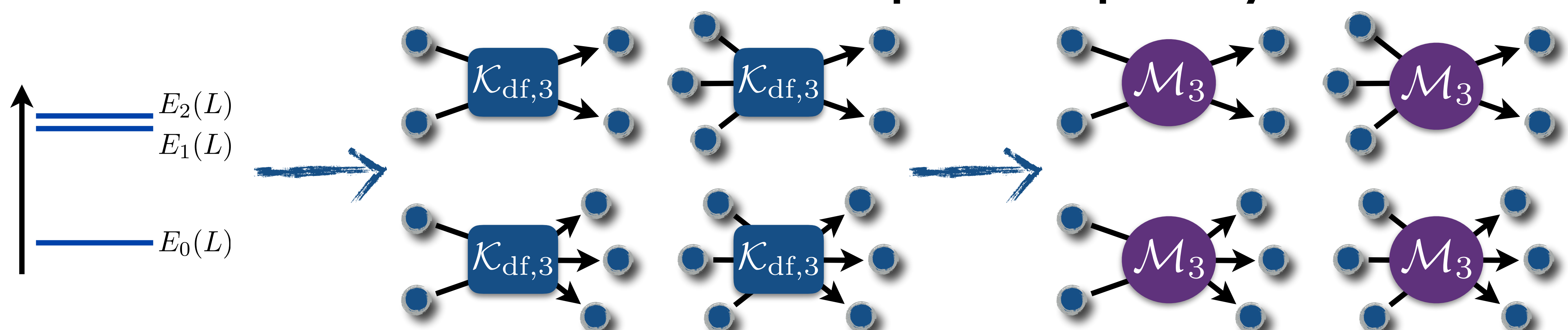
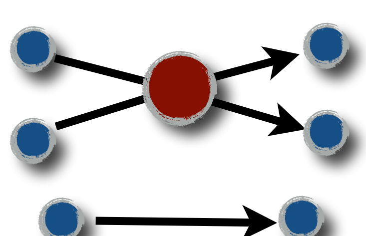
- [x] Result relies on an intermediate three-particle quantity



- [ ] Assumes no sub-channel two-particle resonances



Extension to remove this restriction is underway

- [ ] Derivation assumes identical scalar particles

Briceño, MTH, Sharpe (2017) ○ see also ***Pang, Hammer, Rusetsky*** and ***Döring, Mai***

Usability?

☑ **How do we make the two-particle formalism usable?**

Truncate partial waves

$$\mathcal{M}_2(E_2^*,\theta^*) \approx \sum_{\ell=0}^{N} P_\ell(\cos\theta^*)\mathcal{M}_{2,\ell}(E_2^*) \longrightarrow$$

Single partial wave

$$\mathcal{M}_2(E_2^*,\theta^*) \approx \mathcal{M}_{2,s}(E_2^*) \propto \frac{1}{p^* \cot\delta_0(p^*) - ip^*}$$

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☑ **Is there a three-particle analog?**

$$\mathcal{K}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) \approx \sum_{n=0}^{N} \mathcal{P}_n(\Omega_3', \Omega_3)\mathcal{K}_{\mathrm{df},3,n}(E^*) \longrightarrow \boxed{\mathcal{K}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) \approx \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E^*) \in \mathbb{R}}$$

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$$\mathcal{K}_{\mathrm{df},3}(E^*,\Omega_3',\Omega_3) \approx \mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E^*) \in \mathbb{R}$$

At fixed energy $\mathcal{M}_2(E_2^*,\theta^*)$ / $\mathcal{K}_{\mathrm{df},3}(E^*,\Omega_3',\Omega_3)$ is a smooth function on a compact space.

Further investigation is needed to understand suppression of higher $\mathcal{K}_{\mathrm{df},3,n}(E^*)$

Numerics (keeping only s-wave and $\mathcal{K}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) \approx \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E^*)$)

$$1/\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E^*) = -F_3^{\mathrm{iso}}[E, \vec{P}, L, \mathcal{M}_2^s] \qquad \mathcal{M}_3(E^*, \Omega_3', \Omega_3) = \mathcal{S}\left[\mathcal{D} + \mathcal{L}\frac{1}{1/\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}} + F_{3,\infty}^{\mathrm{iso}}}\mathcal{R}\right]$$

For the numerical approach we restrict attention to… $p^* \cot\delta_0(p^*) = -\frac{1}{a}$, $\vec{P} = 0$

Then the quantization condition is based on $F_3^{\mathrm{iso}}(E, L, a)$

Briceño, Hansen and Sharpe (2018)

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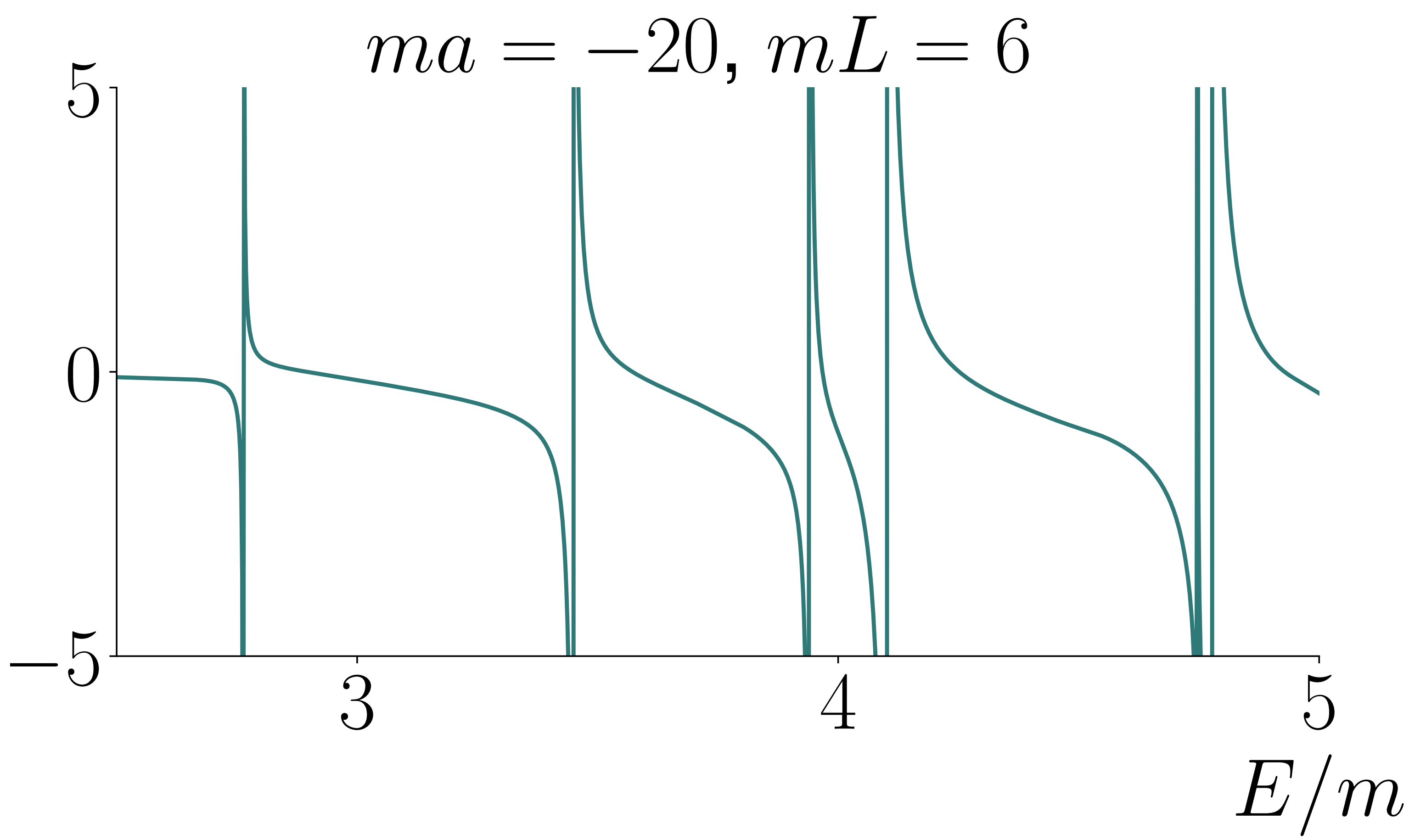


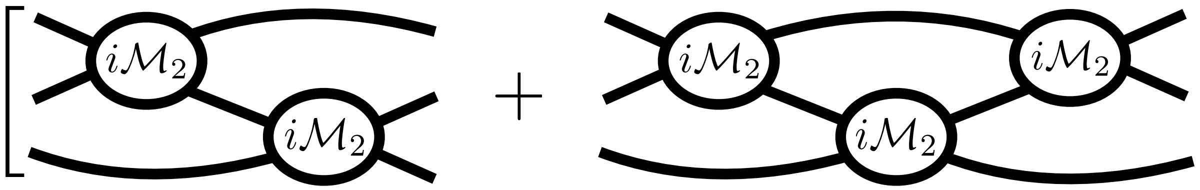
Finite-volume energies wherever these curves intersect $-1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E)$

Briceño, Hansen and Sharpe (2018)

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

- **Provides a useful benchmark: Deviations measure three-particle physics**
- **Meaning for three-to-three scattering is clear**

$$i\mathcal{M}_3 = \mathcal{S}\Big[\quad + \quad + \cdots\Big]$$



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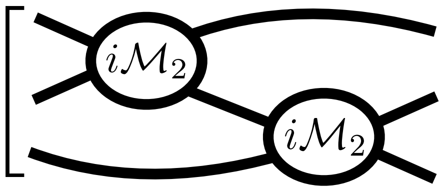
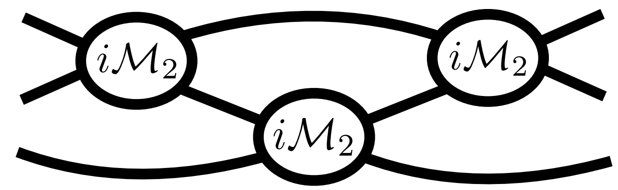
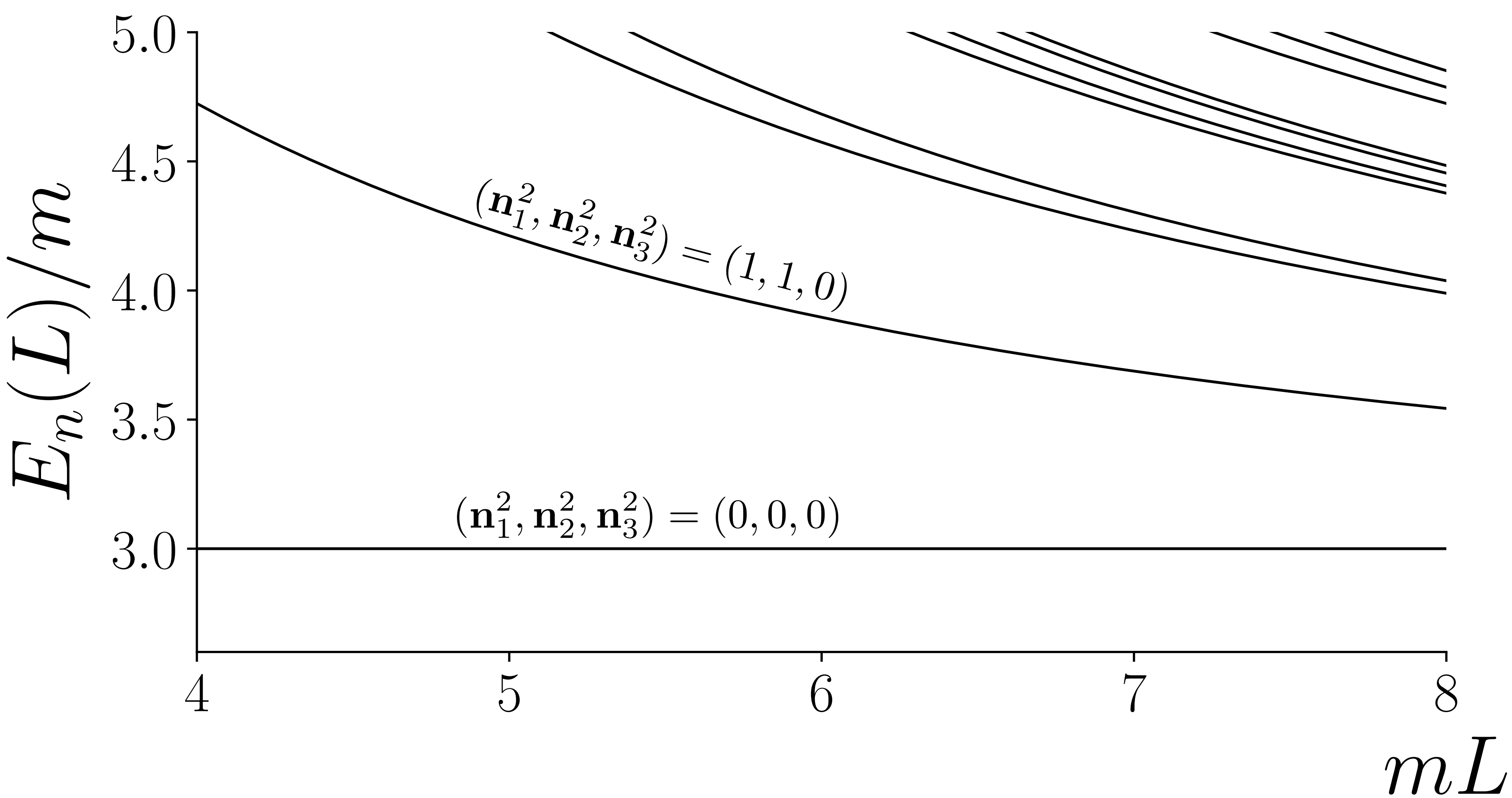
$i\mathcal{M}_3 = \mathcal{S}\Big[$  $+$  $+ \cdots \Big]$

First: **Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)**



Three non-interacting particles

$$E_n(L) = \omega_1 + \omega_2 + \omega_3$$

$$\omega_i = \sqrt{m^2 + (2\pi/L)^2 \mathbf{n}_i^2} \qquad \sum_i \mathbf{n}_i = \mathbf{0}$$

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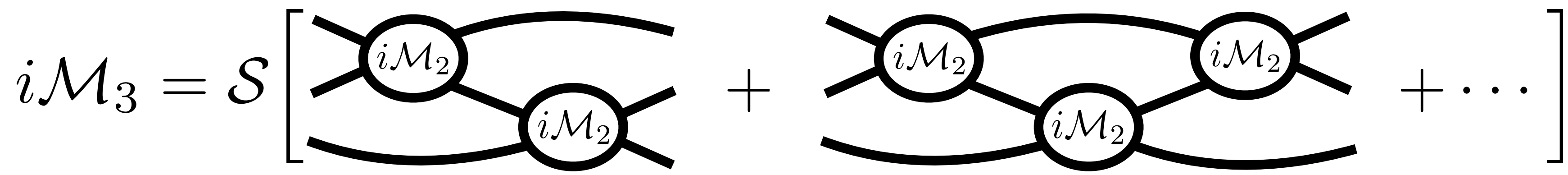

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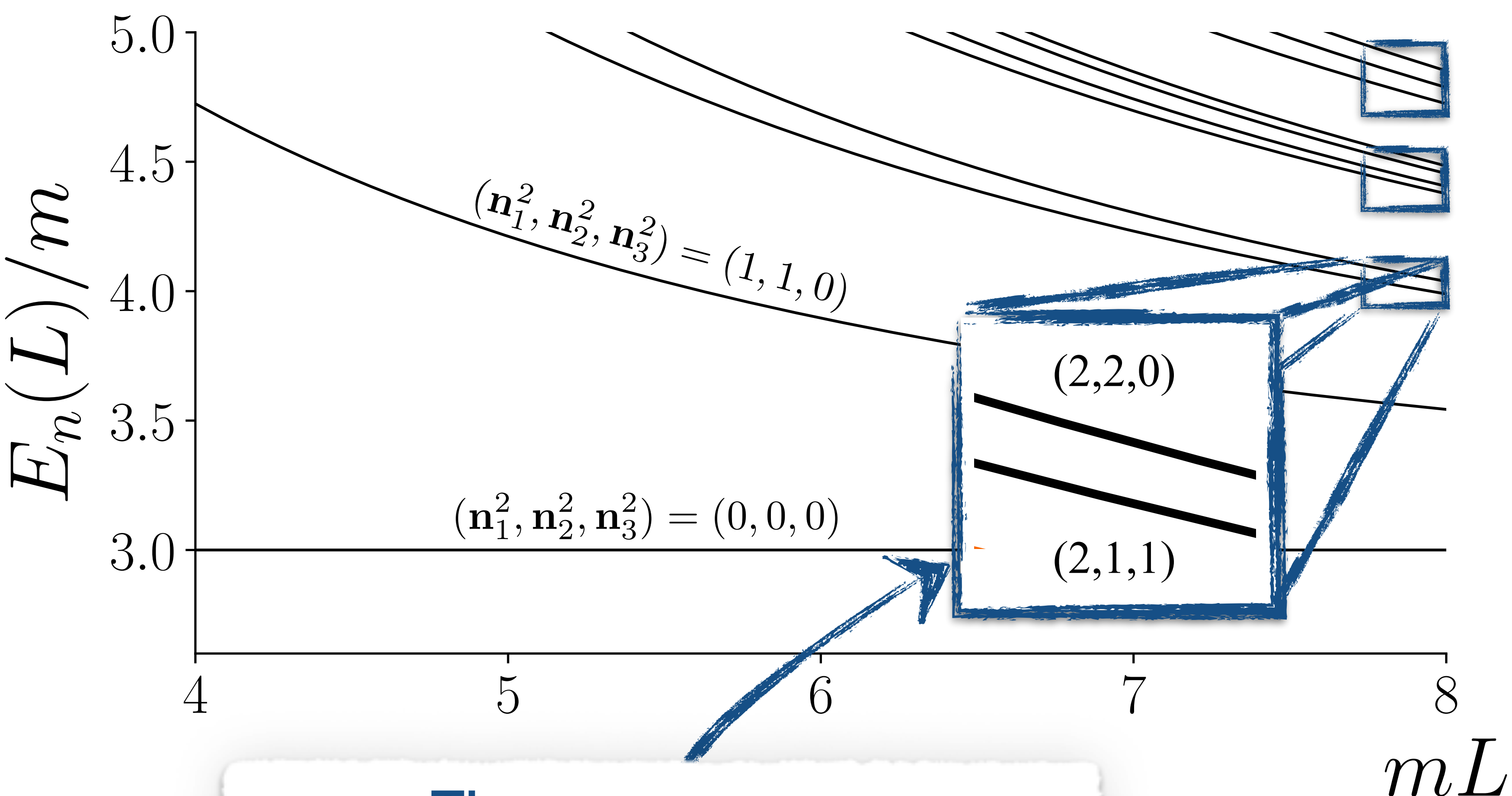

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These two states are degenerate in the NR theory

$$E_n^{\rm NR}(L) = 3m + \frac{2\pi^2}{L^2}(\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2)$$

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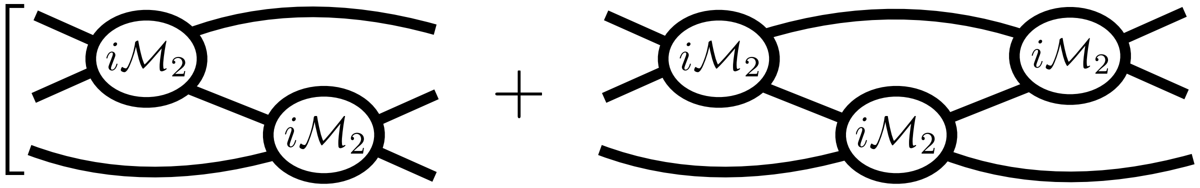
$i\mathcal{M}_3 = \mathcal{S}\Big[$  $+ \cdots \Big]$

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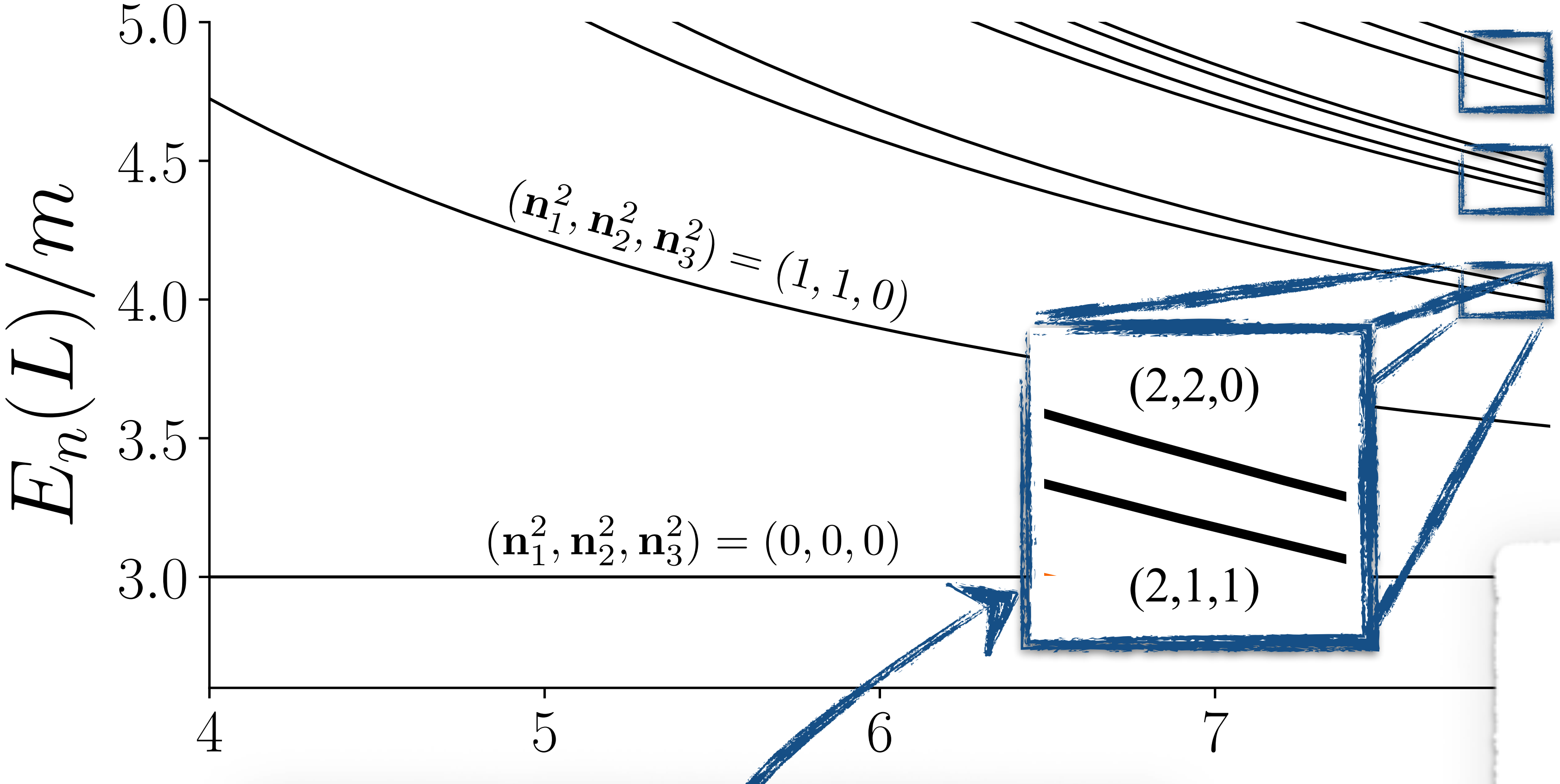

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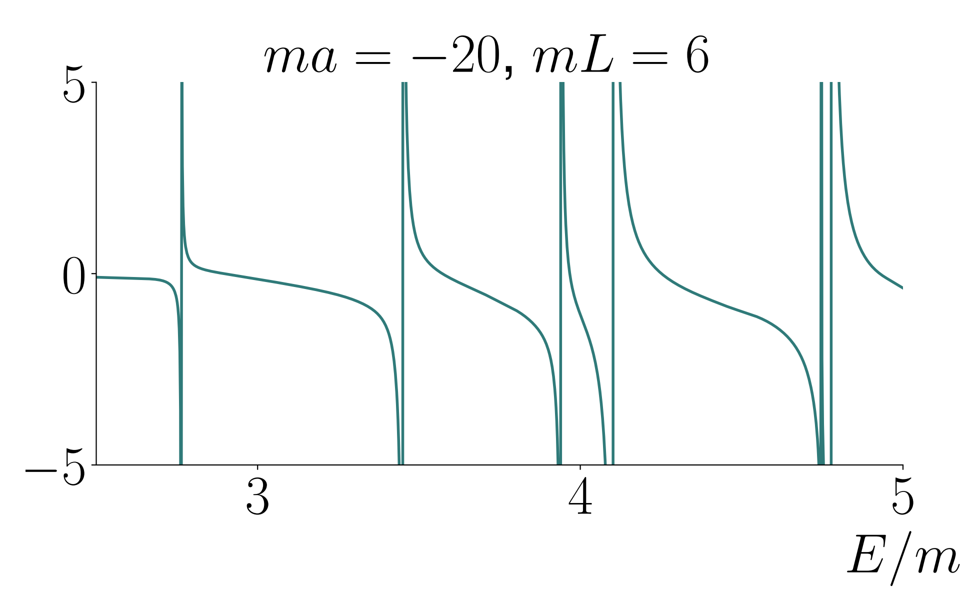
We have already seen these clusters



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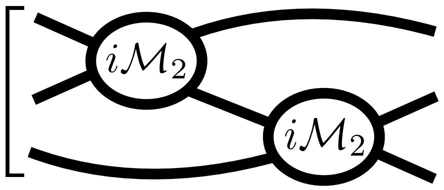
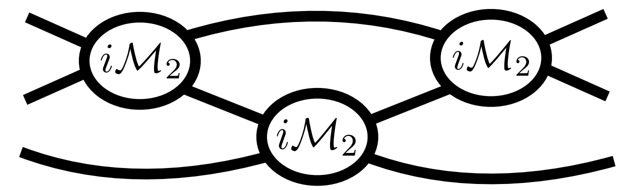
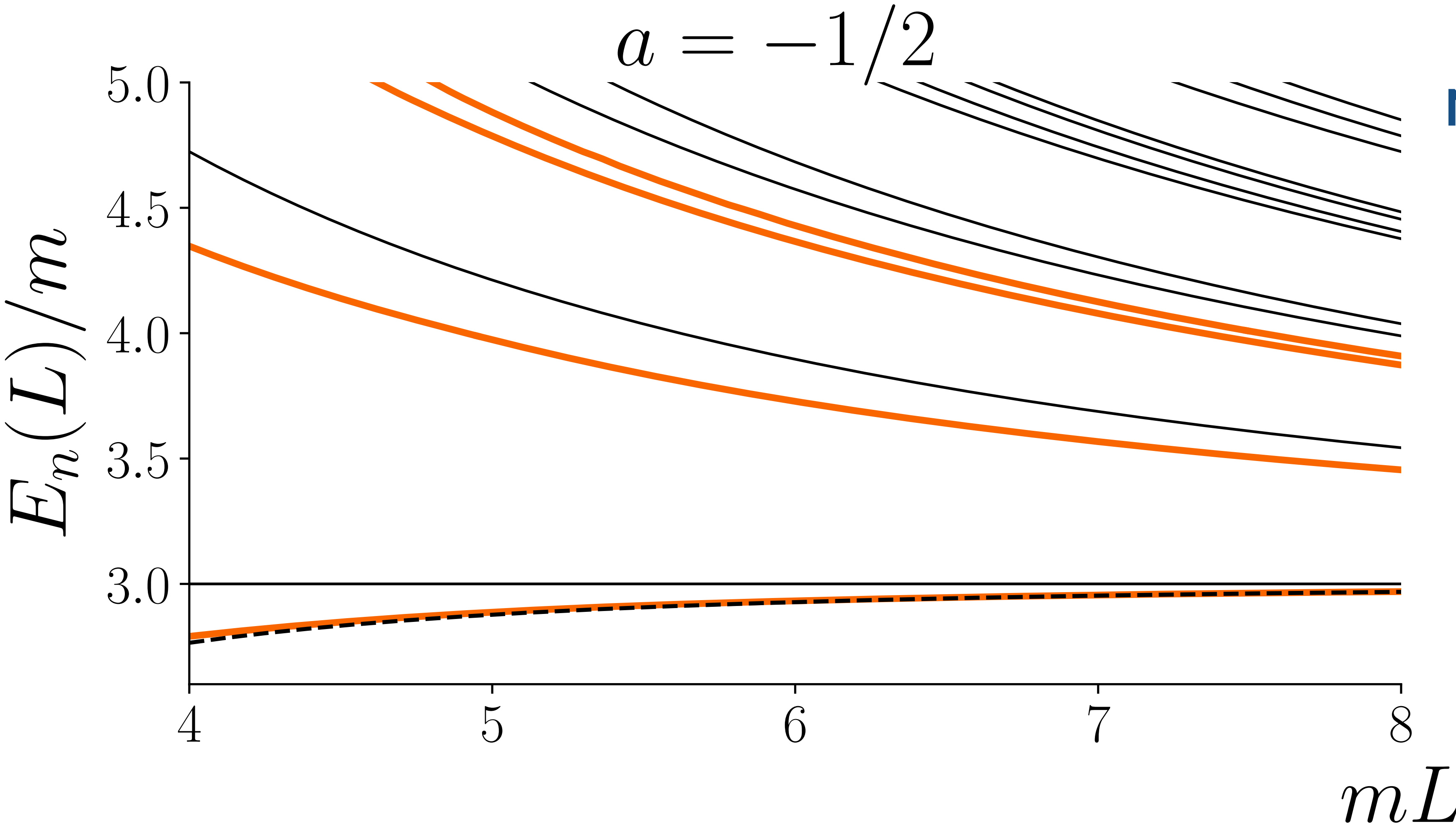
$i\mathcal{M}_3 = \mathcal{S}\Big[$ 

$+$ 

$+\cdots\Big]$



Now we turn on the interactions

$$1/F_3^{\mathrm{iso}}(E, L, a) = 0$$

$$F_3^{\mathrm{iso}} = \frac{1}{L^3}\sum_{\vec{k},\vec{p}}\left[\frac{\widetilde{F}_s}{3} - \widetilde{F}_s \frac{1}{\mathcal{H}}\widetilde{F}_s\right]_{kp}$$

$$\mathcal{H} = 1/(2\omega\mathcal{K}_2) + \widetilde{F}_s + \widetilde{G}_s$$

known functions

- - - - - - $1/L$ **expansion**

Briceño, Hansen and Sharpe (2018)

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

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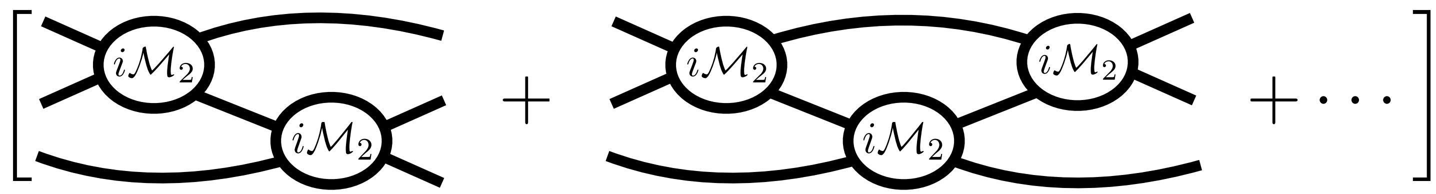
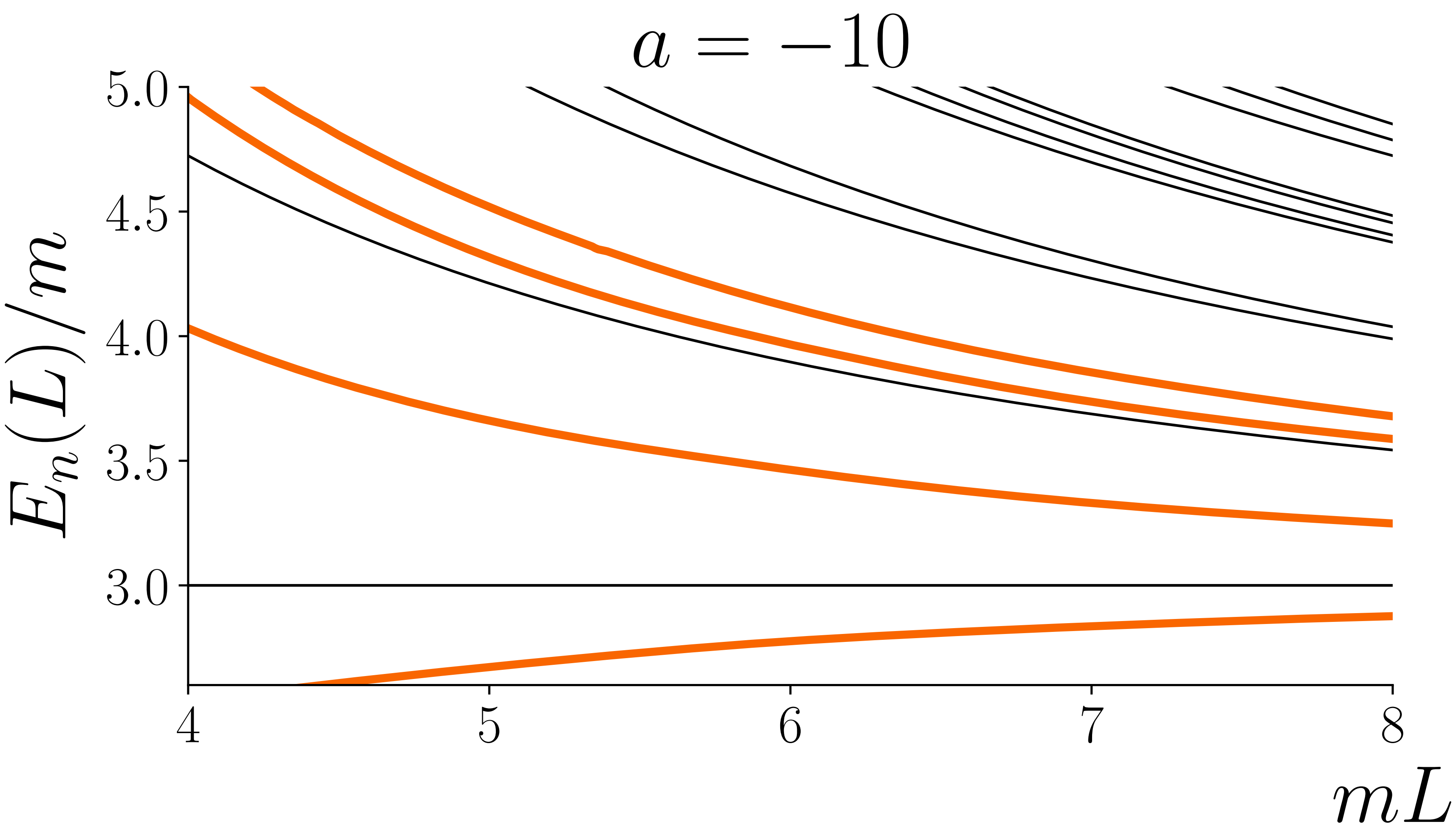
$i\mathcal{M}_3 = \mathcal{S}$





Can also accommodate large *a*

$$1/F_3^{\mathrm{iso}}(E, L, a) = 0$$

$$F_3^{\mathrm{iso}} = \frac{1}{L^3} \sum_{\vec{k},\vec{p}} \left[\frac{\widetilde{F}_s}{3} - \widetilde{F}_s \frac{1}{\mathcal{H}} \widetilde{F}_s \right]_{kp}$$

$$\mathcal{H} = 1/(2\omega \mathcal{K}_2) + \widetilde{F}_s + \widetilde{G}_s$$

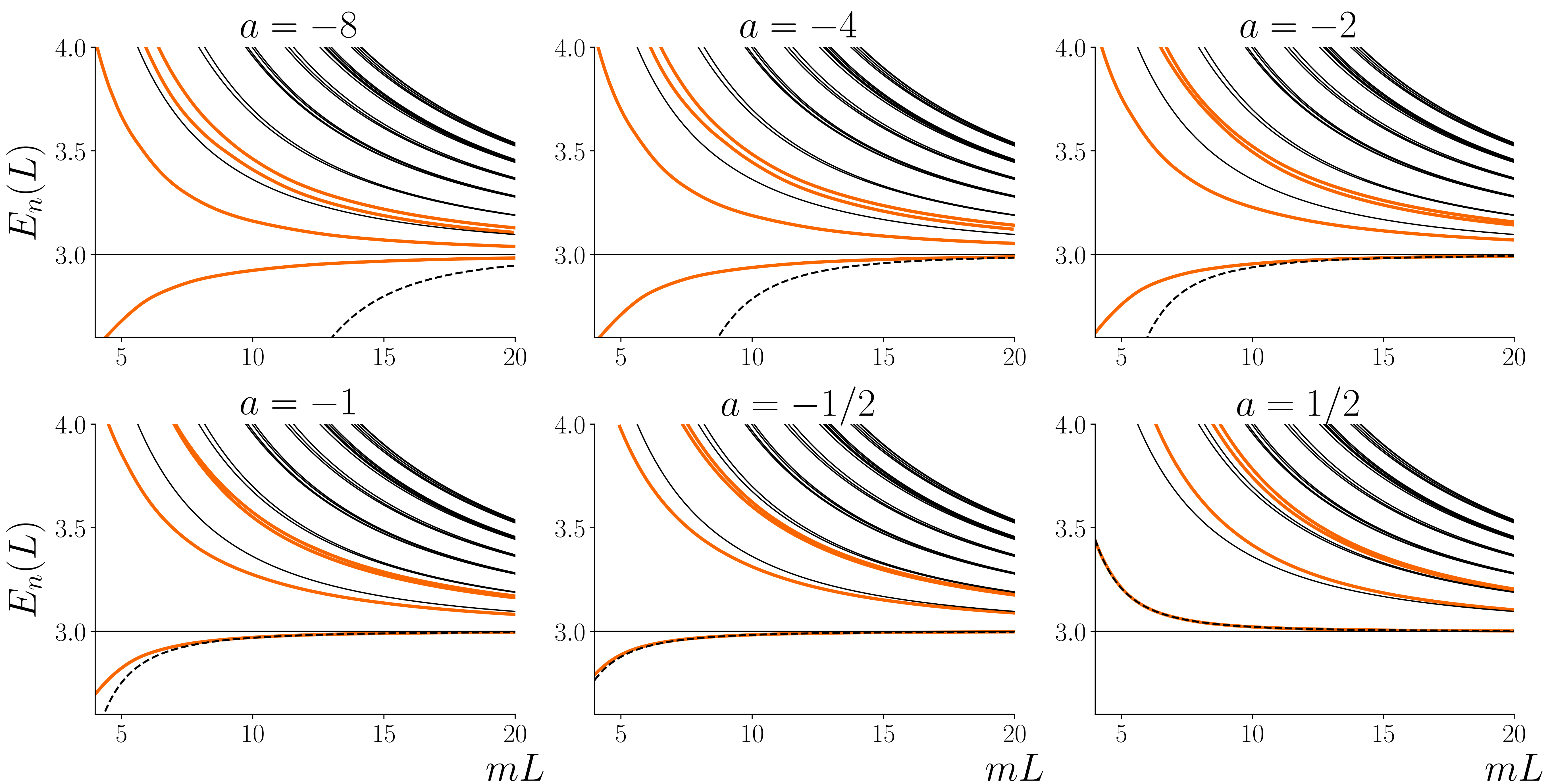
known functions

------- $1/L$ **expansion**

Briceño, Hansen and Sharpe (2018)

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ **solutions**

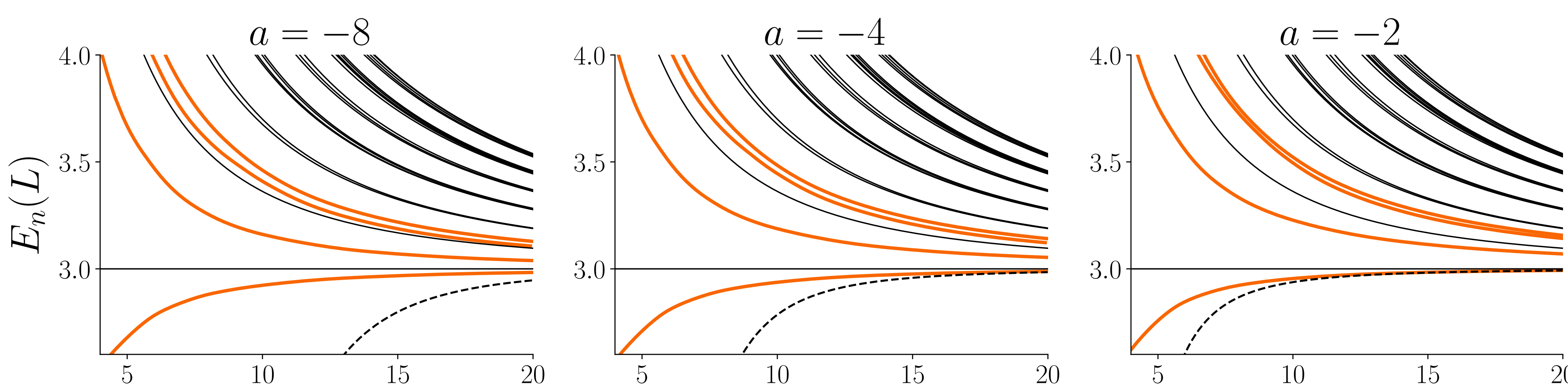
Straightforward to vary *a* and to study large volumes

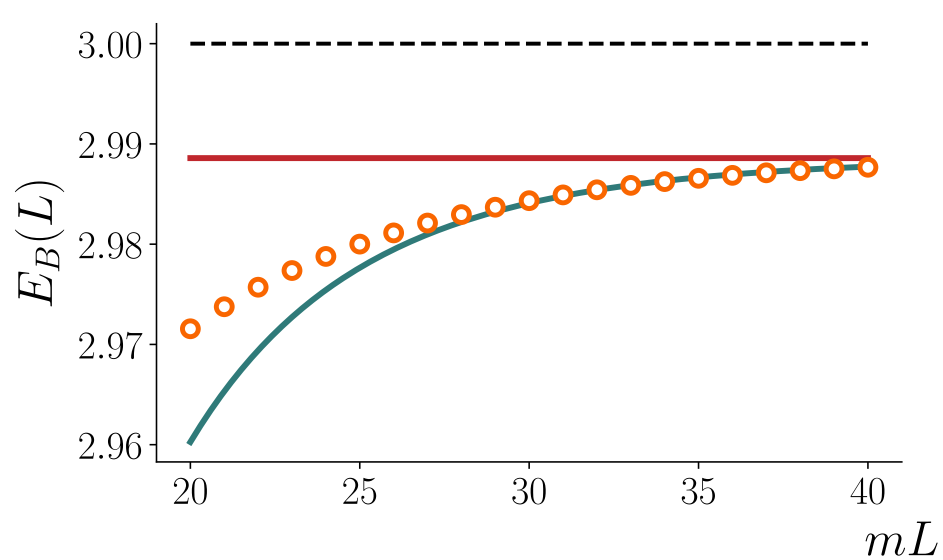


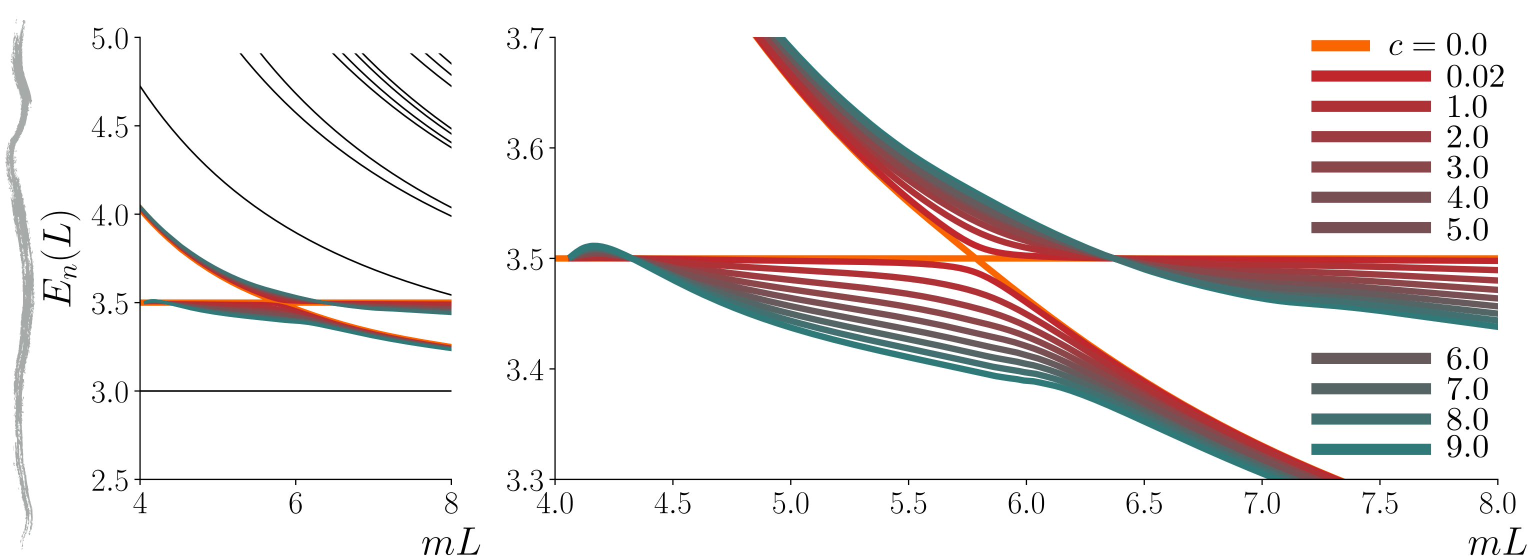
Still lots to do

- ☐ Finish result with intermediate two-particle resonances
- ☐ Extend to non-identical, non-degenerate, multiple channels, spin
- ☐ Study subduction to finite-volume irreps
- ☐ Understand rigorous parametrizations for the infinite-volume observables
- ☐ Convince practitioners that the formalism is mature
- ☐ Reliably measure finite-volume spectra
- ☐ Extract three-particle scattering from LQCD

Big picture: making progress, but not quite there yet

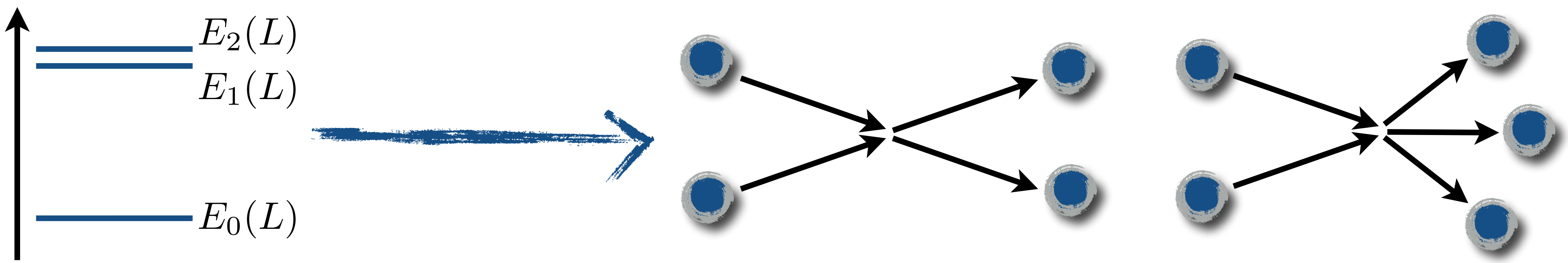




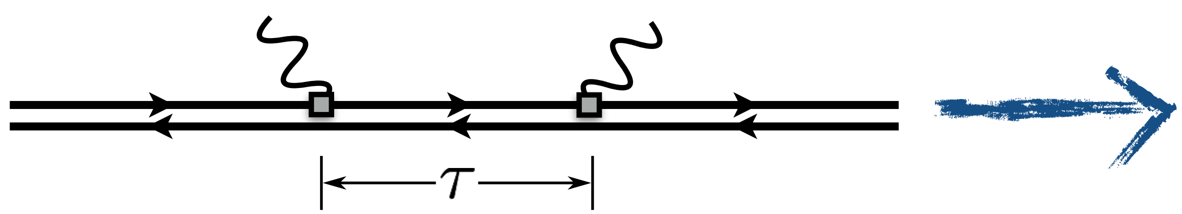


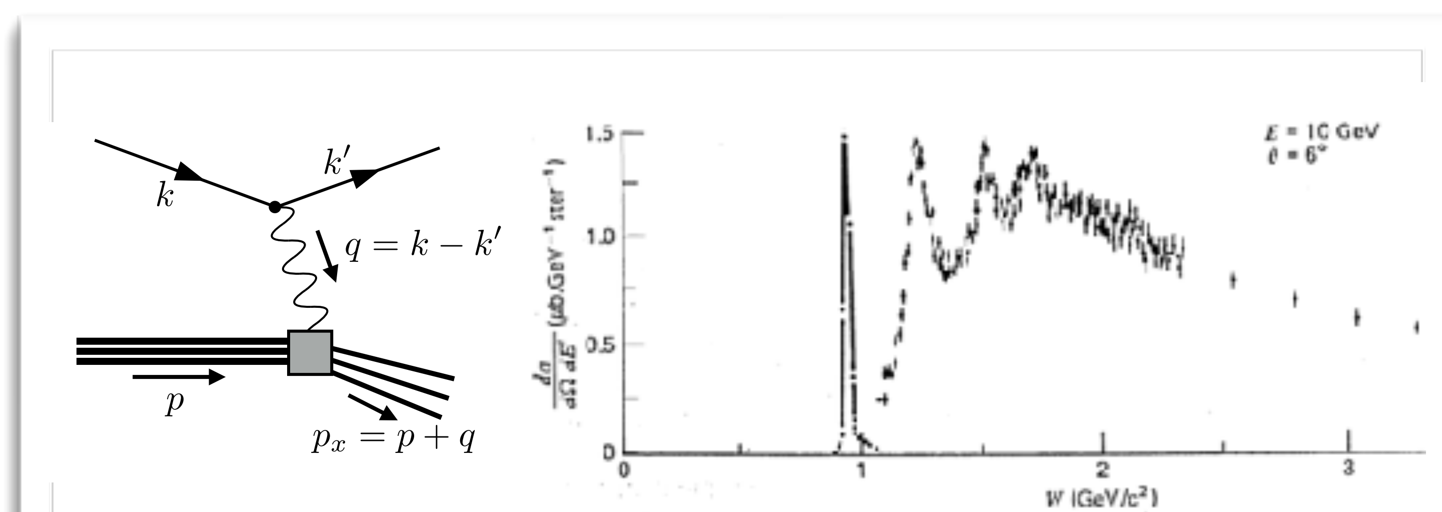
Outline

☑ **Three-particle scattering**



☐ **n-particle inclusive rates**

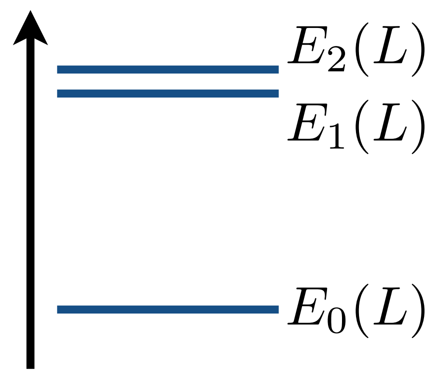


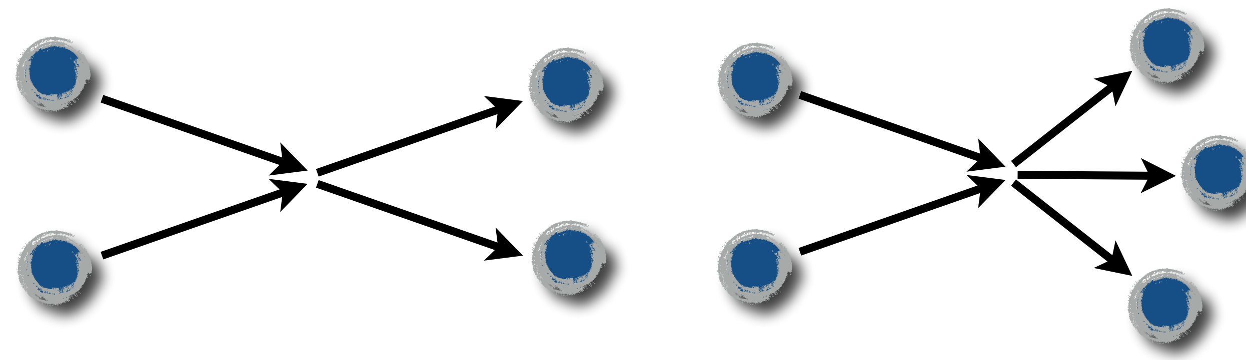


taken from Halzen and Martin

Outline

☑ **Three-particle scattering**





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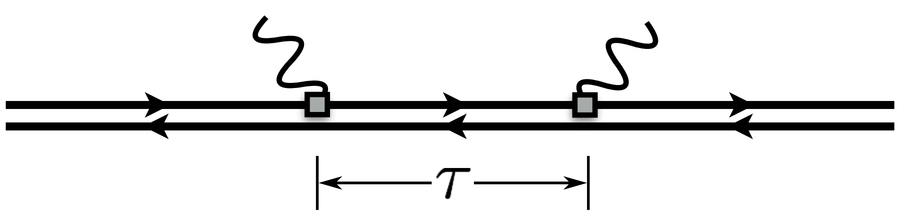

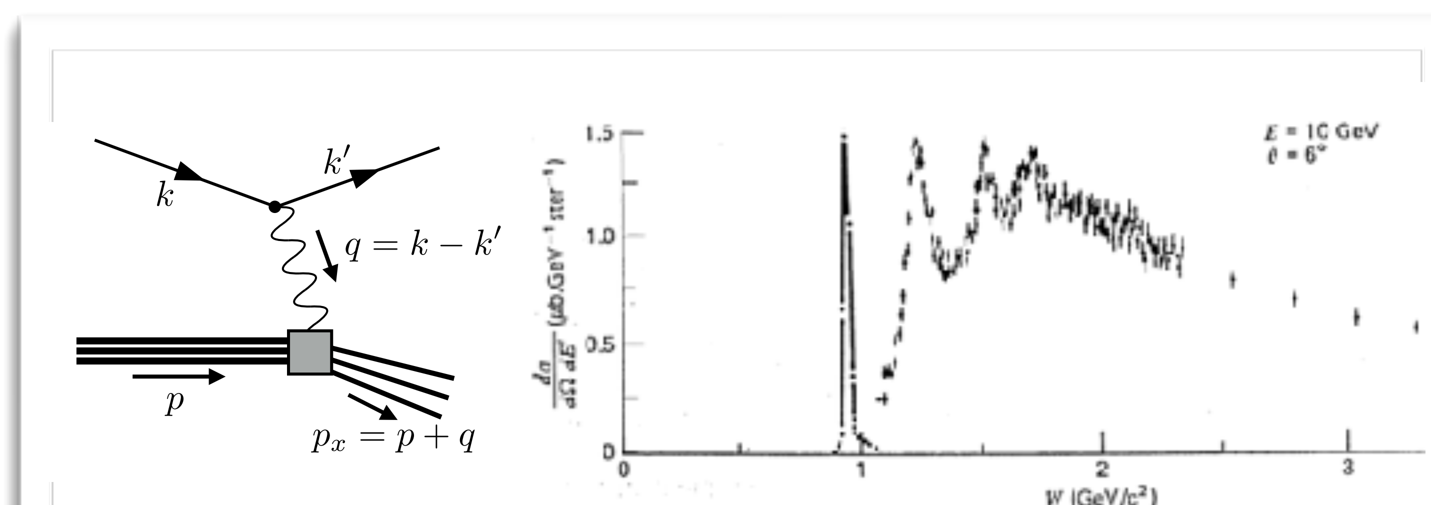

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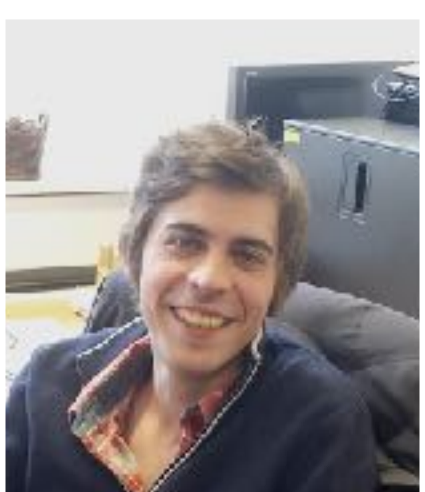
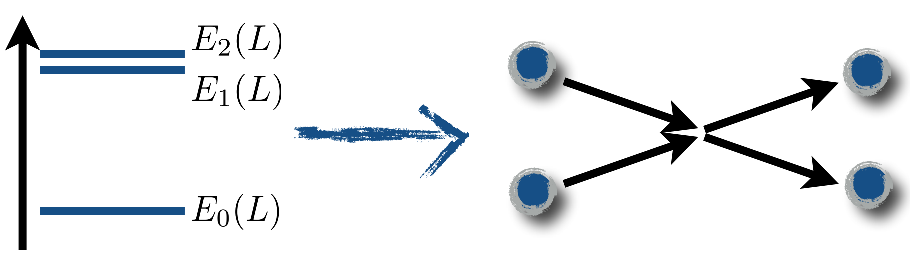
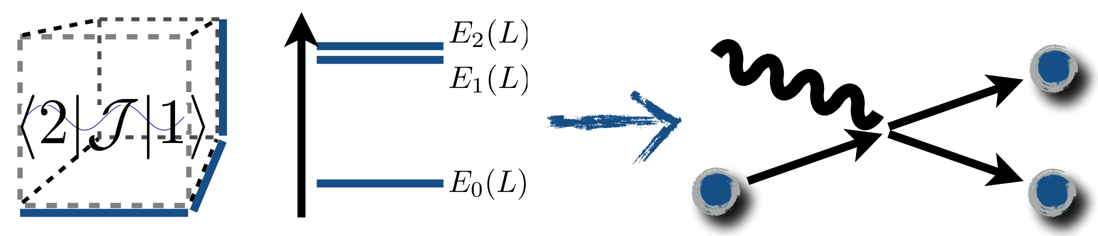


Based on MTH, H. Meyer & D. Robaina, (1704.08993)

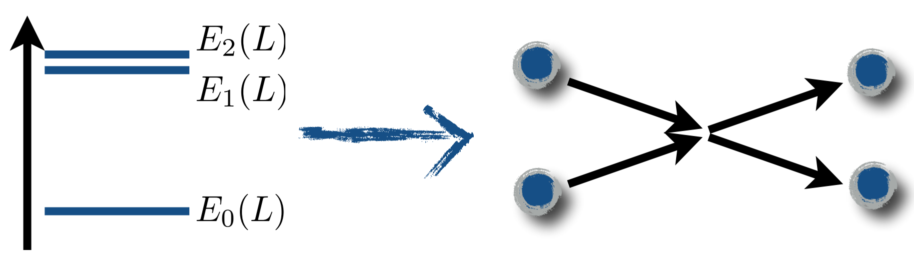
Rates instead of amplitudes…

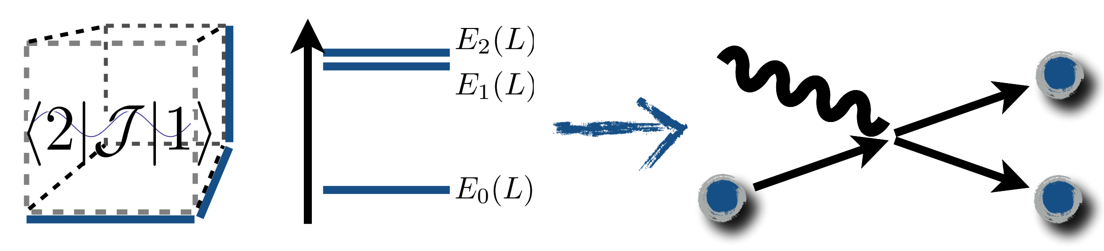




Finite-volume quantities are used to determine scattering and transition **amplitudes**

Rates instead of amplitudes…



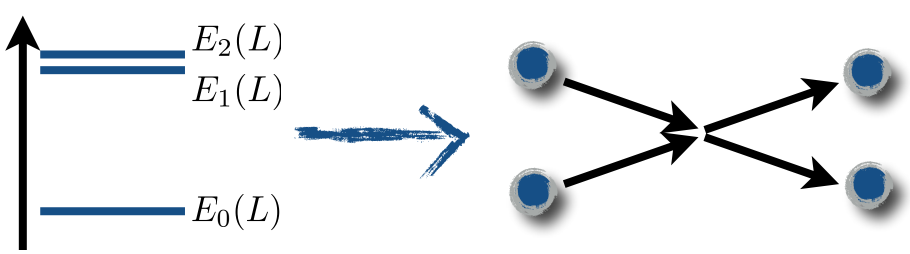


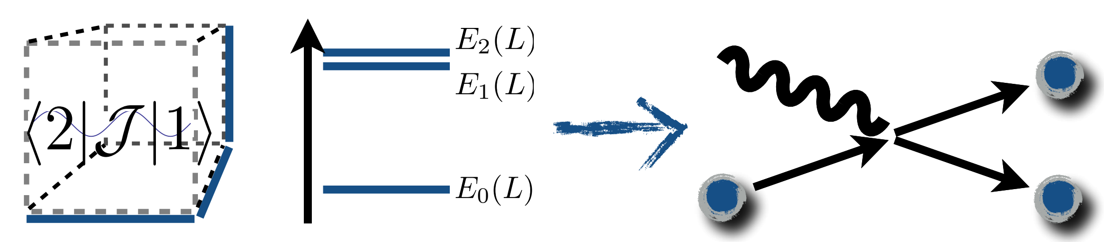
Finite-volume quantities are used to determine scattering and transition **amplitudes**

As the energy increases and more channels open,
this becomes increasingly challenging

$E_n(L) \longleftarrow$ **Contains information about all open channels with given QCD quantum numbers** $\pi\pi, \pi\pi\pi\pi, K\overline{K}, \cdots$

Rates instead of amplitudes...





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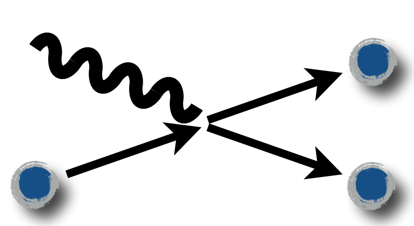
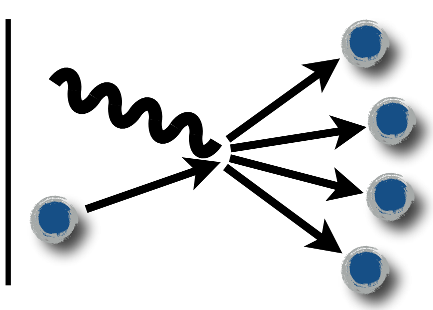
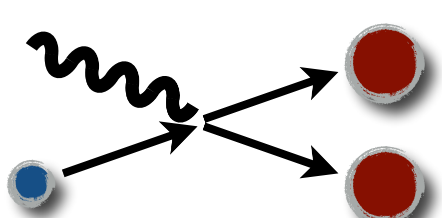
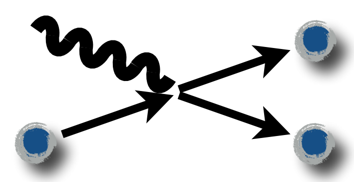
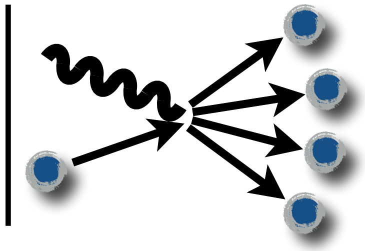
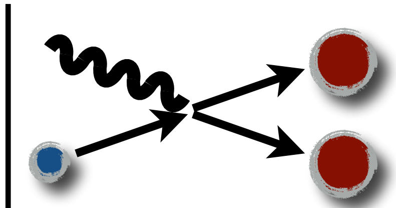
$E_n(L) \longleftarrow$ **Contains information about all open channels with given QCD quantum numbers** $\pi\pi, \pi\pi\pi\pi, K\overline{K}, \cdots$

Suppose we attempt an "easier" task: Total production rates

$\sigma_{\pi\gamma^* \to X} \equiv \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2$

$+ \int d\Phi \Big|$  $\Big|^2 + \cdots \longleftarrow$ **Sum over all open channels with given QCD quantum numbers**

The hadronic tensor

$\sigma_{\pi\gamma^*\to X} \equiv \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \cdots$

This rate is related to the matrix element of the current…

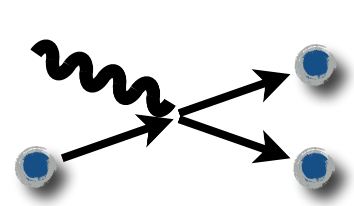
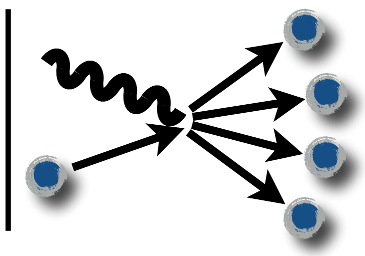
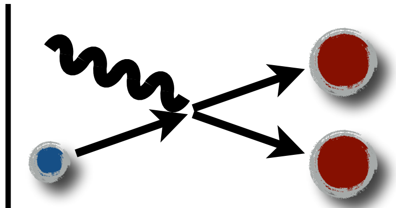
$$\sigma_{\pi\gamma^*\to X} \propto \sum_\alpha \int d\Phi_\alpha |\langle E, \mathbf{p}, \alpha, \mathrm{out}|\mathcal{J}(0)|\pi\rangle|^2$$

$\pi\pi,\ \pi\pi\pi\pi,\ K\overline{K}, \cdots$

proportional because of missing kinematic factors

n-particle phase space integral

The hadronic tensor

$\sigma_{\pi\gamma^*\to X} \equiv \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \cdots$

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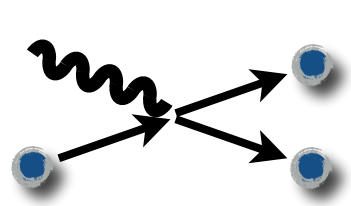
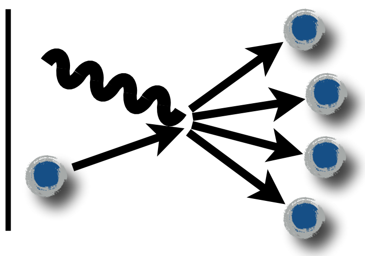
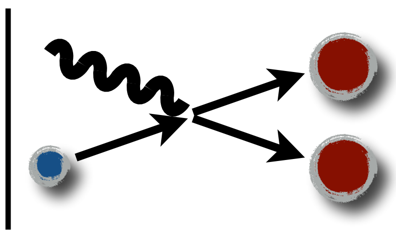
$$\propto \int_{\text{all states, }(P',\alpha')} (2\pi)^4 \delta^4(P'-P)\ \langle \pi|\mathcal{J}^\dagger(0)|E',\mathbf{p}',\alpha'\rangle\langle E',\mathbf{p}',\alpha'|\mathcal{J}(0)|\pi\rangle$$

integrate over all energies but include a delta

The hadronic tensor

$$\sigma_{\pi\gamma^*\to X} \equiv \int d\Phi \left| \right.$$  $$\left. \right|^2 + \int d\Phi \left| \right.$$  $$\left. \right|^2 + \int d\Phi \left| \right.$$  $$\left. \right|^2 + \cdots$$

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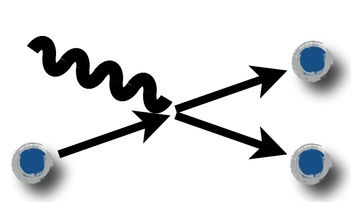
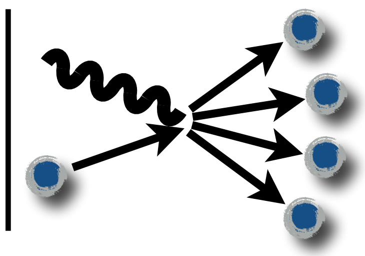
$$\propto \int d^4x\ e^{iqx}\langle\pi|e^{i\hat{P}\cdot x}\mathcal{J}^\dagger(0)e^{-i\hat{P}\cdot x}\ \int_{\text{all states, } (P',\alpha')} |E',\mathbf{p}',\alpha'\rangle\langle E',\mathbf{p}',\alpha'|\mathcal{J}(0)|\pi\rangle$$

the delta is rewritten using

$$\int dx e^{ipx} = 2\pi\delta(p)$$

The hadronic tensor

$$\sigma_{\pi\gamma^*\to X} \equiv \int d\Phi \left| \right.$$

$$\left. \right|^2 + \int d\Phi \left| \right.$$

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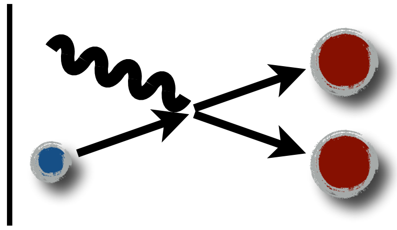
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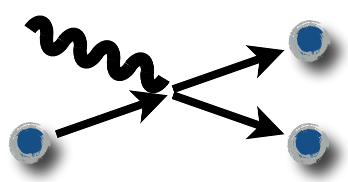
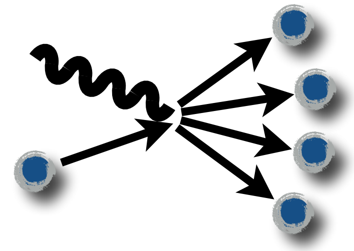
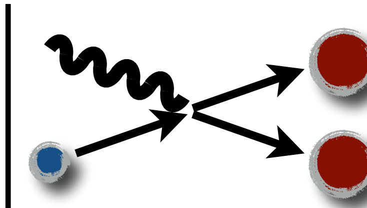
the sum over states is just the identity, I

$$\propto \int d^4x\ e^{iqx} \langle \pi|\mathcal{J}^\dagger(x)\mathcal{J}(0)|\pi\rangle$$

the current is translated

The hadronic tensor

$\sigma_{\pi\gamma^*\to X} \equiv \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \cdots$

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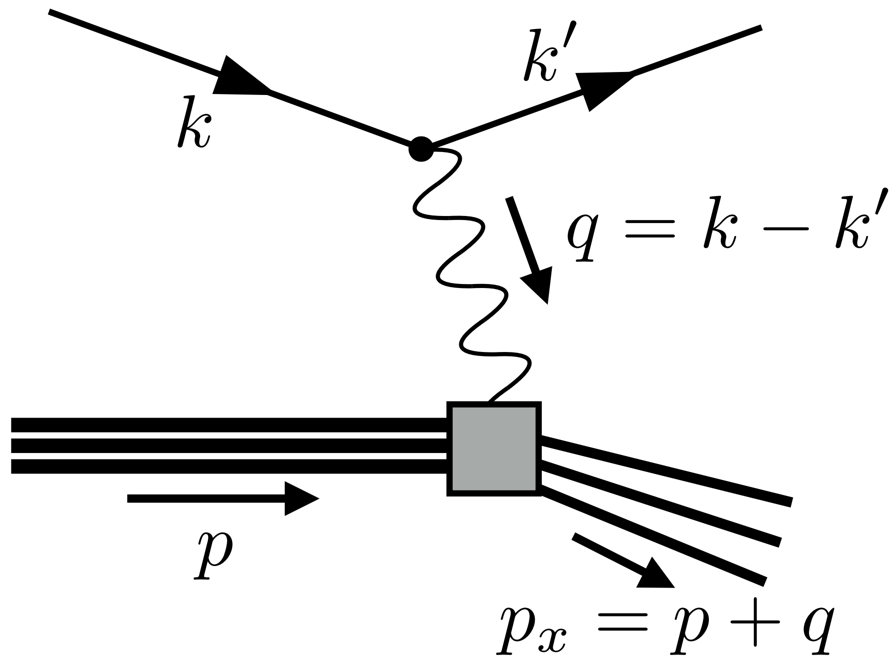
$$\propto \int d^4x\ e^{iqx}\langle\pi|\mathcal{J}^\dagger(x)\mathcal{J}(0)|\pi\rangle$$

This motivates the hadronic tensor...

$$W_{\mu\nu}(p,q) \equiv \int d^4x\ e^{iqx}\langle\pi,p|\mathcal{J}^\dagger_\mu(x)\mathcal{J}_\nu(0)|\pi,p\rangle$$

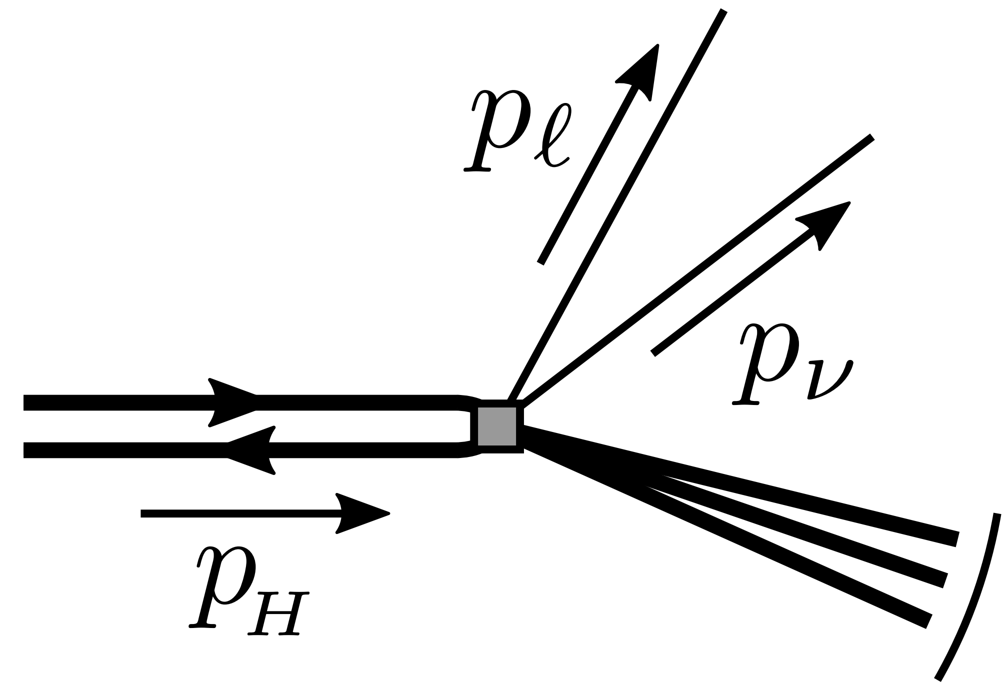
Potential applications

Onset of deep-inelastic scattering



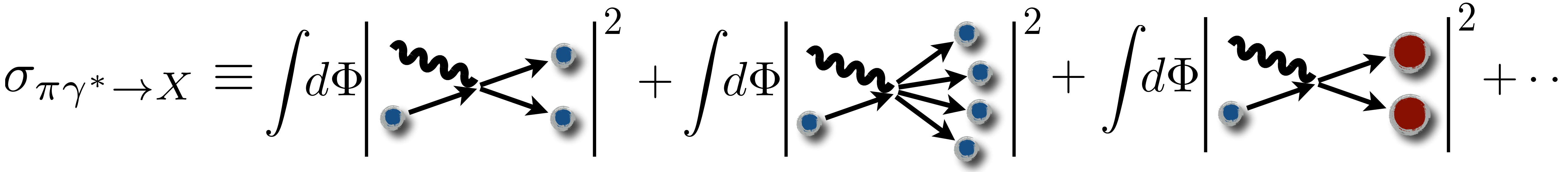
$$\langle N | J_\mu(q) J_\nu(0) | N \rangle$$

Heavy-flavor decays



$$\langle D | J_\mu(q) J_\nu(0) | D \rangle$$

The idea is to describe systems where many hadrons are produced and they are not individually detected



See also… **Liu, Dong** (hep-ph/9306299), **Liu** (arXiv:1703.04690), **Hashimoto** (arXiv:1703.01881), **Chambers *et al.*** (arXiv:1703.01153)

Can we calculate this using LQCD?

$$W_{\mu\nu}(p,q) \equiv \int d^4x\ e^{iqx} \langle \pi, p | \mathcal{J}^\dagger_\mu(x) \mathcal{J}_\nu(0) | \pi, p \rangle$$

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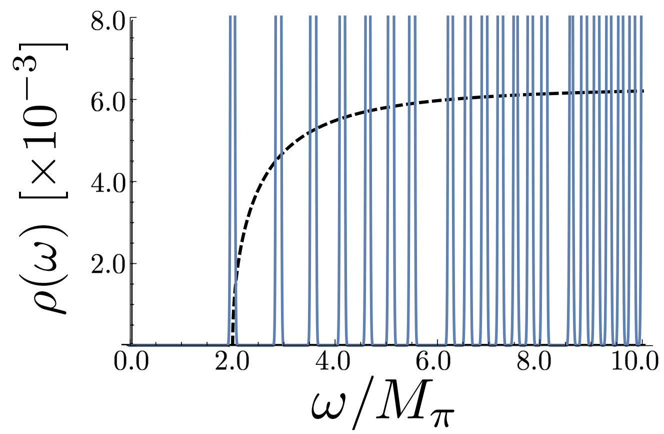
The main obstacles are finite volume and Euclidean time

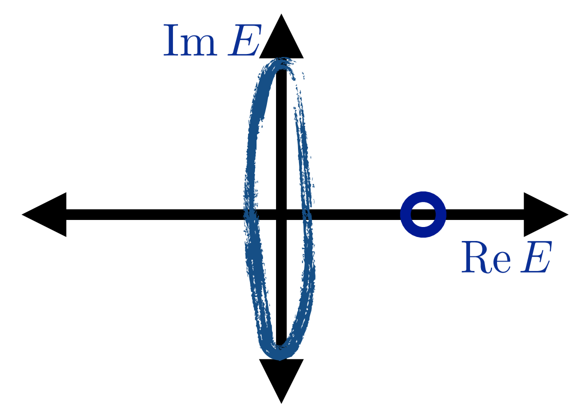
$$\sum_\alpha \int d\Phi_\alpha \; |p_x, \alpha, \text{out}\rangle\langle p_x, \alpha, \text{out}| \;\neq\; \sum_n |E_n, \mathbf{p}_x, L\rangle\langle E_n, \mathbf{p}_x, L|$$

$$e^{i\omega t} \neq e^{i\omega\tau}$$

Finite volume



Euclidean time



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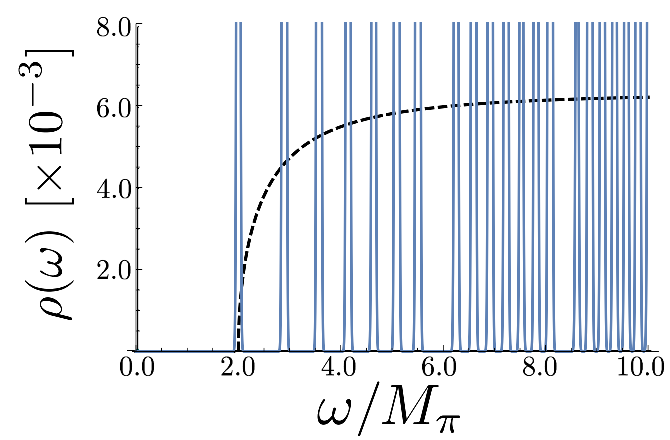
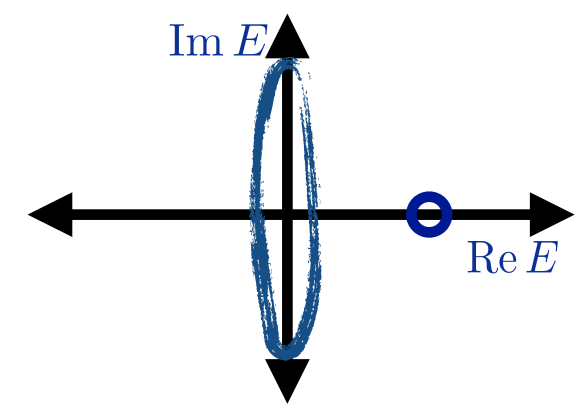
Finite volume

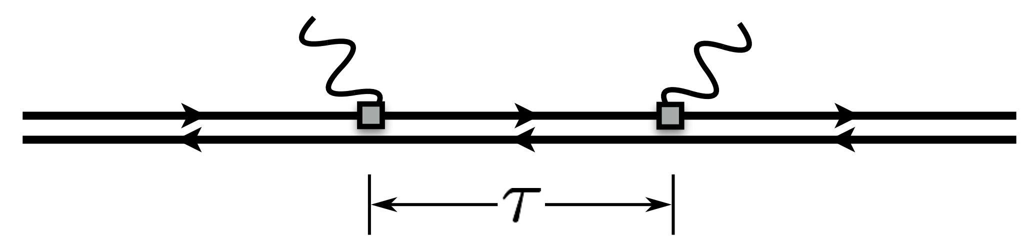


Euclidean time



What is the closest lattice quantity?

$$G_{\mu\nu}(\tau, \mathbf{q}, L) \propto \frac{\langle \pi_{\mathbf{p}}(\tau_f) \mathcal{J}_\mu^\dagger(\tau, \mathbf{q}) \mathcal{J}_\nu(0) \pi_{\mathbf{p}}(\tau_i) \rangle_{\mathrm{conn}}}{\langle \pi_{\mathbf{p}}(\tau_f) \pi(\tau_i) \rangle}$$



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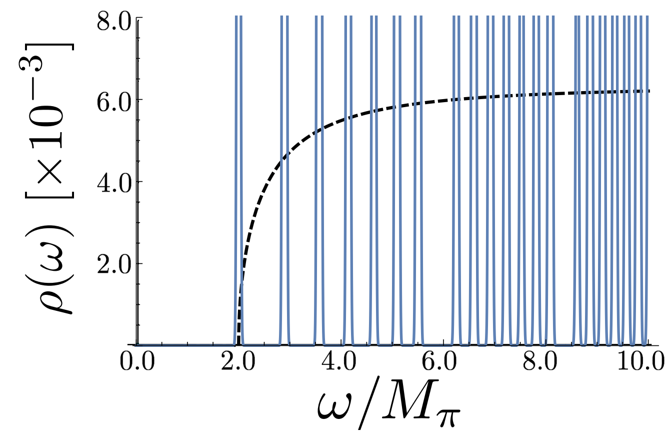
The main obstacles are finite volume and Euclidean time

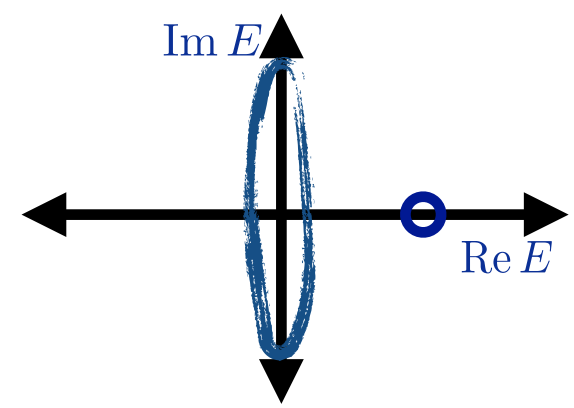
$$\boxed{\sum_\alpha \int d\Phi_\alpha \, |p_x, \alpha, \text{out}\rangle\langle p_x, \alpha, \text{out}| \neq \sum_n |E_n, \mathbf{p}_x, L\rangle\langle E_n, \mathbf{p}_x, L|}$$

$$\boxed{e^{i\omega t} \neq e^{i\omega\tau}}$$

Finite volume



Euclidean time



What is the closest lattice quantity?

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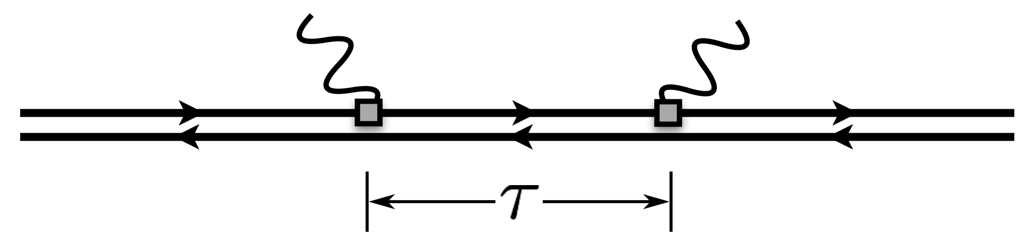

$$\boxed{G_{\mu\nu}(\tau, \mathbf{q}, L) = e^{-\omega_{\mathbf{p}}\tau} \langle \pi, \mathbf{p} | \mathcal{J}_\mu(\tau, \mathbf{q}) \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle_L + \cdots}$$

Our goal...

(a) This is what we have...

$$G_{\nu\nu}(\tau, \mathbf{q}, L) = \sum_n e^{-E_n(L)\tau} |\langle E_n(L), \mathbf{p}_x | \mathcal{J}_\nu(0) | \pi, \mathbf{p}\rangle_L|^2$$

(b) This is what we want...

$$W_{\nu\nu}(p, q) = \int_{\text{all states}, E'} (2\pi)\delta(E - E')\ |\langle E', \mathbf{p}_x, \text{out} | \mathcal{J}_\nu(0) | \pi, \mathbf{p}\rangle|^2$$

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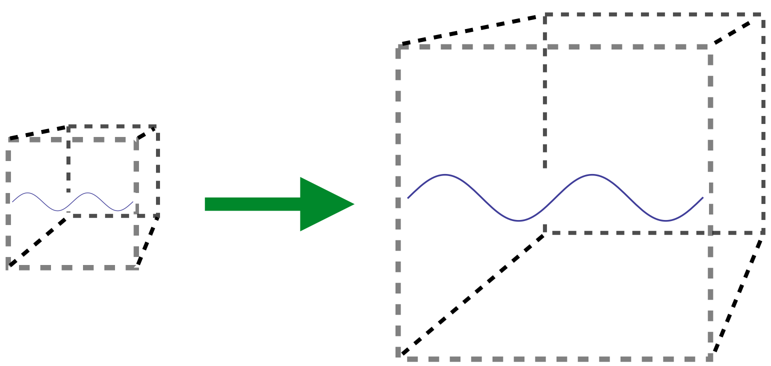
To go from *(a)* to *(b)* one must first take the infinite-volume limit



$$\boxed{G_{\nu\nu}(\tau, \mathbf{q}) = \int_{\text{all states}, E'} e^{-E'\tau} |\langle E', \mathbf{p}_x, \text{out} | \mathcal{J}_\nu(0) | \pi, \mathbf{p}\rangle|^2}$$

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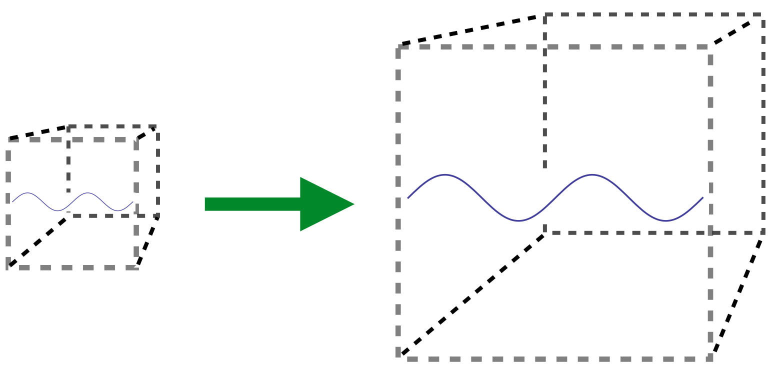

$$\boxed{G_{\nu\nu}(\tau,\mathbf{q}) = \int_{\text{all states},E'} e^{-E'\tau} |\langle E',\mathbf{p}_x,\text{out}|\mathcal{J}_\nu(0)|\pi,\mathbf{p}\rangle|^2}$$

Then replace $e^{-E'\tau}$ **with** $(2\pi)\delta(E-E')$

The importance of infinite-volume

Suppose we take some (large) finite volume…

$$G_{\nu\nu}(\tau, \mathbf{q}, L) = \sum_n e^{-E_n(L)\tau} |\langle E_n(L), \mathbf{p}_x | \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle_L|^2$$

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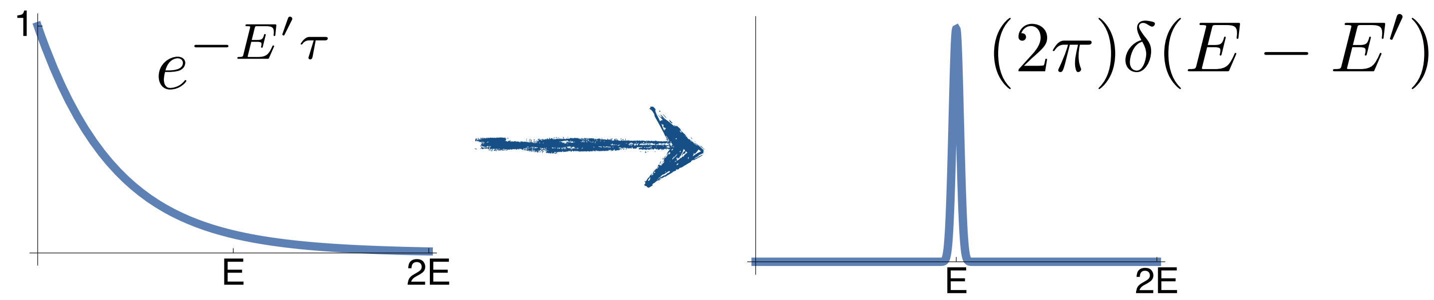

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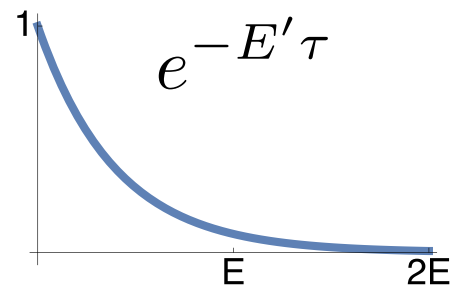
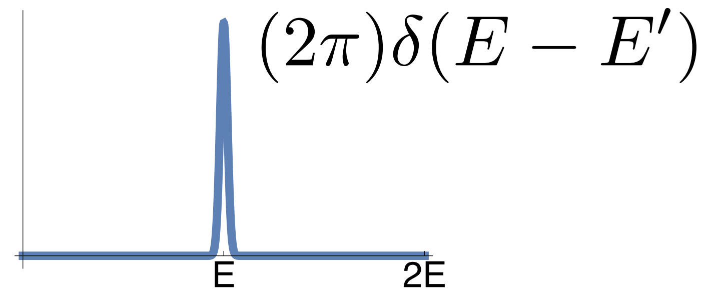
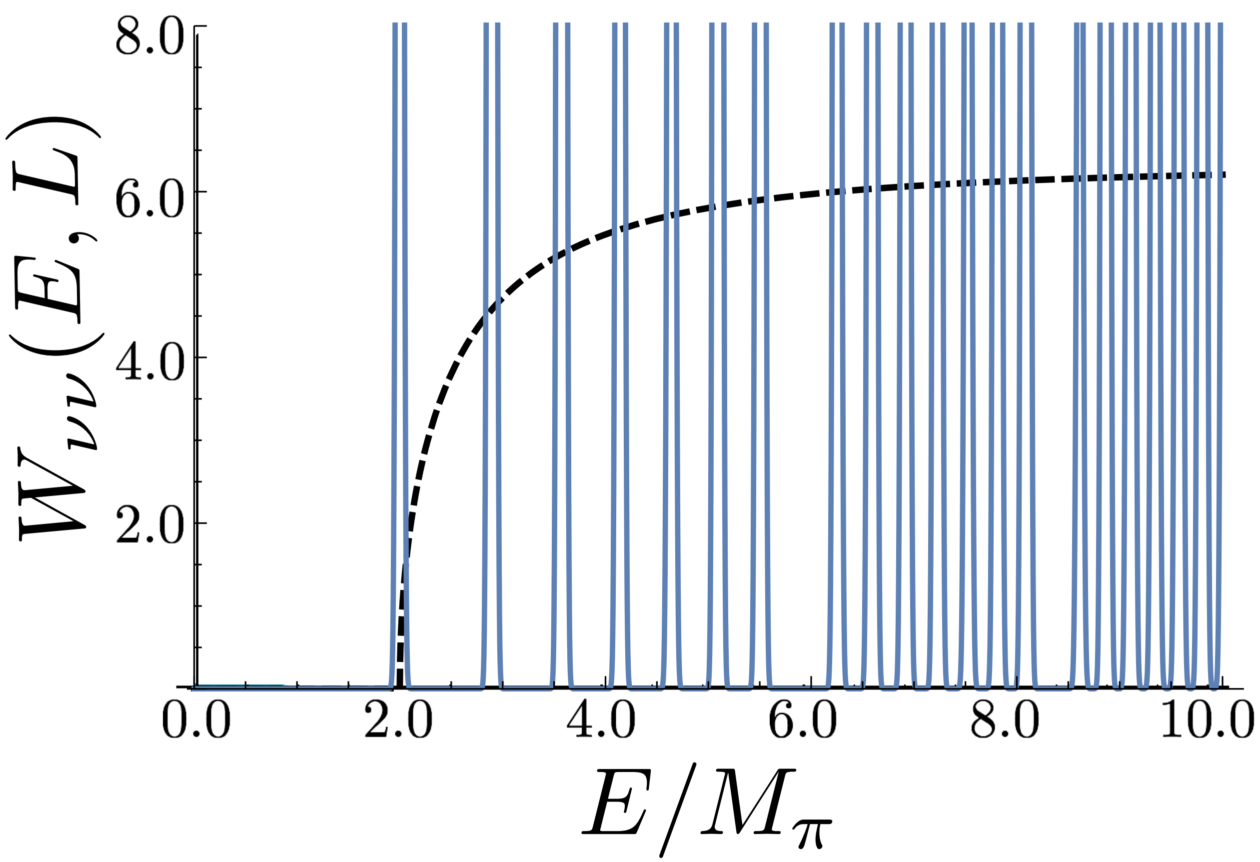




This does not give a useful estimate



For a fixed volume the sum over delta functions does not directly give a useful estimate of the infinite-volume target quantity

Our goal (cont.)...

Suppose we could send $L \to \infty$

$$\lim_{L\to\infty} G_{\nu\nu}(\tau, \mathbf{q}, L) = \int_{\text{all states}, E'} e^{-E'\tau} |\langle E', \mathbf{p}_x, \text{out}|\mathcal{J}_\nu(0)|\pi, \mathbf{p}\rangle|^2$$

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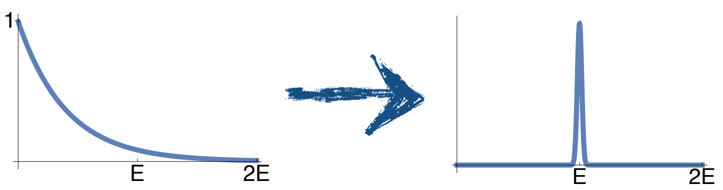
Then the remaining step is an inverse Laplace transform

$$G(\tau) = \int_0^\infty dE' e^{-E'\tau} \rho(E') \xrightarrow[?]{} \rho(E)$$



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This amounts to forming clever linear combinations...

$$q_1(E)e^{-E' a_\tau} + q_2(E)e^{-E' 2a_\tau} + q_3(E)e^{-E' 3a_\tau} + \cdots \approx \delta(E - E')$$

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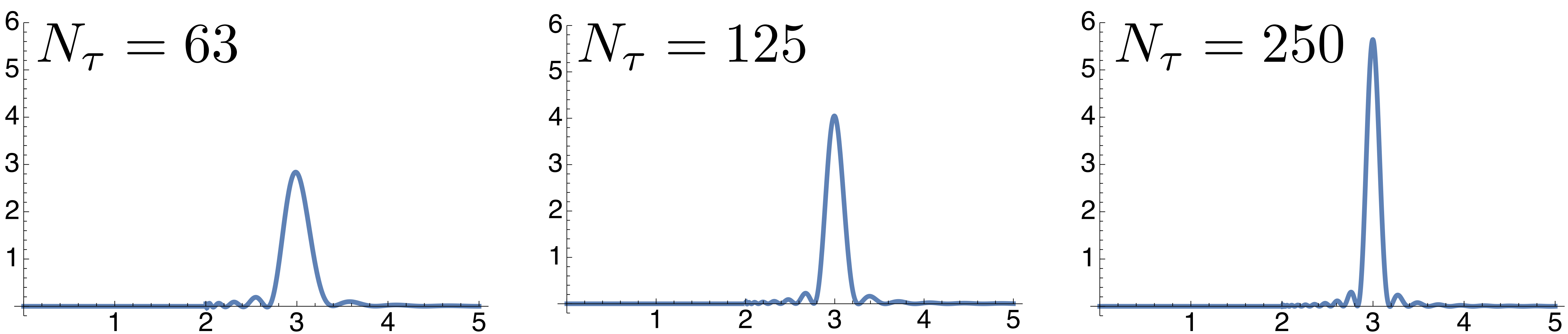
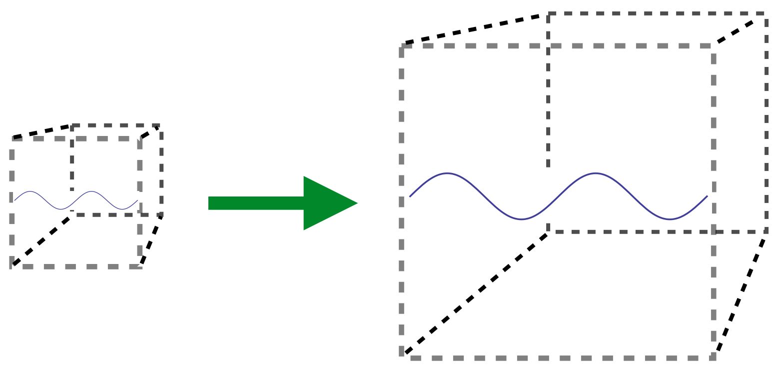
In fact this is possible…



Practical limitations

One cannot strictly take the infinite-volume limit…



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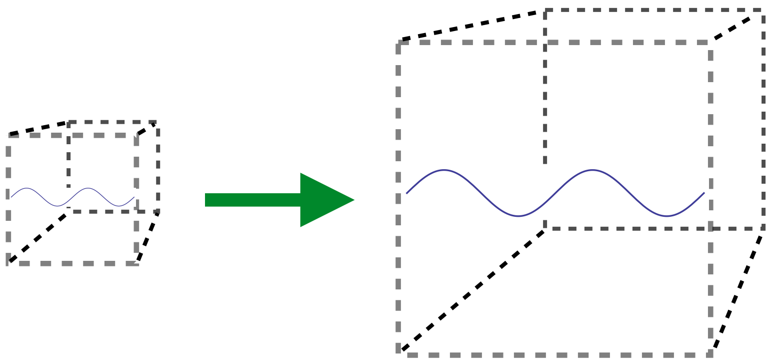

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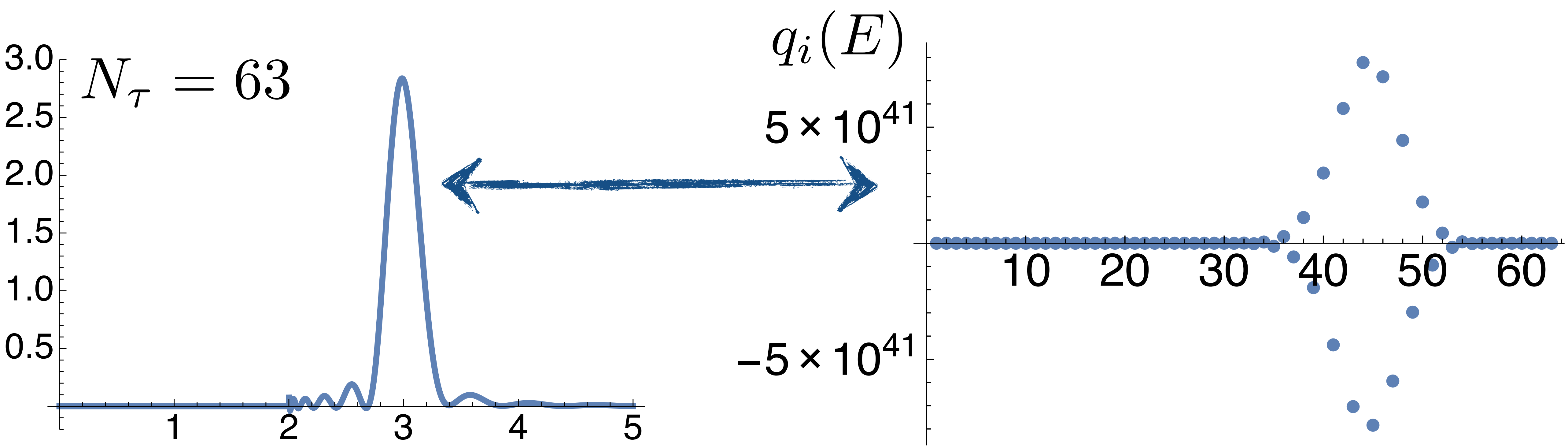

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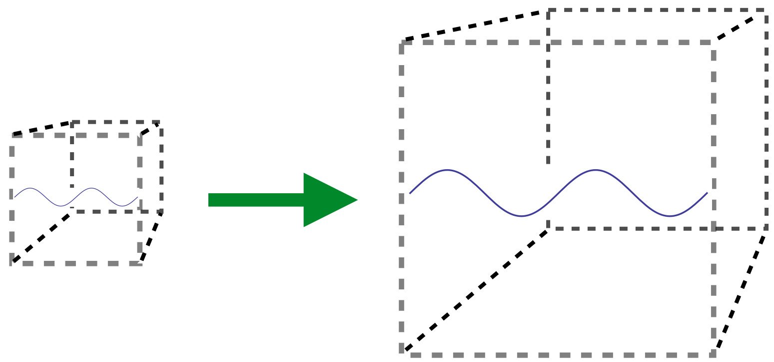

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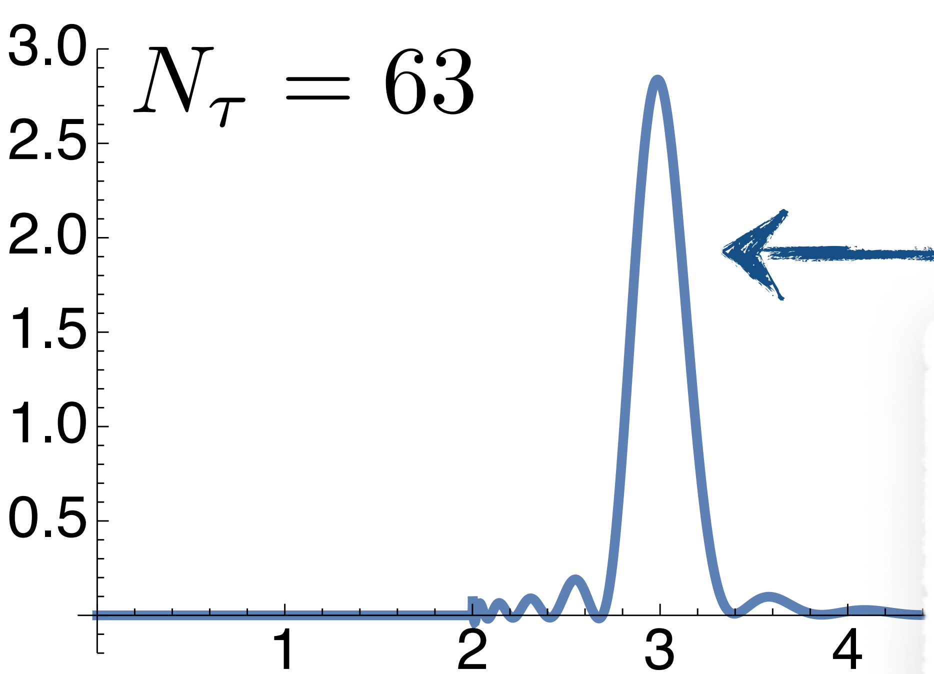

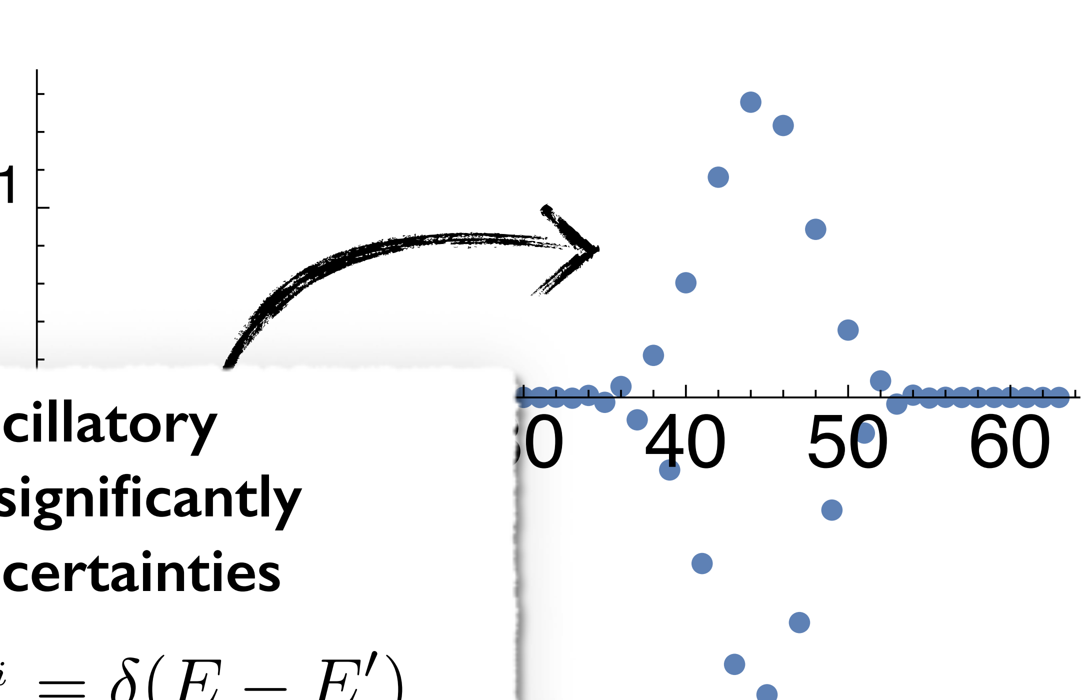

Large, oscillatory coefficients significantly enhance uncertainties

$$\sum_i q_i(E)e^{-E'\tau_i} = \delta(E-E')$$

Two big problems…

$$G_{\mu\nu}(\tau, \mathbf{q}, L) = \int_0^\infty dE_x \; W_{\mu\nu}(p, q; L) \; e^{-E_x \tau}$$

The inverse Laplace transform
is numerically ill-defined

For any *L*, numerical values of ρ
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$$G(\tau) = \int_0^\infty d\omega \; \rho(\omega) \; e^{-\omega\tau} \longrightarrow \widehat{\rho}(\overline{\omega}) = \int_0^\infty d\omega \; \widehat{\delta}_\Delta(\overline{\omega}, \omega) \; \rho(\omega)$$

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- The correlation matrix is used to stabilize the inverse
- The quality of the data determines the **resolution**

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Two birds with one stone

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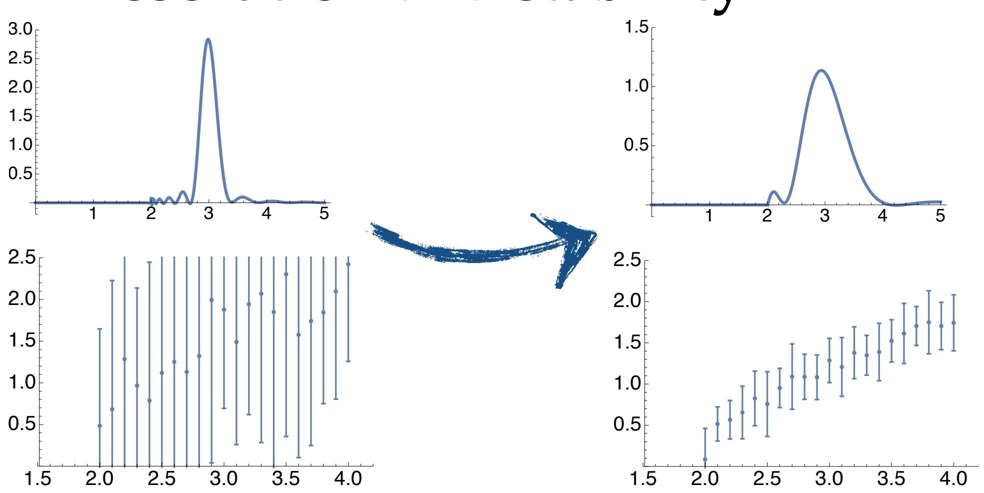
Backus-Gilbert tackles the inverse problem by balancing **resolution** and **stability**



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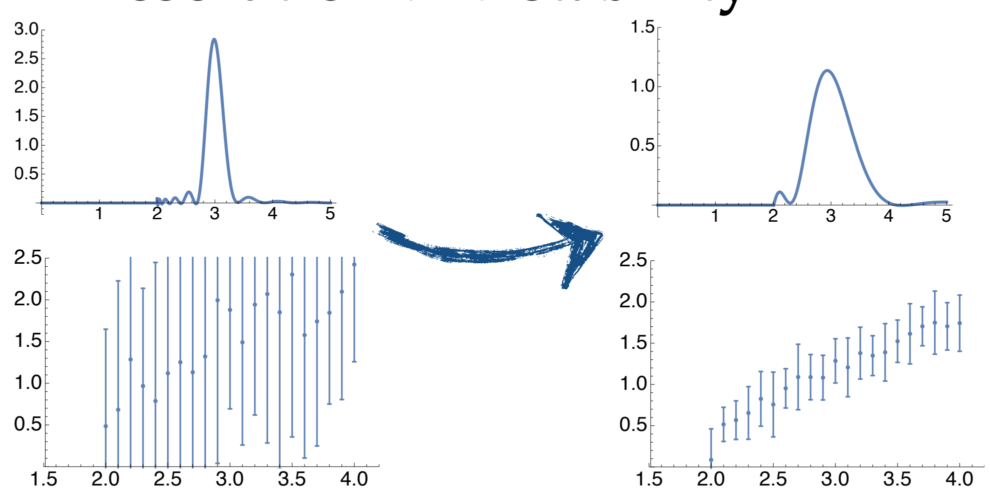
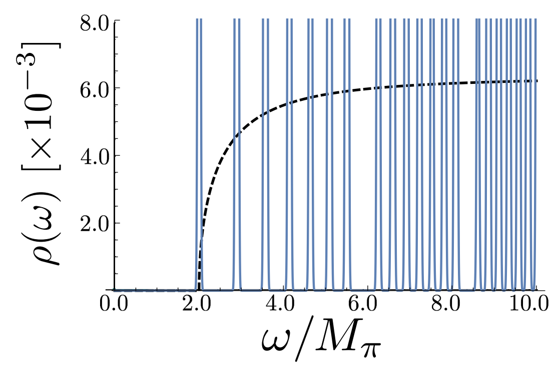


Smeared spectral function gives a ***better estimate*** of the infinite-volume observable



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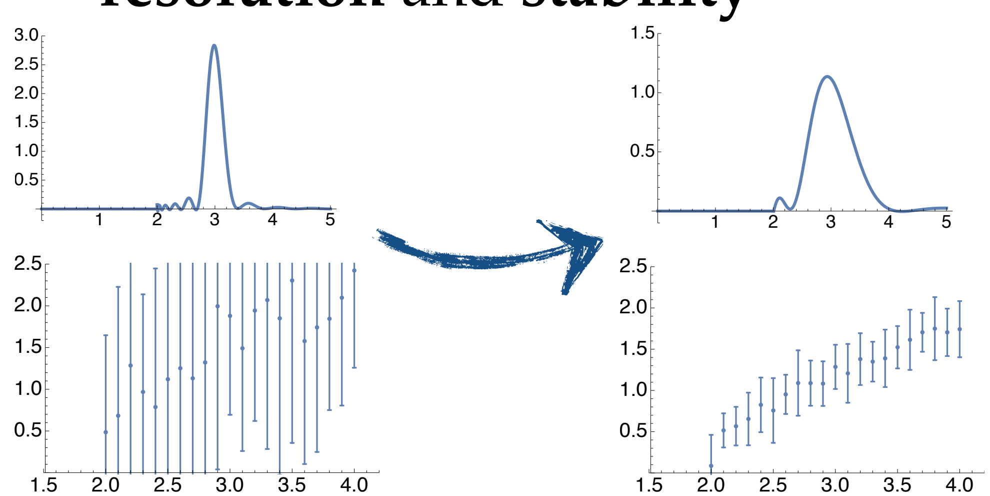

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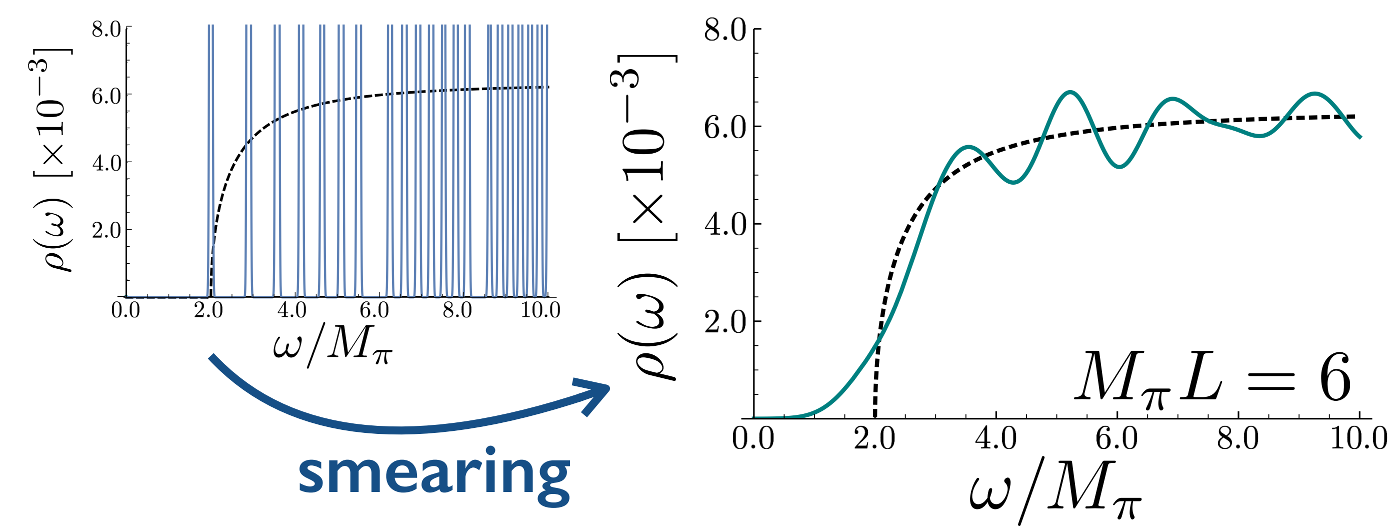

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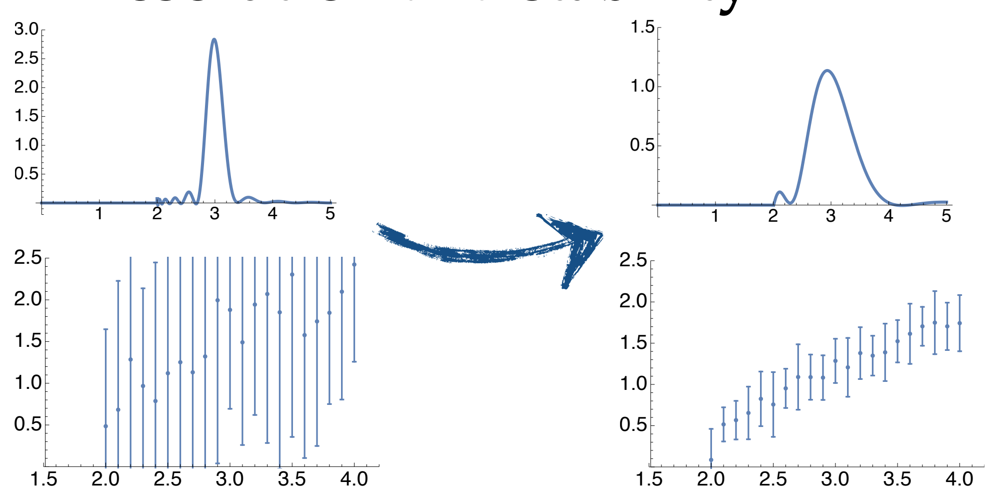

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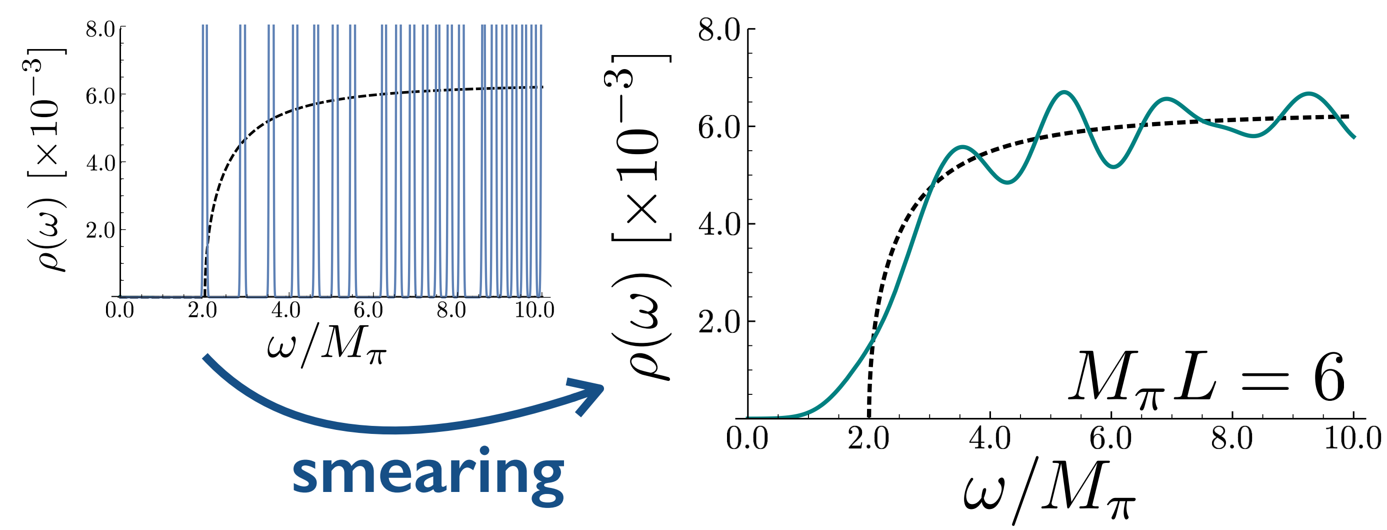

Target quantity is formally recovered in an ordered double limit

$$W_{\mu\nu}(p, q) = \lim_{\Delta \to 0} \lim_{L \to \infty} \widehat{W}_{\mu\nu}(p, q, L, \Delta)$$

The approach in full…

normalization factors

$$R_i \equiv \int_0^\infty dE' e^{-E'\tau_i}$$

spread matrix

$$W_{ij}(E) = \int_0^\infty dE' e^{-E'\tau_i} (E - E')^2 e^{-E'\tau_j}$$

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$$q_i(E) = \frac{[W(E)+\lambda S]^{-1}\cdot R}{R\cdot[W(E)+\lambda S]^{-1}\cdot R}$$

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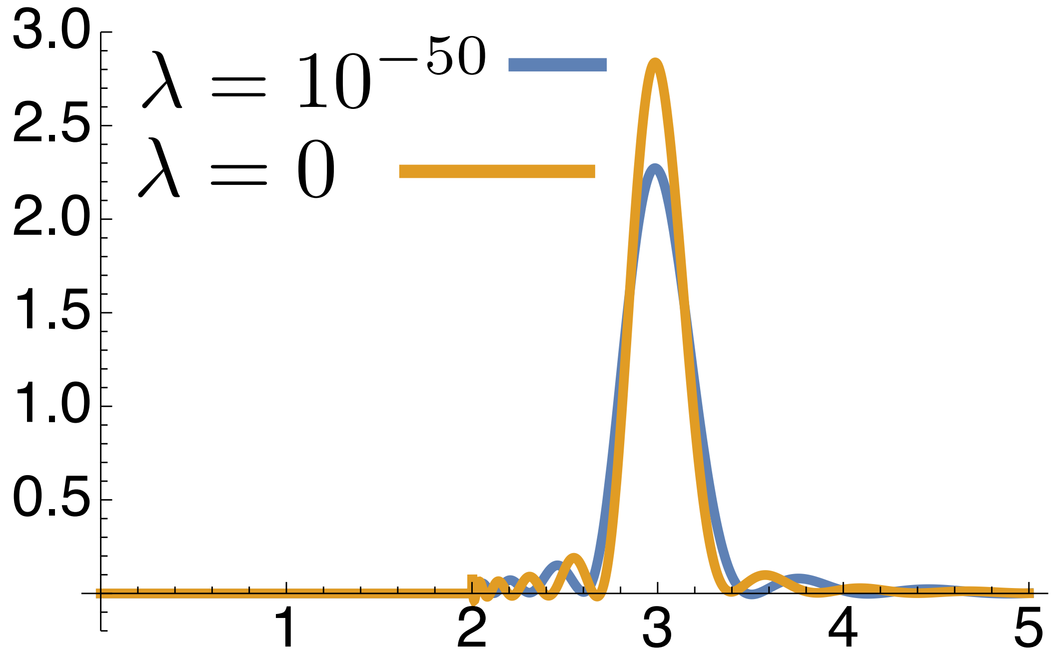

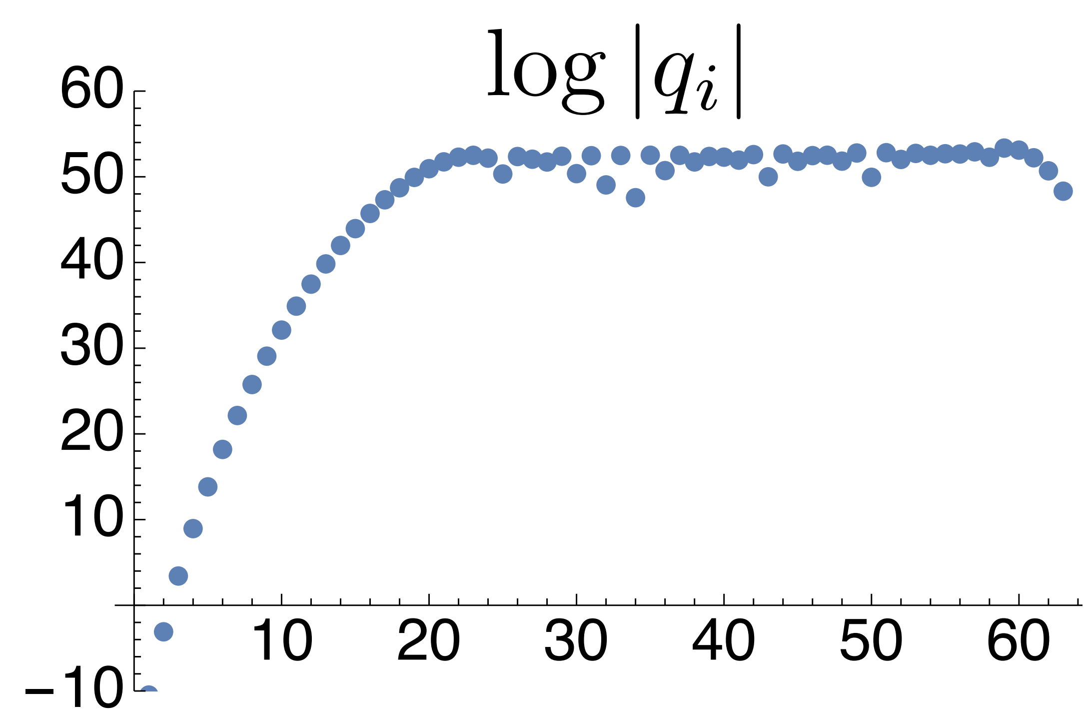

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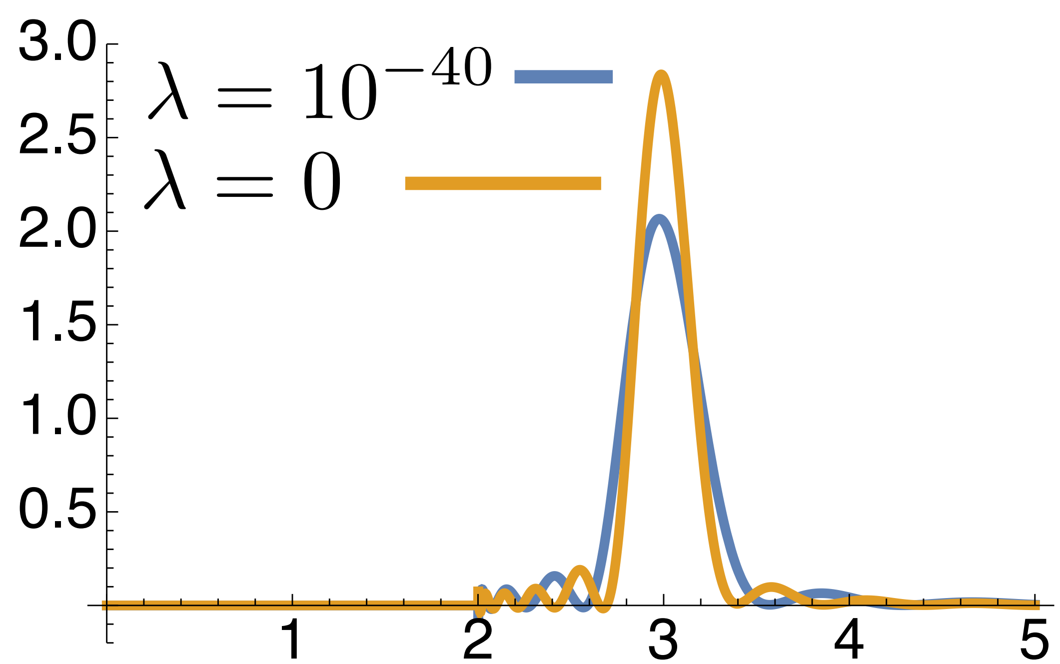

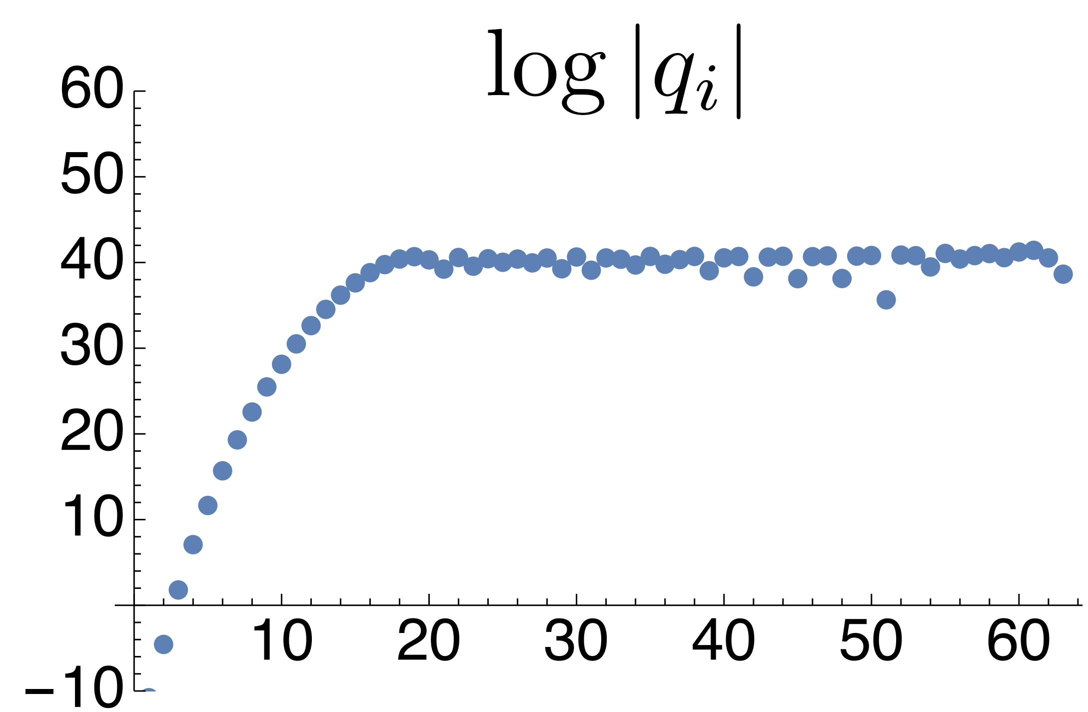

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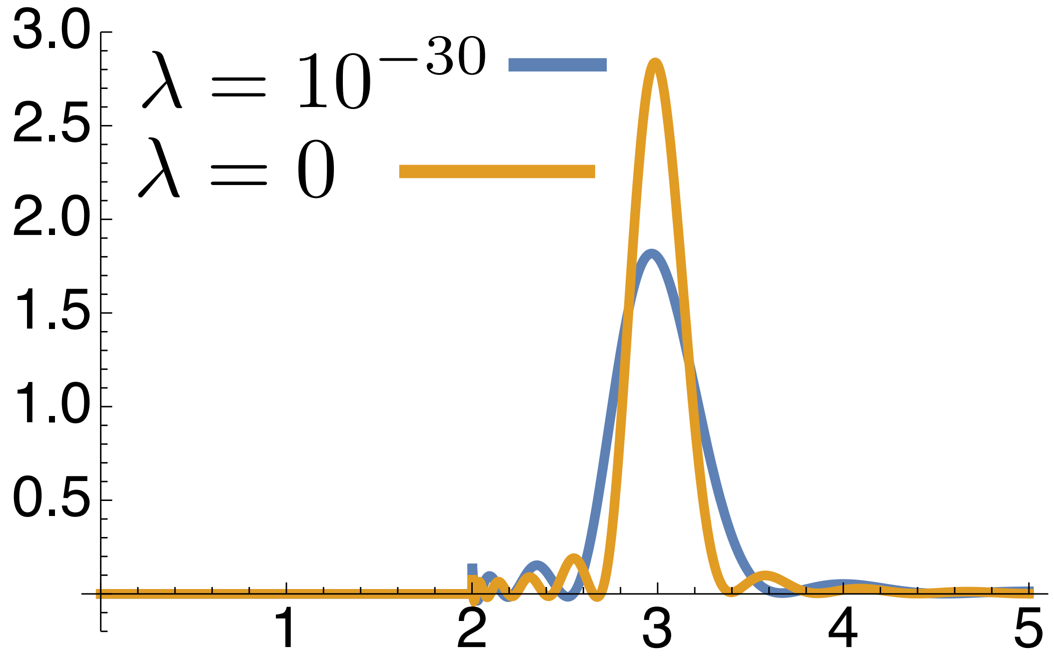

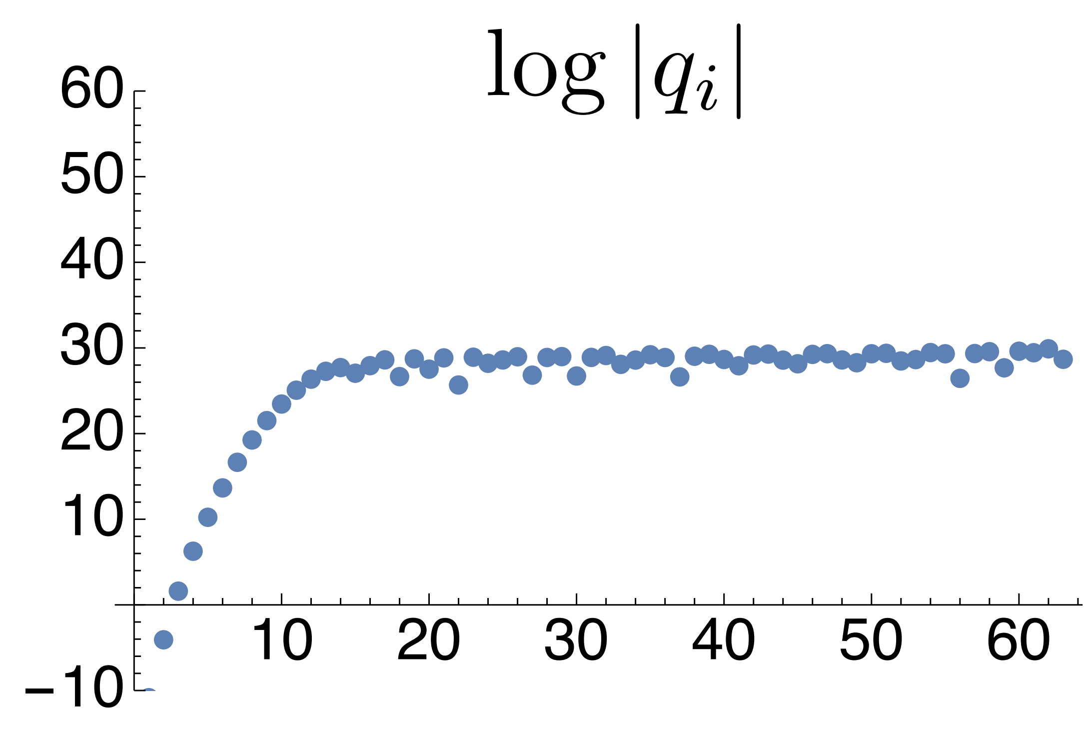

The approach in full…

normalization factors

$$R_i \equiv \int_0^\infty dE' e^{-E'\tau_i}$$

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$$W_{ij}(E) = \int_0^\infty dE' e^{-E'\tau_i}(E - E')^2 e^{-E'\tau_j}$$

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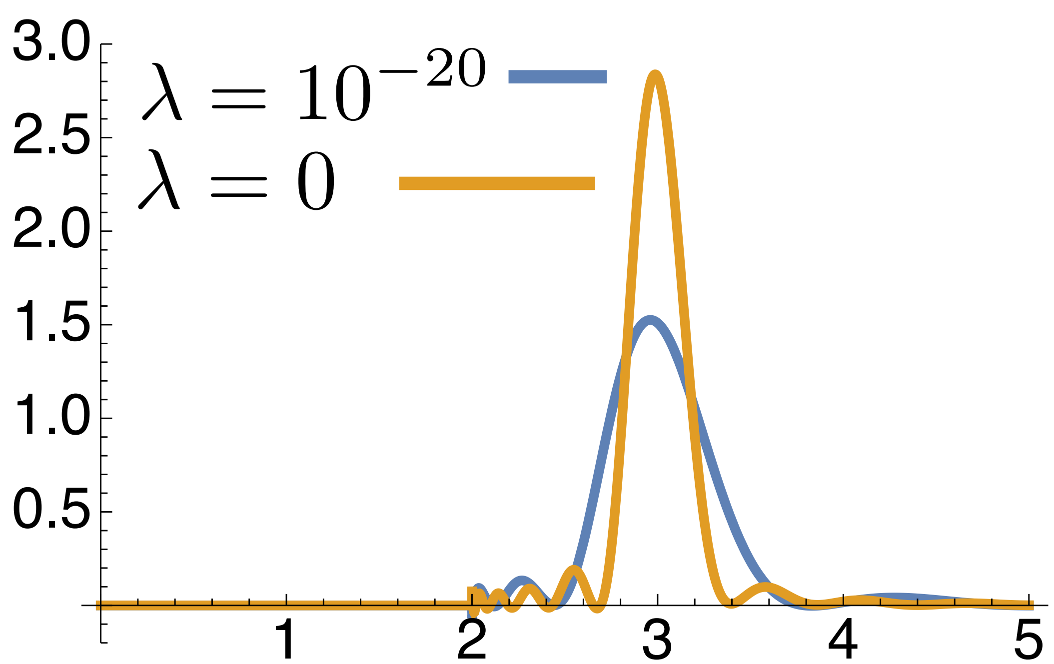

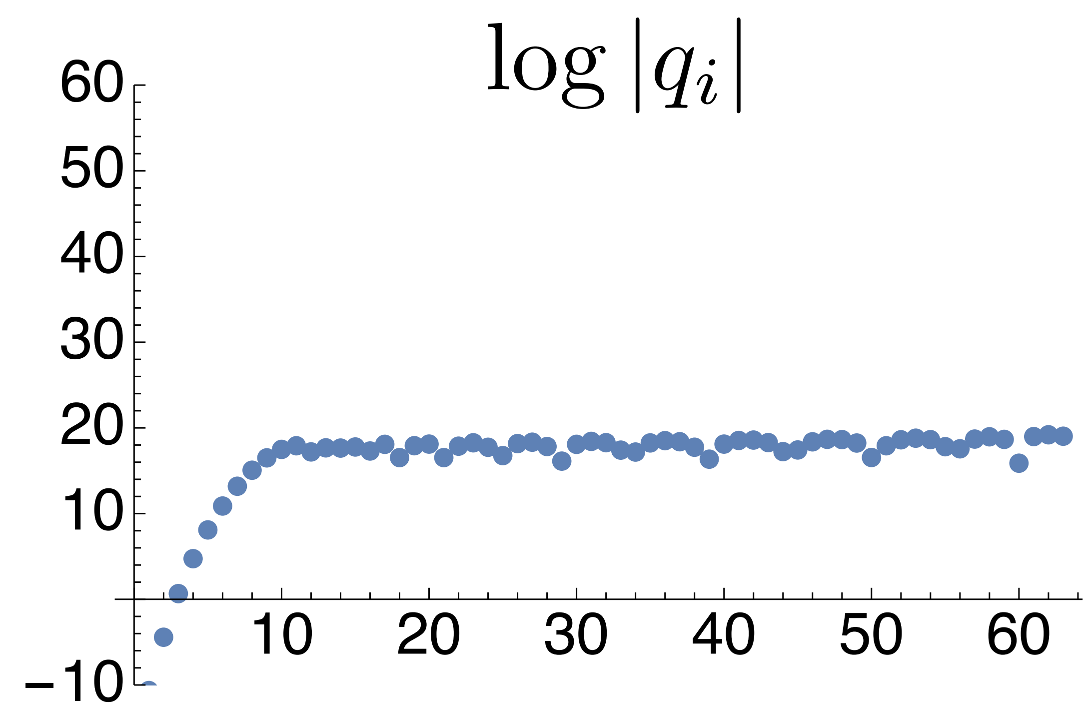

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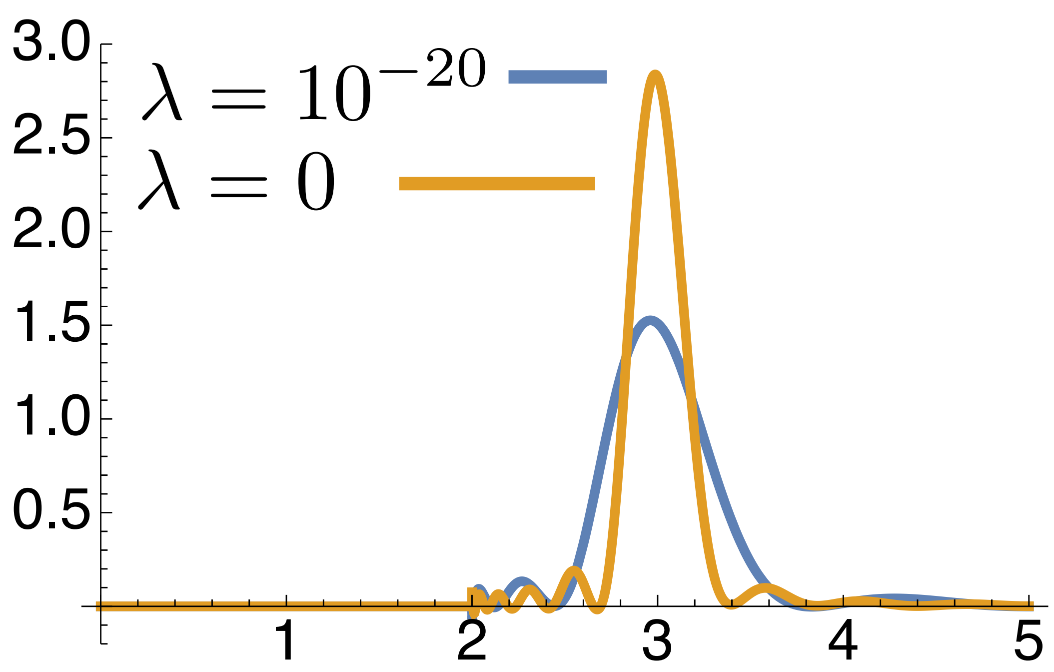

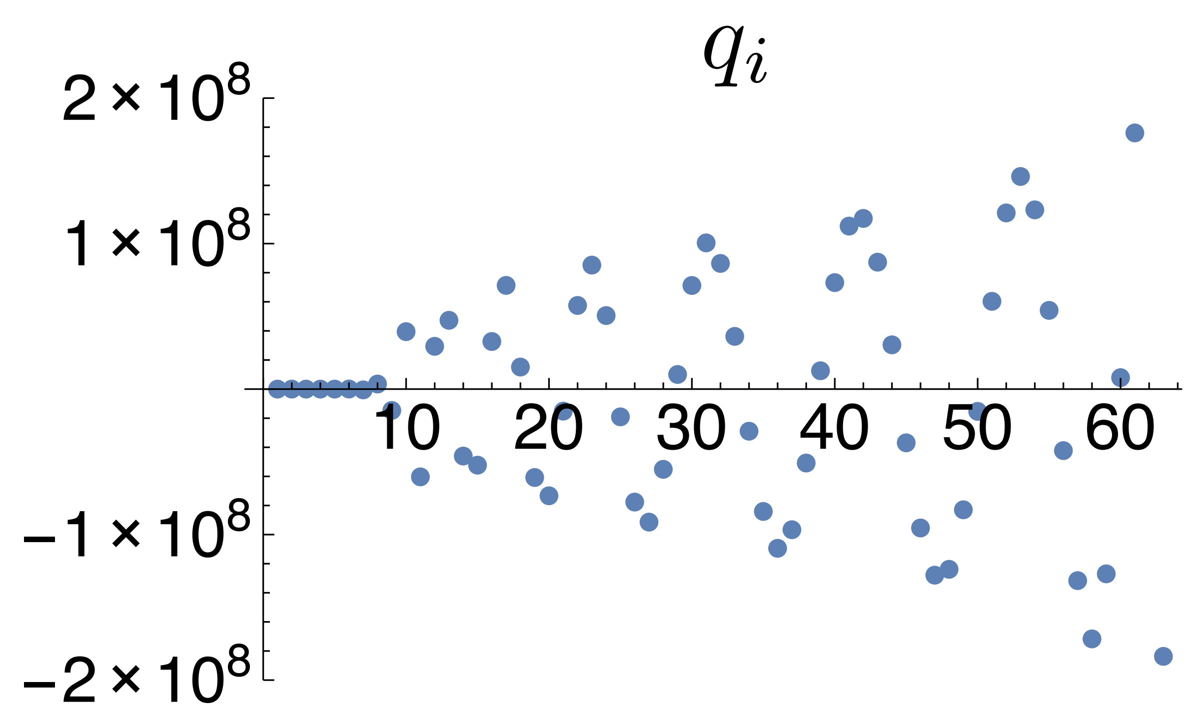

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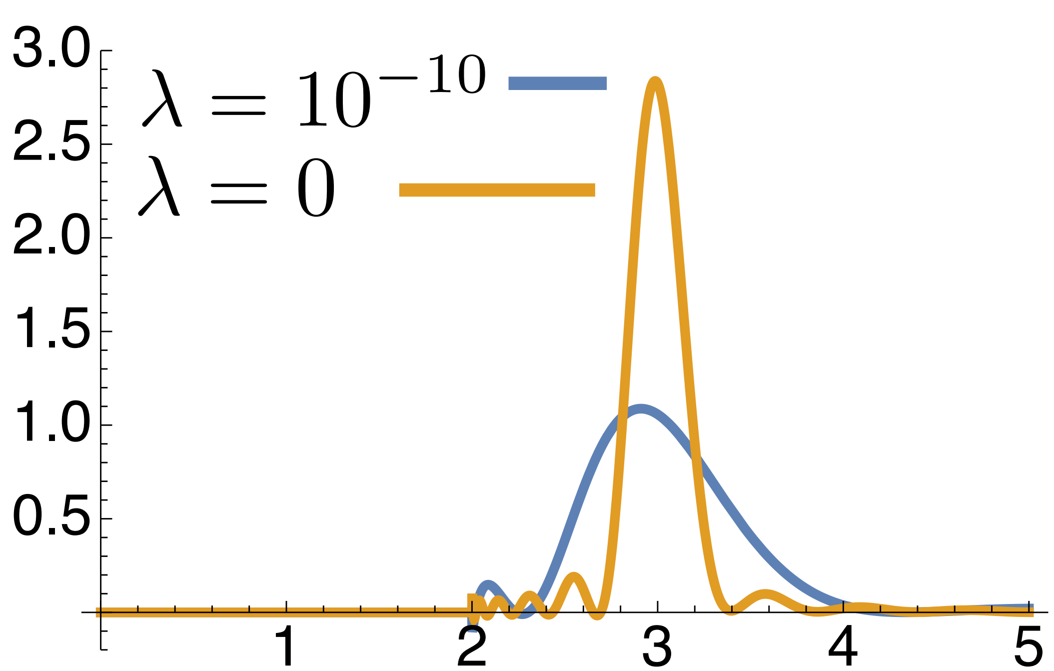

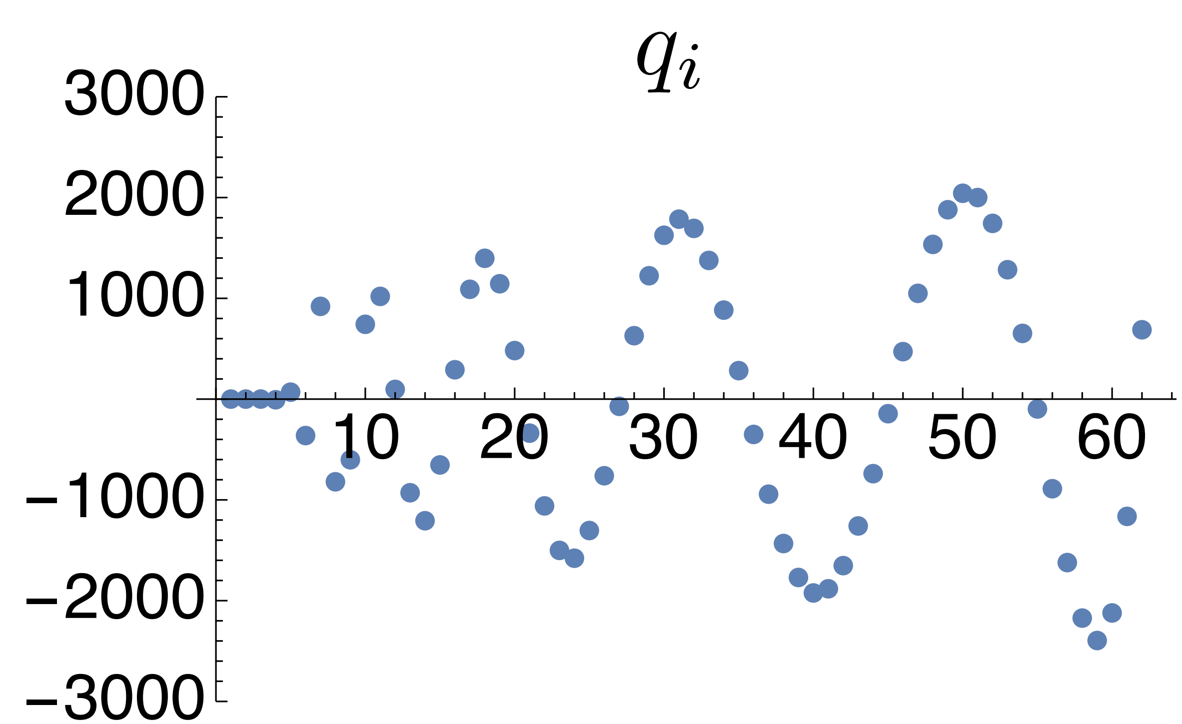

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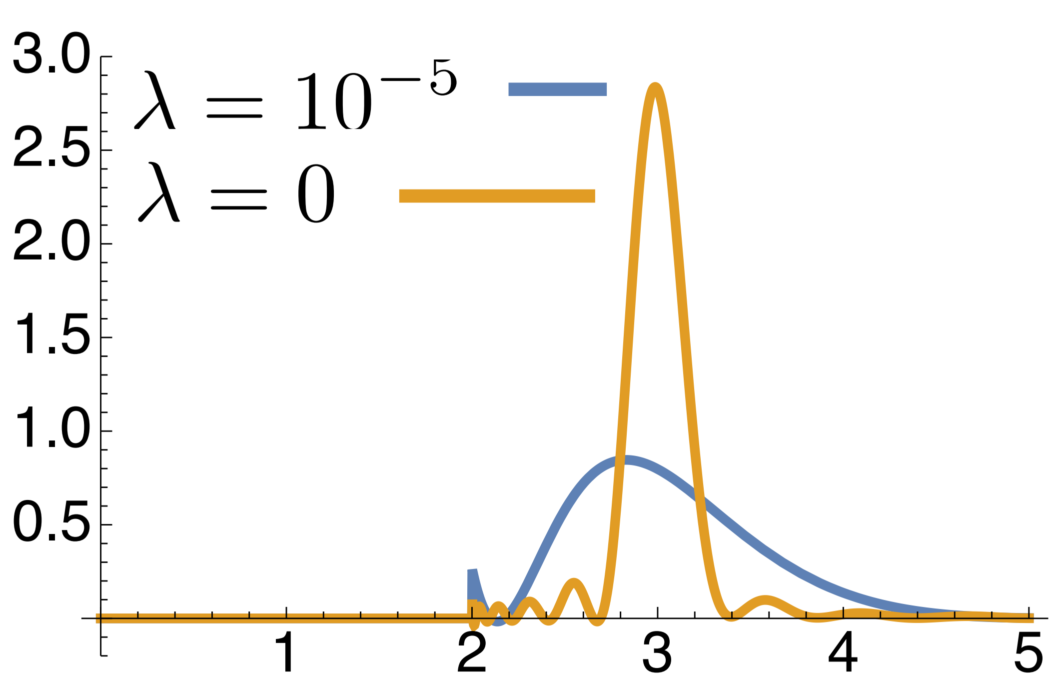

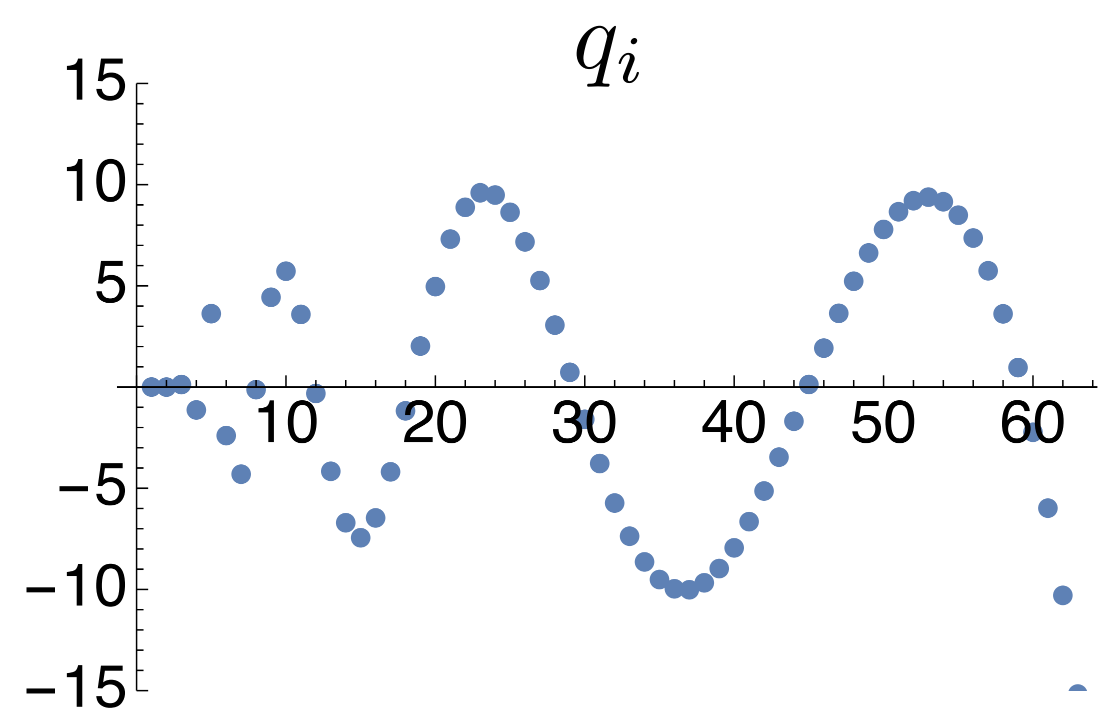

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To gain intuition on this consider a simple example

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free, two-particle spectral function

Gaussian resolution function

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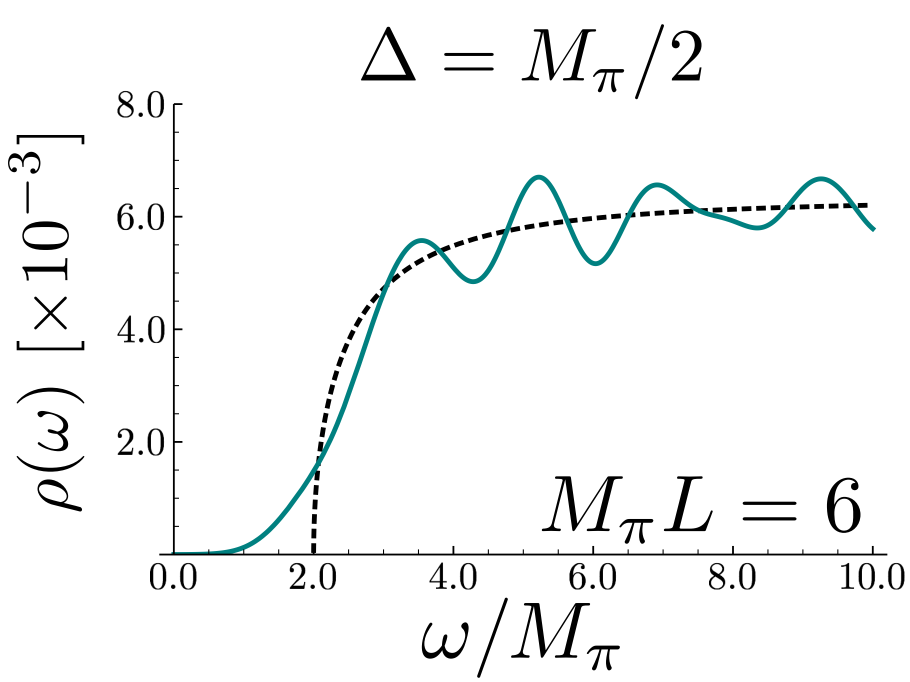

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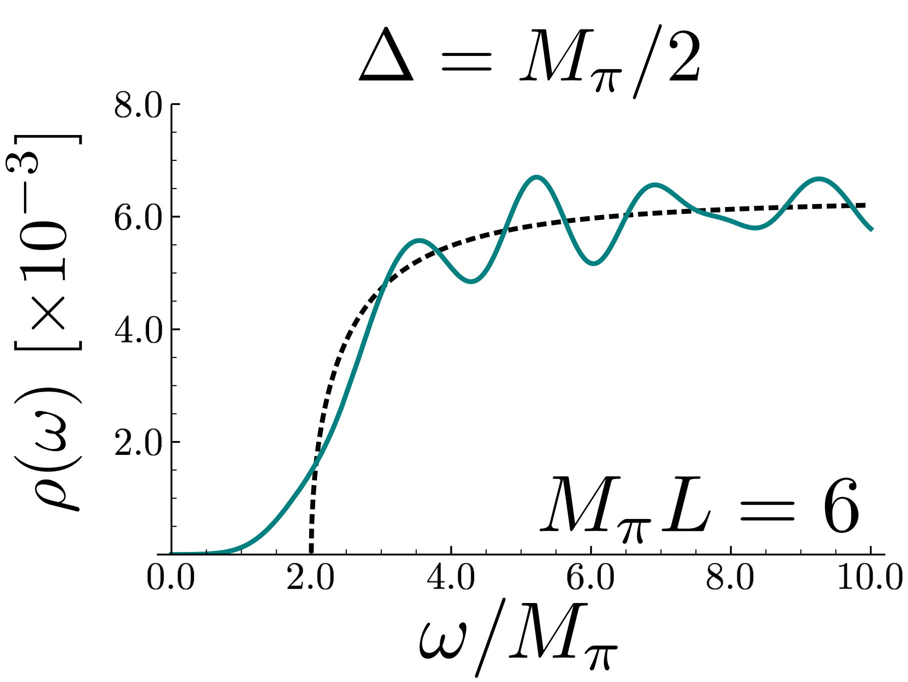

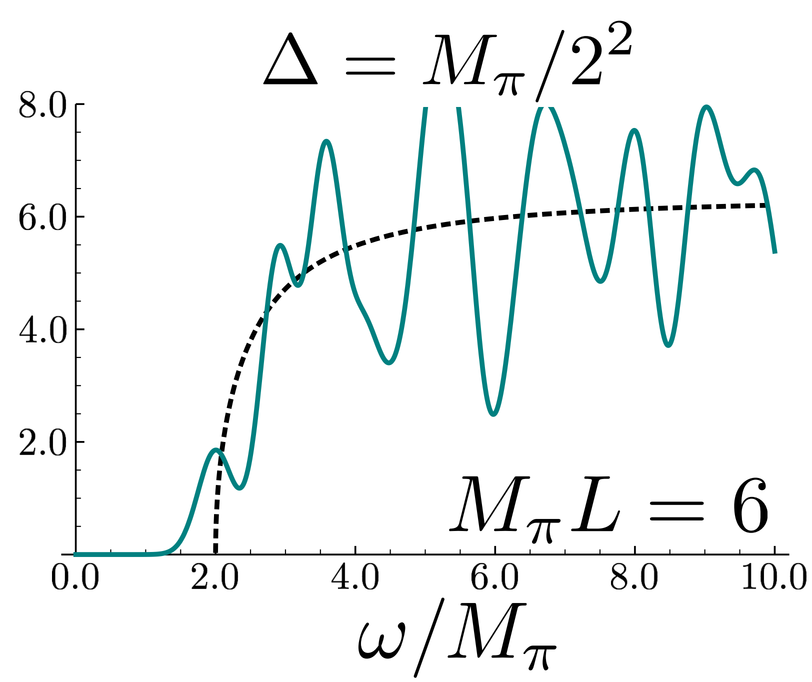

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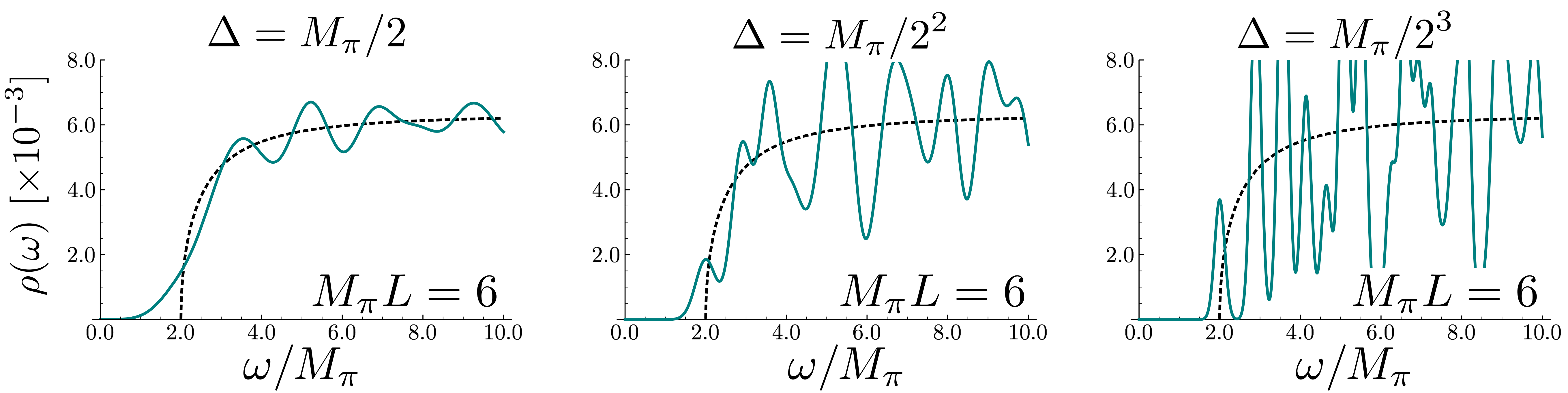

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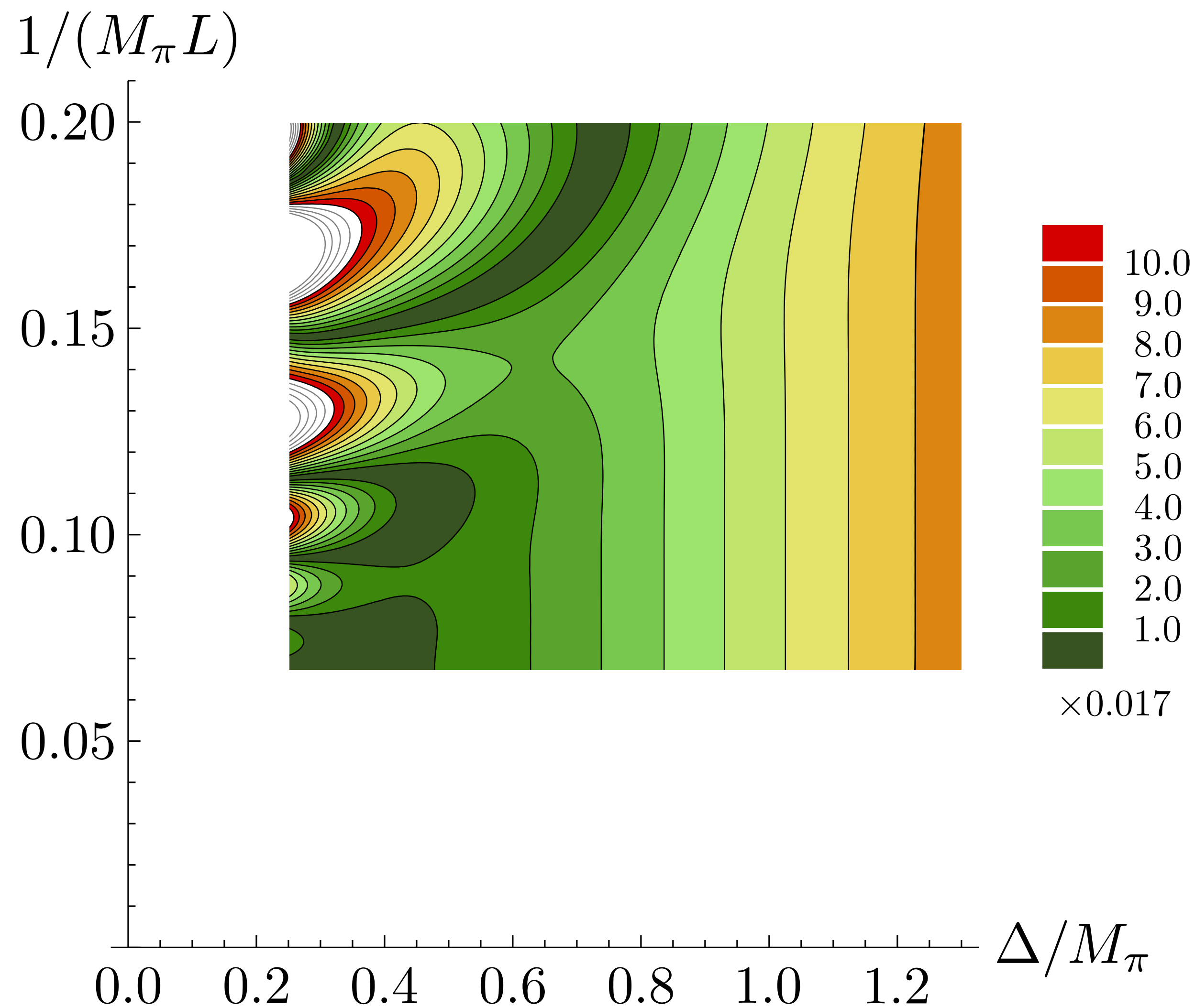

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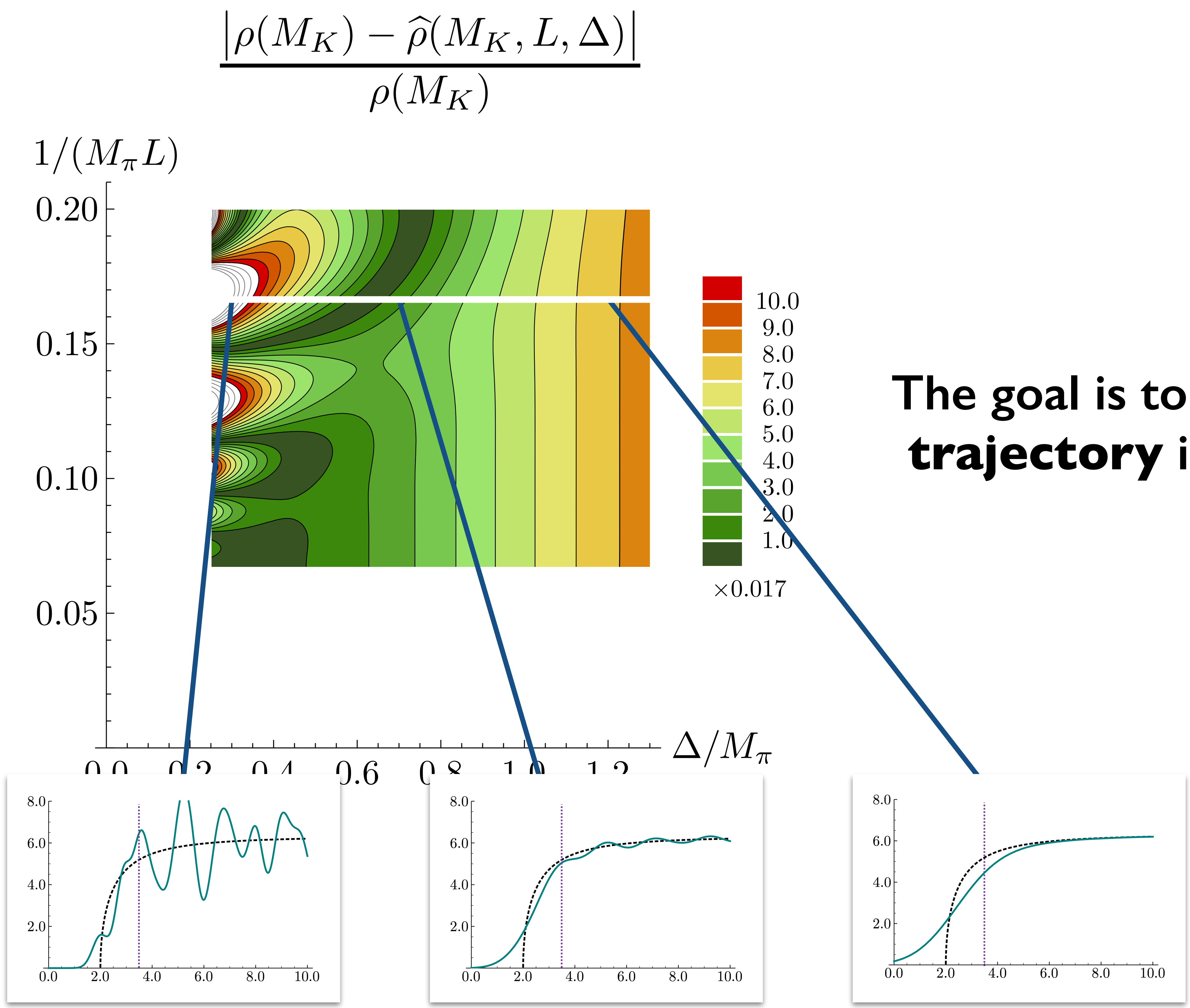

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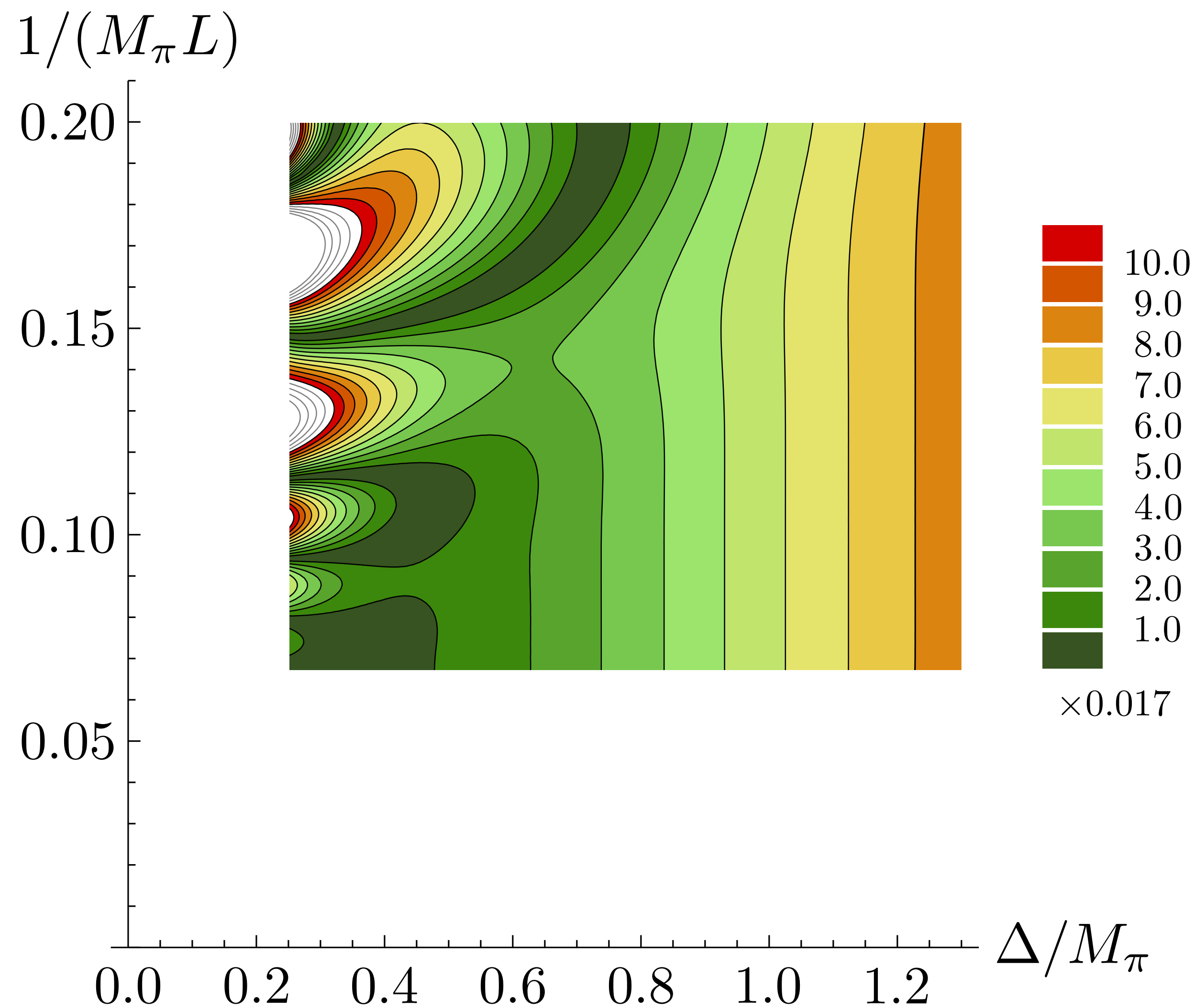
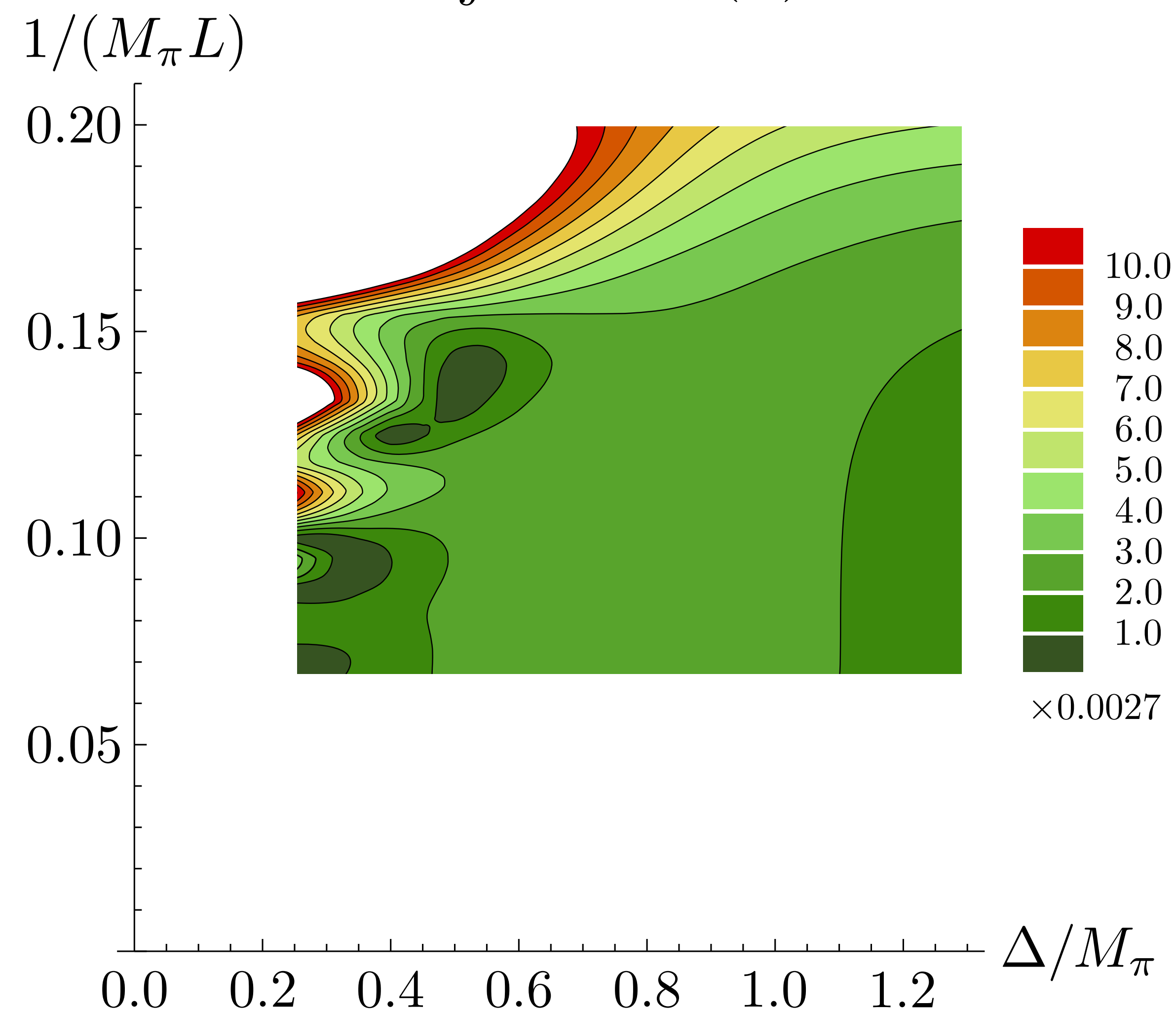


$$\frac{\left|\int^{4M_\pi} d\omega\rho(\omega) - \int^{4M_\pi} d\omega\widehat{\rho}(\omega, L, \Delta)\right|}{\int^{4M_\pi} d\omega\rho(\omega)}$$



Integrated spectral functions are generally **more precise** and less dependent on the choice of (Δ,1/L)

Testing the method…

To test the idea we studied a toy system with two open channels

$\phi \to \pi\pi\pi,\ K\overline{K}$ $\quad 3M_\pi < 2M_K$ $\quad \Gamma_\phi \propto \langle\phi|\mathcal{H}(q)\mathcal{H}(0)|\phi\rangle$

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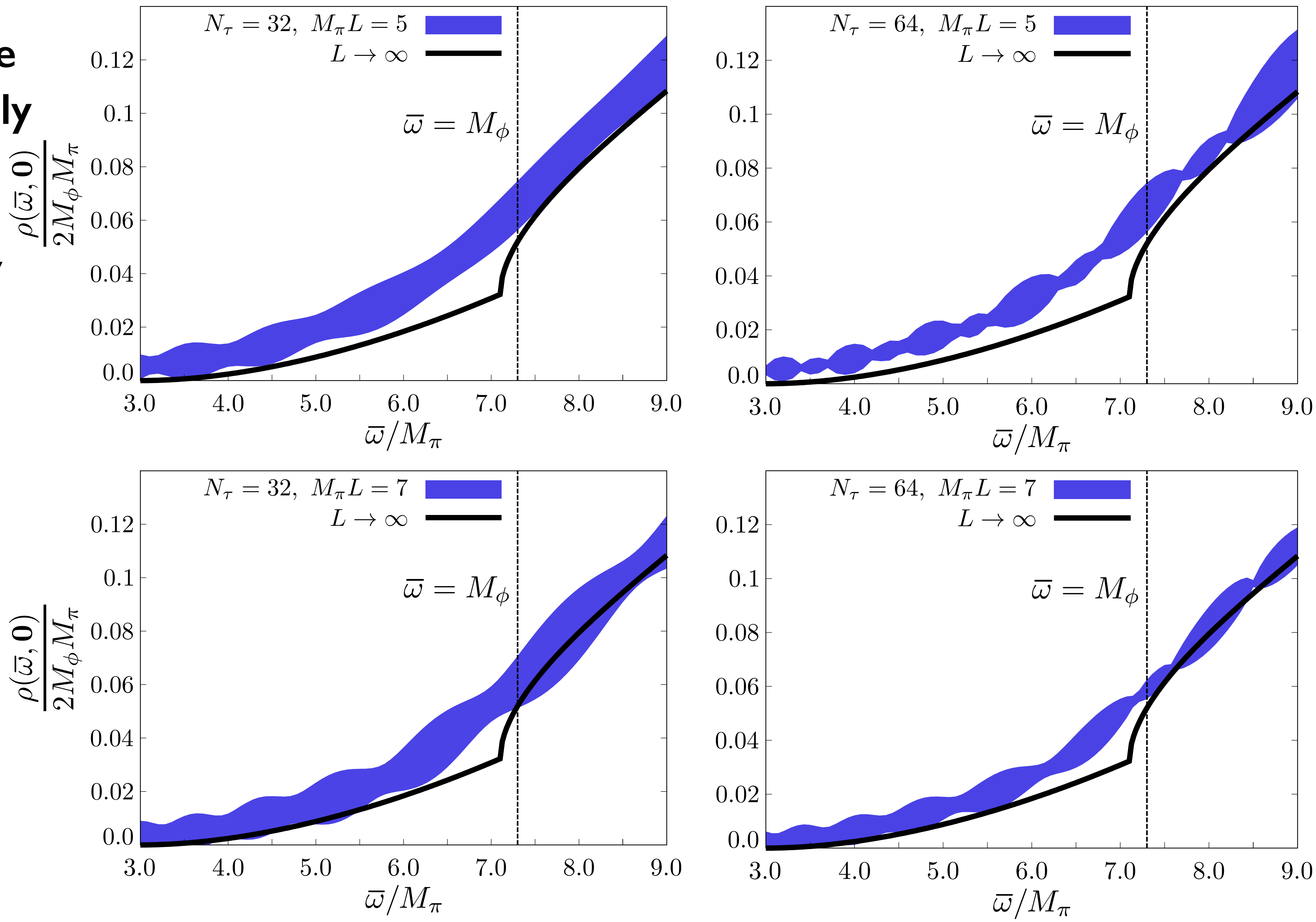

figures by D. Robaina

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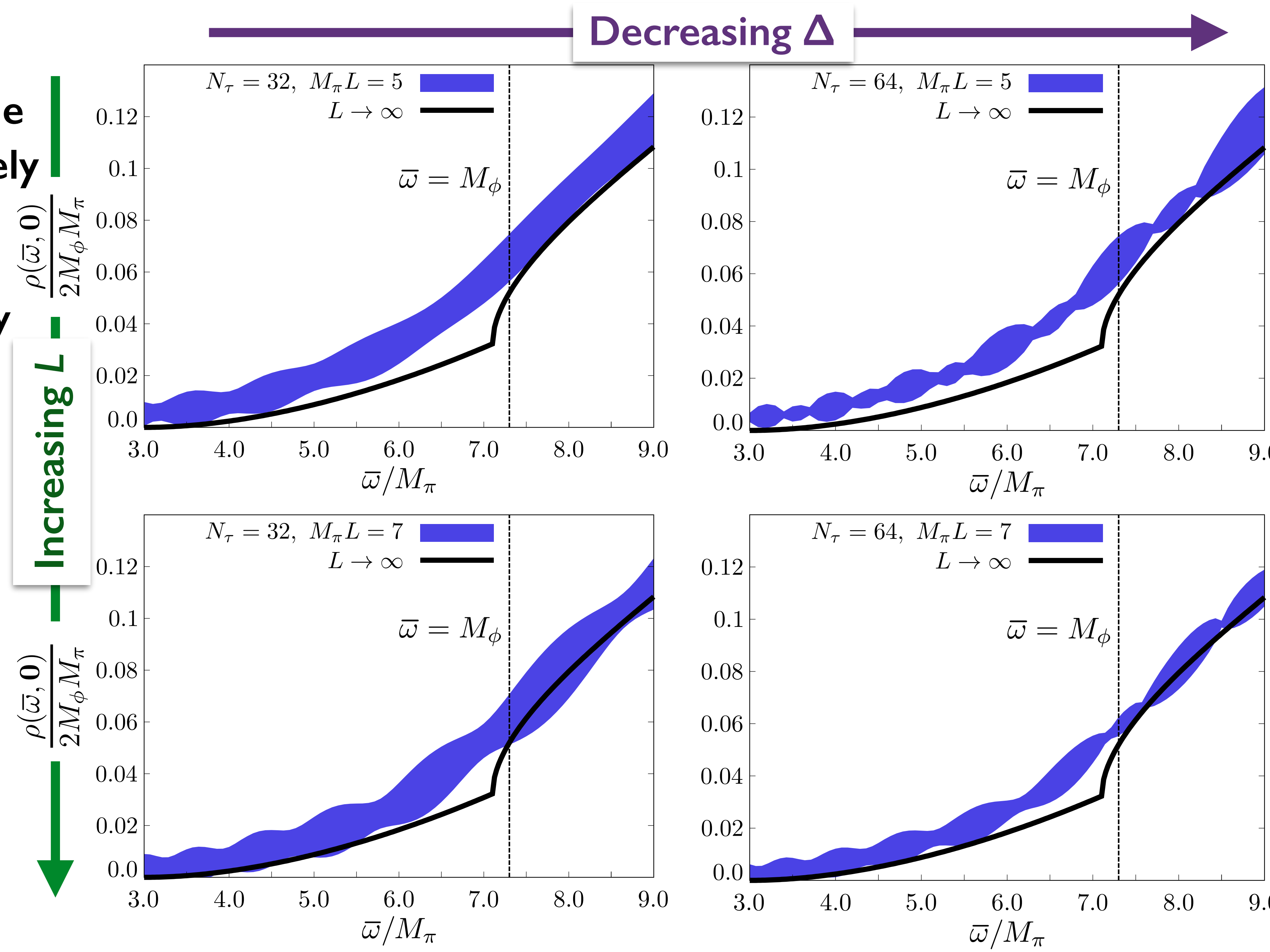

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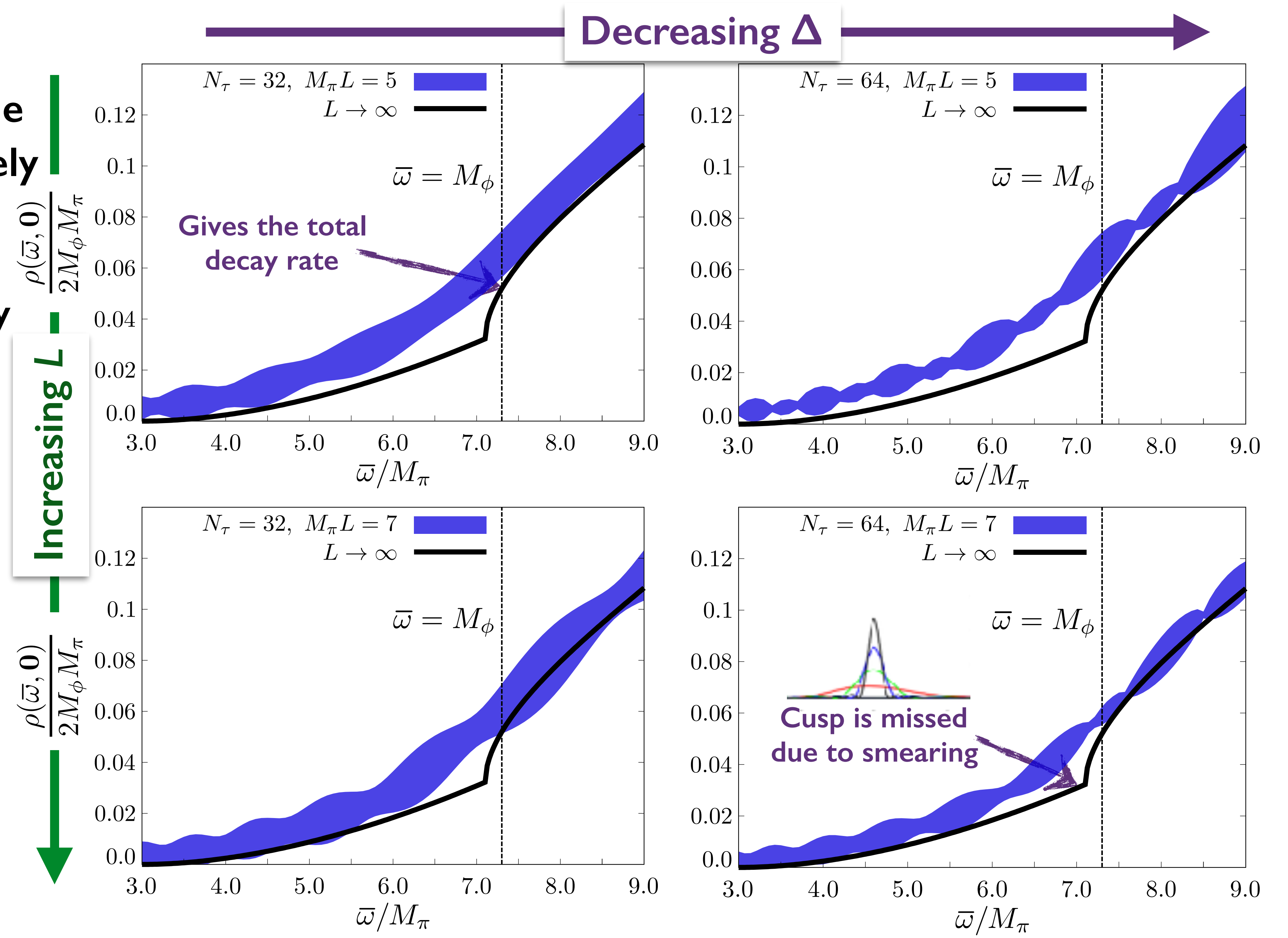

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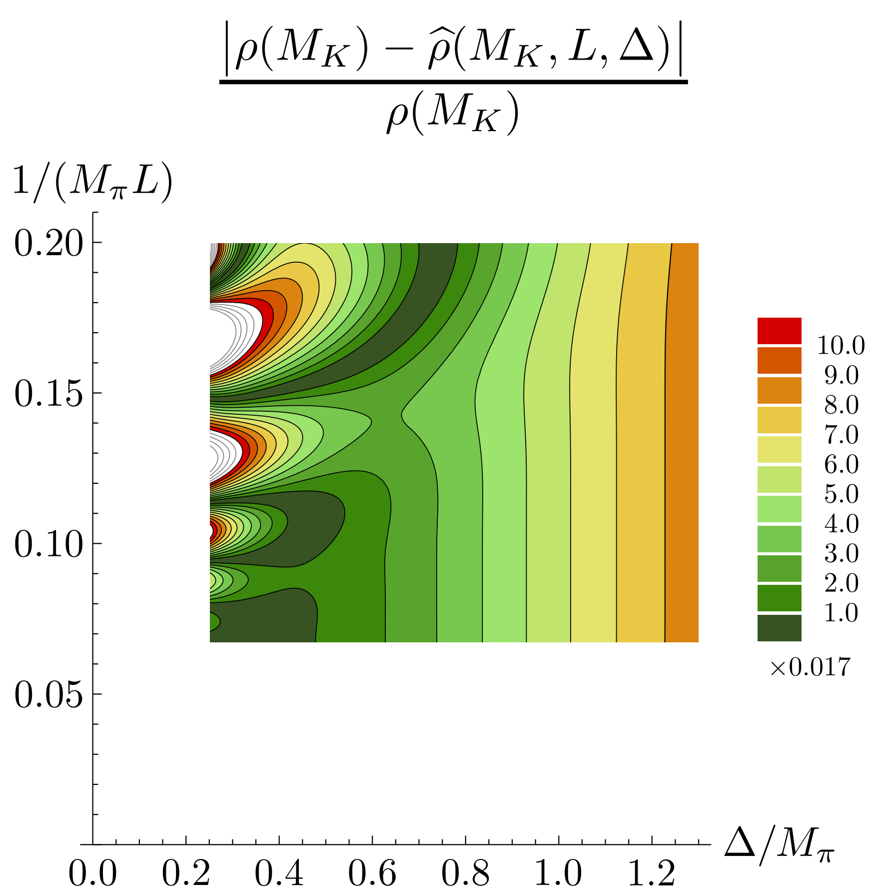
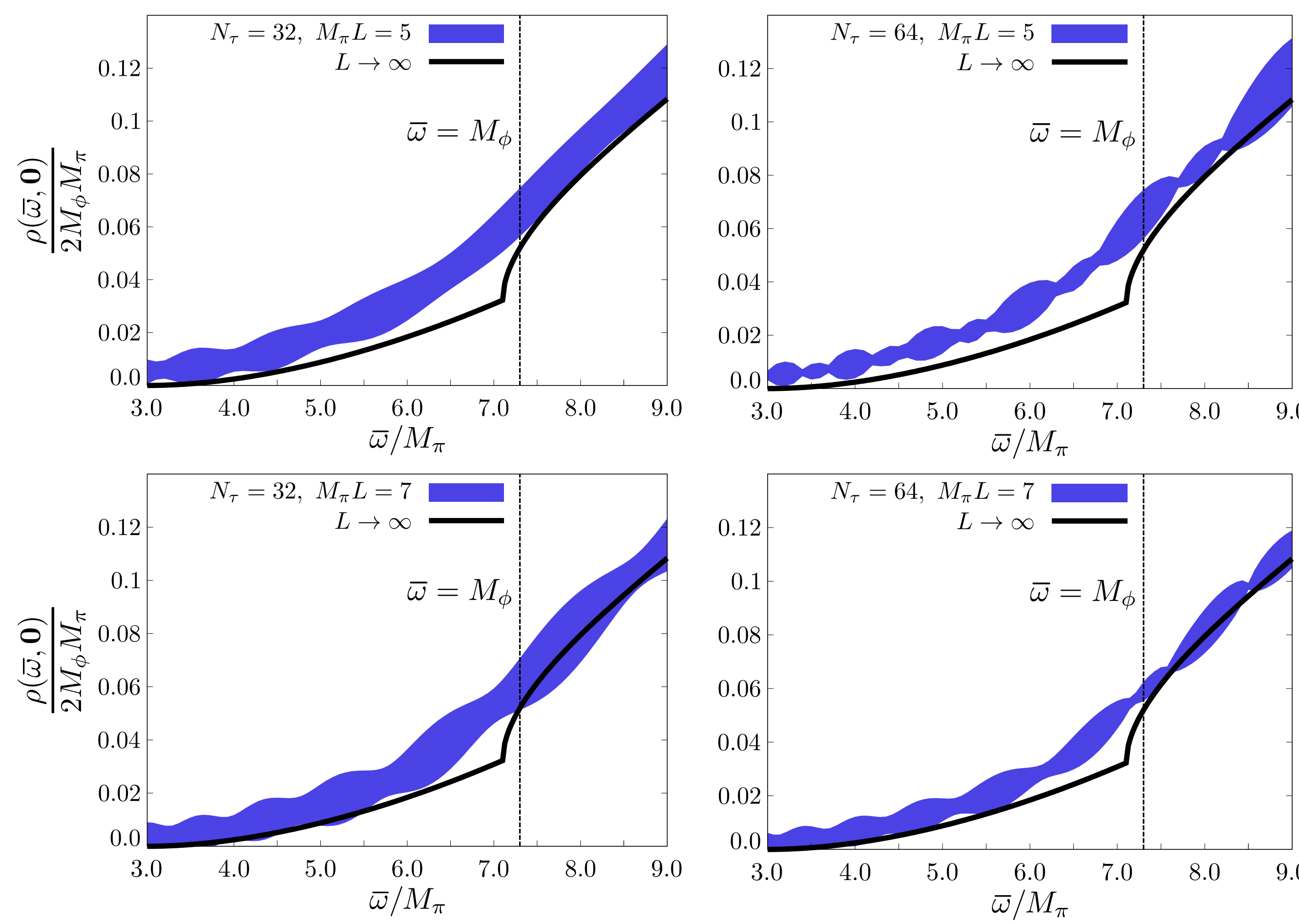
Still lots to do

- [ ] Develop formal strategies for estimating the ordered double limit
- [ ] Test the approach in a numerical lattice calculation

- [ ] Extend the approach to multiple incoming hadrons, other observables
- [ ] Investigate the role of QED
- [ ] Explore combining this with the exclusive ideas (Lüscher and Lellouch-Lüscher)

Big Picture: New idea that needs to be explored





Thanks for listening!