



Multi-hadron observables from lattice QCD

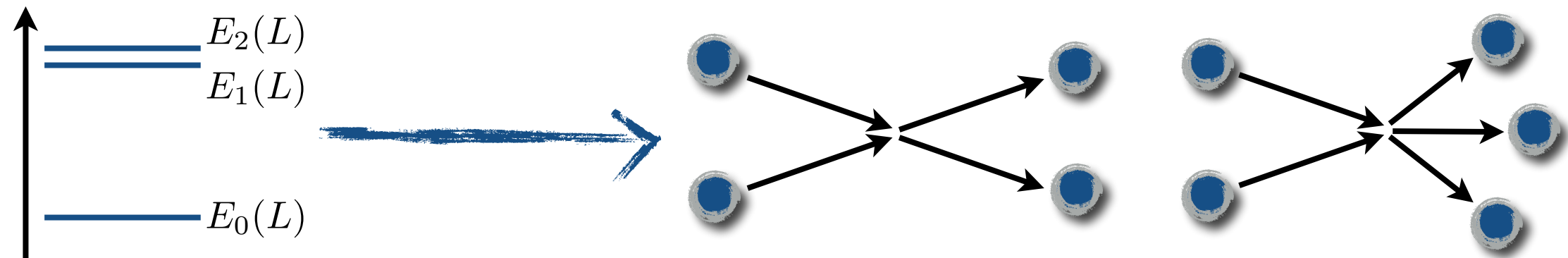
Maxwell T. Hansen

May 25th, 2018

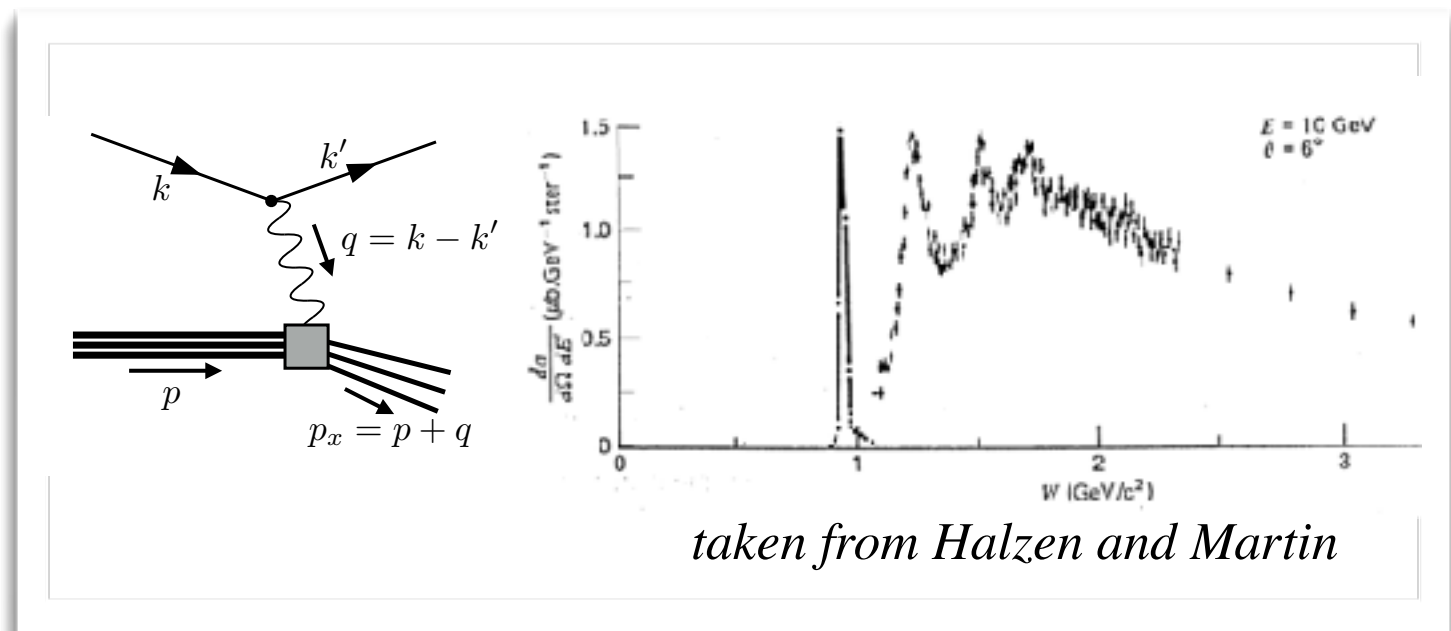
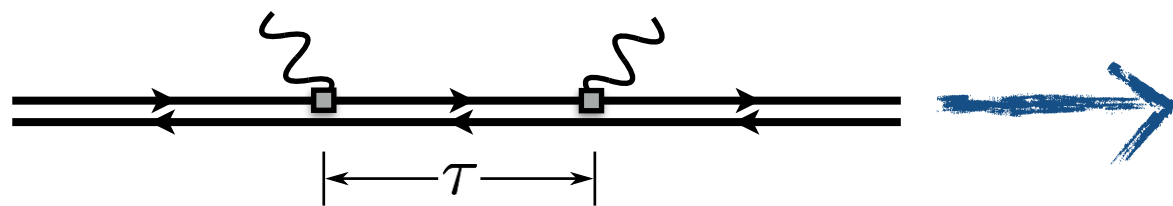


Outline

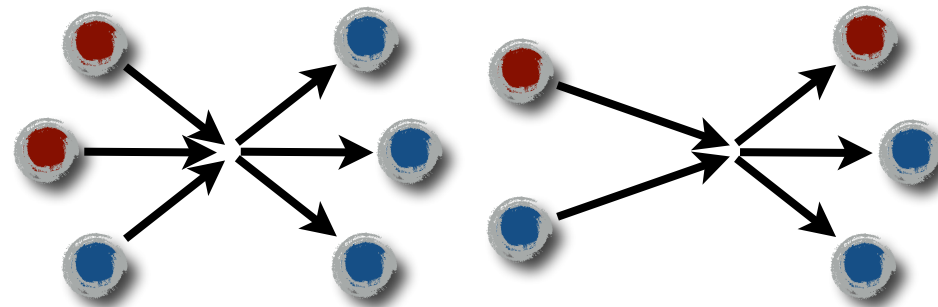
□ Three-particle scattering



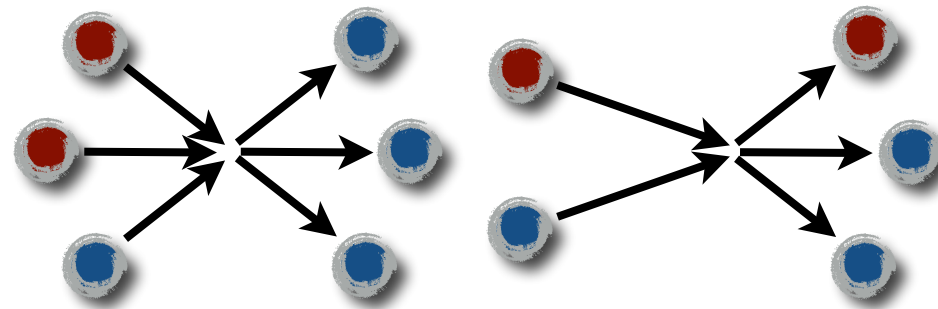
□ n-particle inclusive rates



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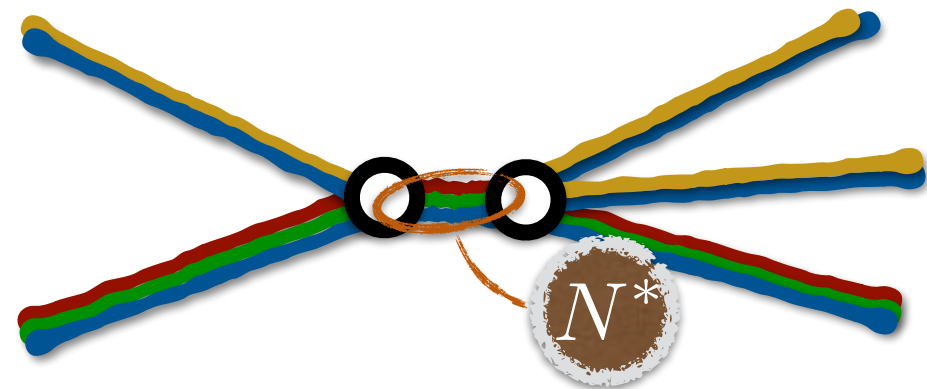


Potential applications...

Studying three-particle resonances

$$\omega(782), a_1(1420) \rightarrow \pi\pi\pi$$

$$N(1440) \rightarrow N\pi, N\pi\pi$$



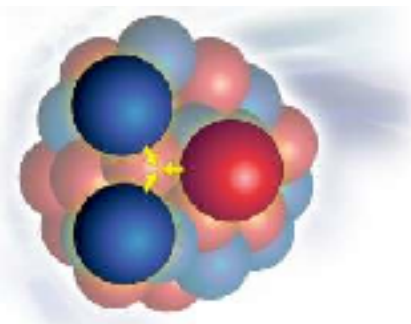
Calculating weak decays, form factors and transitions

$$K \rightarrow \pi\pi\pi$$

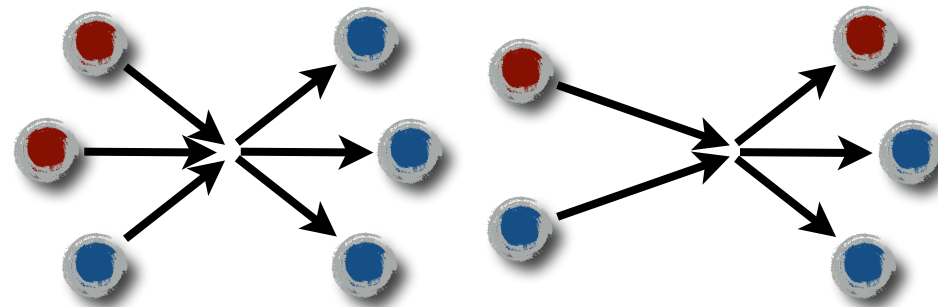
$$N\gamma^* \rightarrow N\pi\pi$$

Determining three-body interactions

NNN three-body forces needed as EFT input for studying larger nuclei and nuclear matter



The aim is to derive a formalism for studying relativistic two- and three-particle systems from lattice QCD

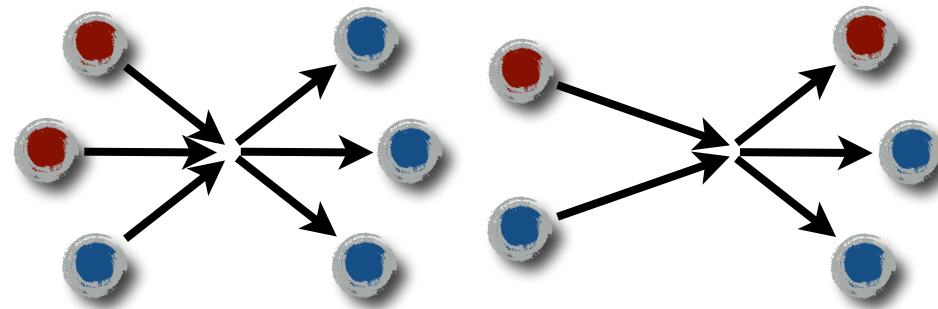


Other motivations...

The Lüscher and Lellouch-Lüscher formalism only applies for

$$\sqrt{s} = E_{\text{cm}} < \text{multi-particle threshold}$$

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$$\sqrt{s} = E_{\text{cm}} < \text{multi-particle threshold}$$

$$\pi K \rightarrow \pi K$$

$$\sqrt{s} = 2m_{\pi} + m_K$$

$$\pi\pi \rightarrow \pi\pi$$

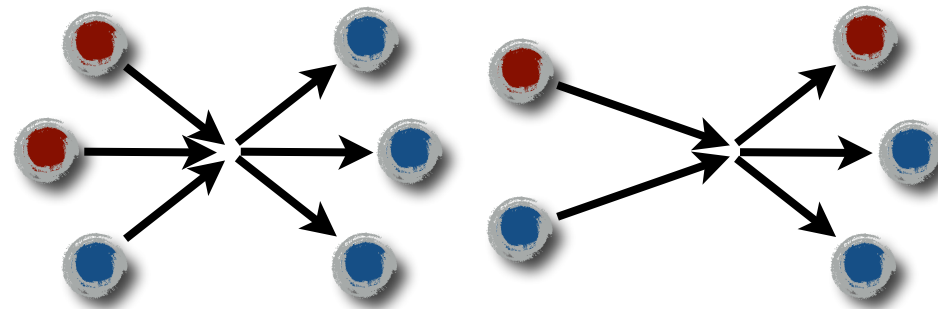
only up to

$$\sqrt{s} = 4m_{\pi}$$

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Other motivations...

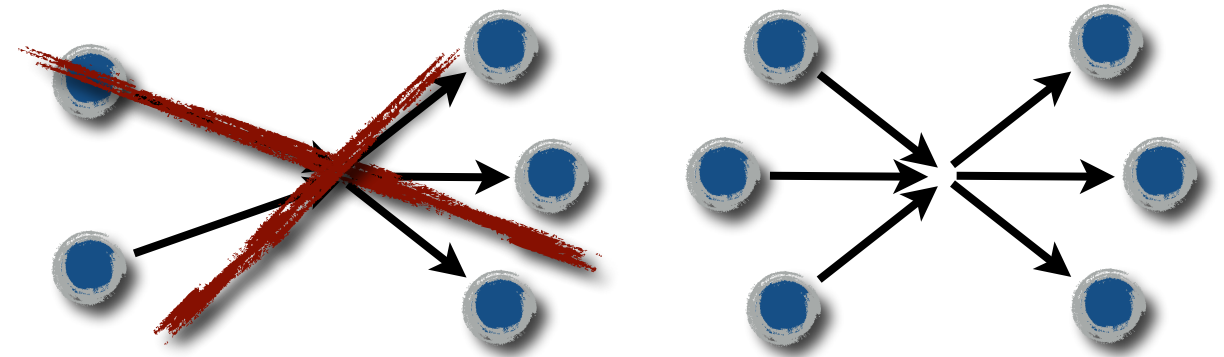
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$\pi K \rightarrow \pi K$		$\sqrt{s} = 2m_{\pi} + m_K$
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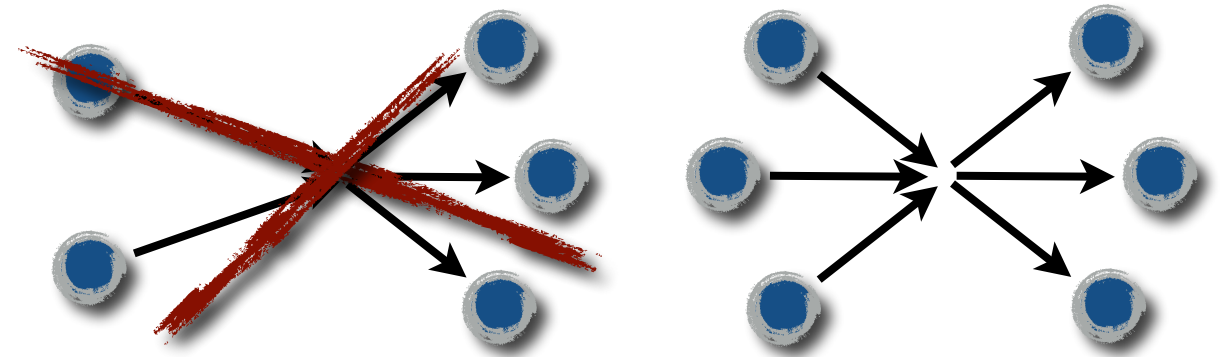
3-particle formalism is a step on the way to n -particle formalism

We focus here on
identical scalar particles



For now we turn off two-to-three scattering using a symmetry

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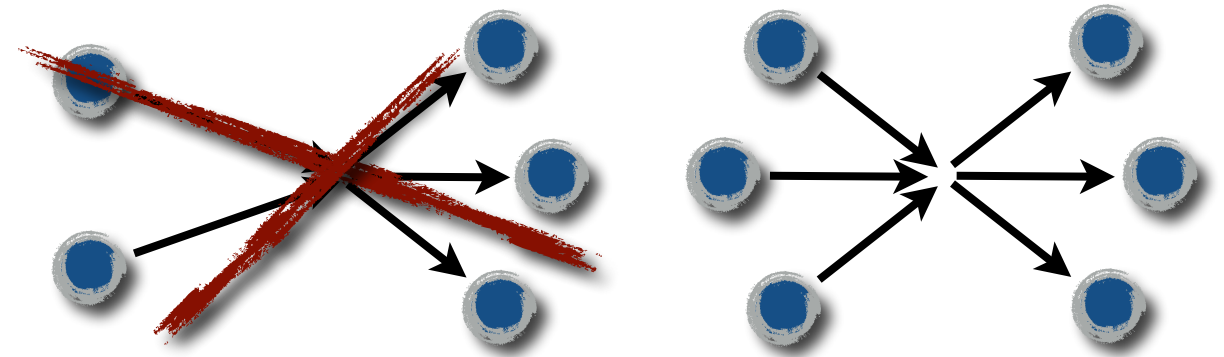


For now we turn off two-to-three scattering using a symmetry

Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\rightarrow 3} \equiv$ fully connected correlator with
six external legs amputated and projected on shell

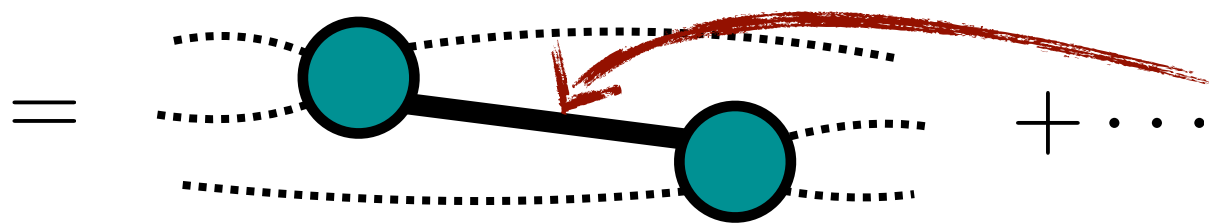
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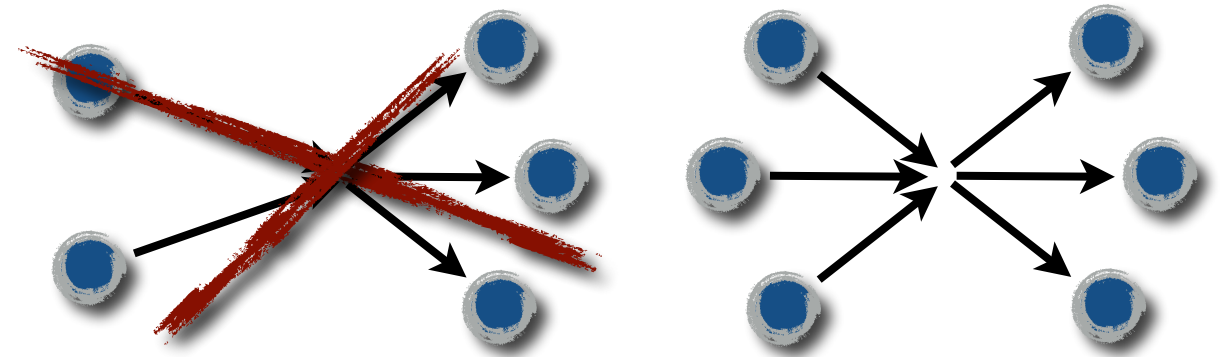
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Certain external momenta
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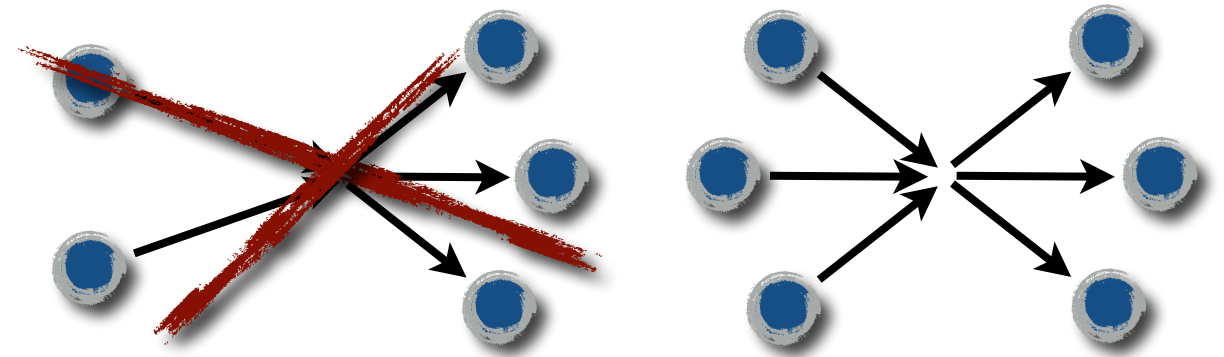
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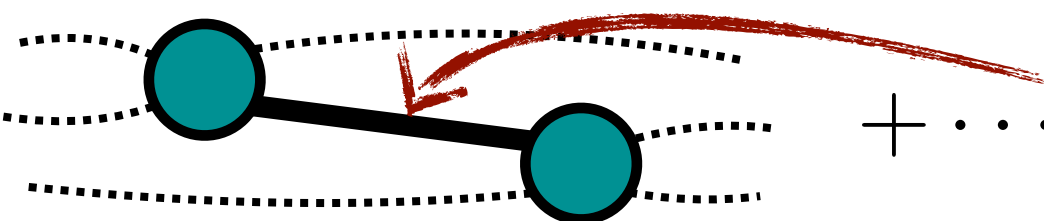
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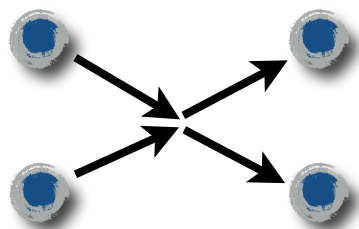
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$=$  $+ \dots$ **Certain external momenta
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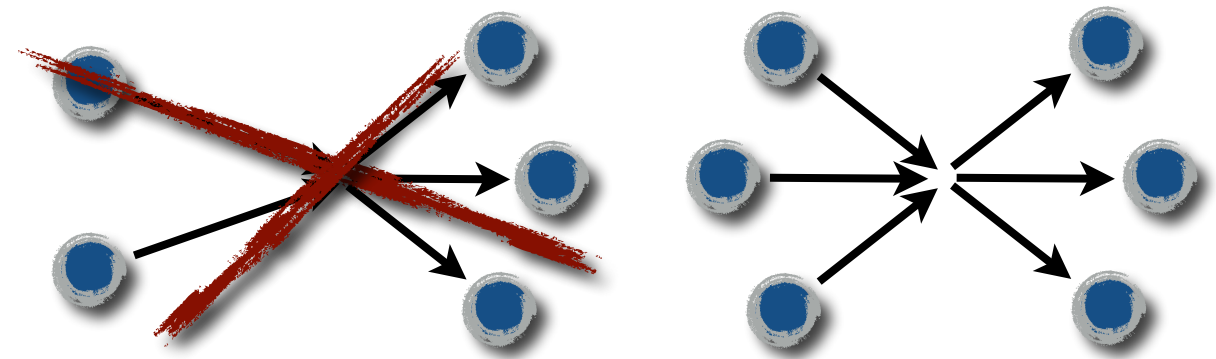


12 momentum
components

-10 Poincaré generators

2 degrees of freedom

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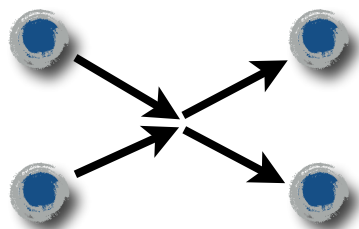
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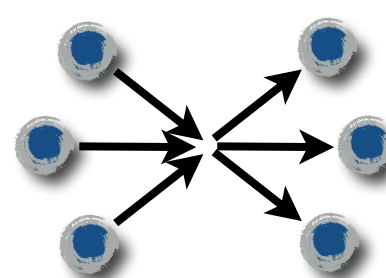
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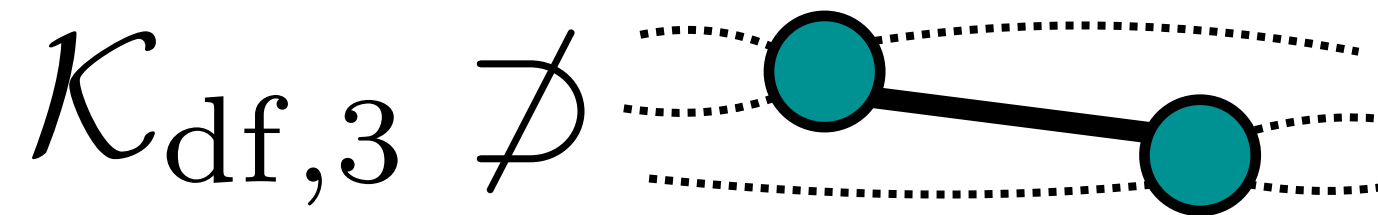


18 momentum
components
-10 Poincaré generators

8 degrees of freedom

How can we extract a singular, eight-coordinate function using finite-volume energies?

Spectrum depends on a modified quantity with singularities removed



PV pole prescription

df stands for “divergence free”

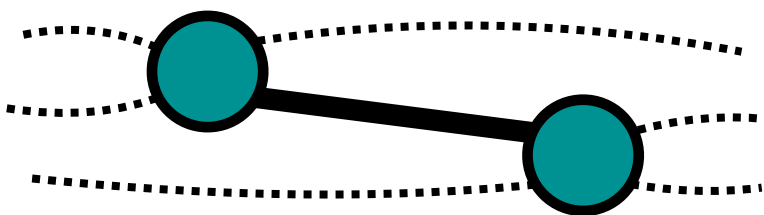
Same degrees of freedom as $\mathcal{M}_3$

Smooth, real function (easier to extract)

Relation to $\mathcal{M}_3$ is known (depends only on on-shell $\mathcal{M}_2$)

How can we extract a **singular, eight-coordinate function** using finite-volume energies?

Spectrum depends on a modified quantity with singularities removed

$$\mathcal{K}_{\text{df},3} \not\equiv$$


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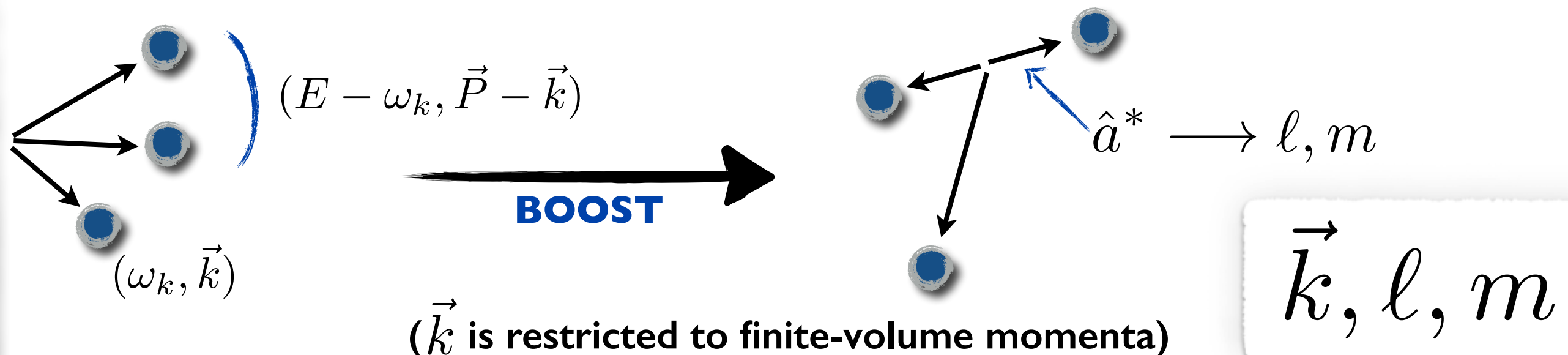
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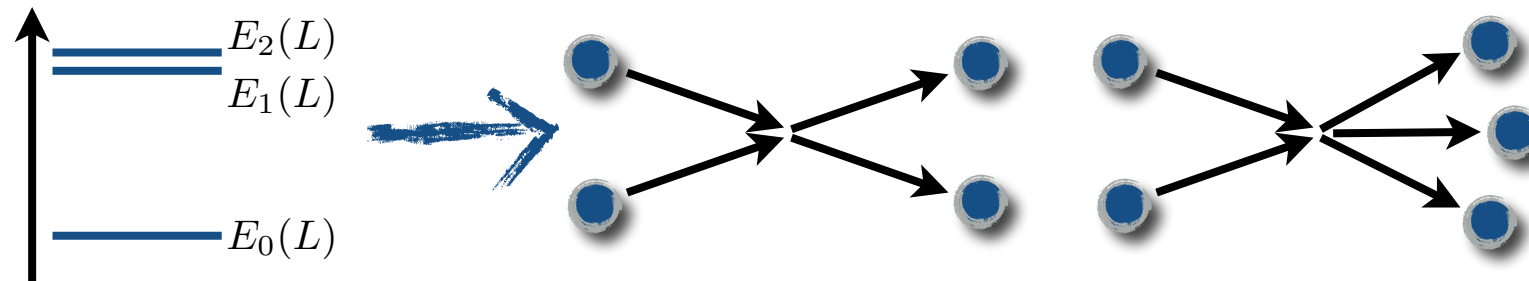
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Degrees of freedom encoded in an extended matrix space



Finite-volume correlators



$$C_L(P) \equiv \int_L d^4x \, e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Total 4-momentum

$$P = (E, \mathbf{P}) = (E, 2\pi \mathbf{n}_P / L)$$

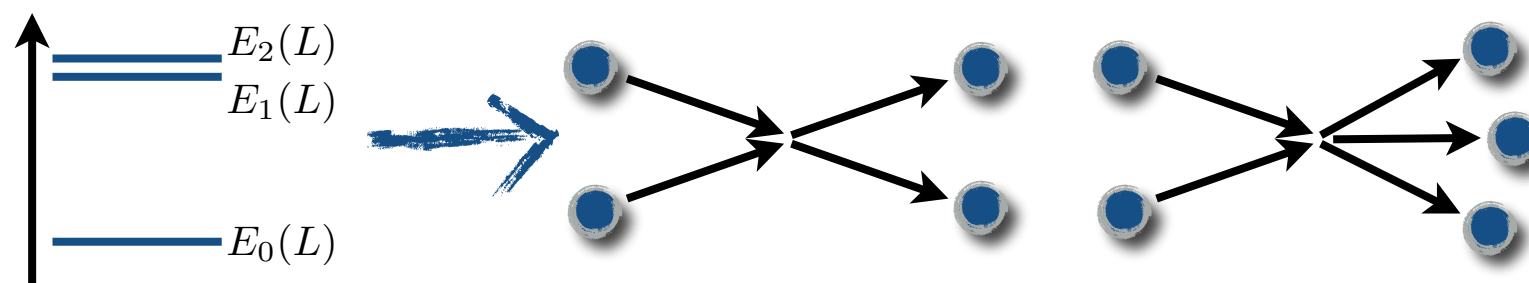
c.m. frame energy: $E^{*2} = E^2 - \mathbf{P}^2$

n-particle interpolator

e.g. $\pi(\mathbf{p})\pi(-\mathbf{p})$ **or** $\bar{q}\Gamma q$

(only quantum numbers relevant)

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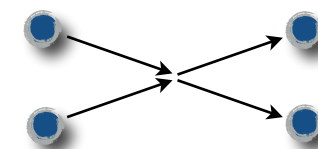
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Focus on a window of energies to isolate particular on-shell states

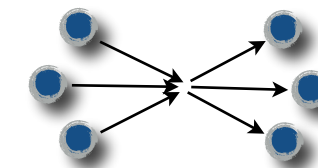
Two particles: $\boxed{\mathbb{Z}_2 \checkmark}$ $0 < E^* < 4m$

$\boxed{\cancel{\mathbb{Z}_2}}$ $m < E^* < 3m$



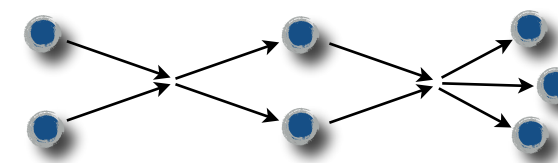
Three particles

$\boxed{\mathbb{Z}_2 \checkmark}$ $m < E^* < 5m$



Two and three particles

$\boxed{\cancel{\mathbb{Z}_2}}$ $m < E^* < 4m$

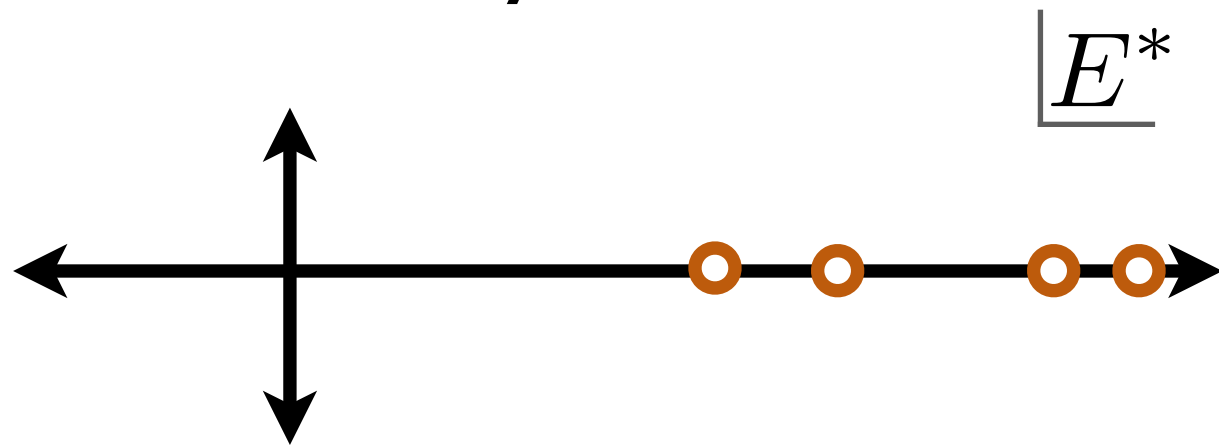


Of poles and branch cuts

$$C_L(P) \equiv \int_L d^4x \, e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

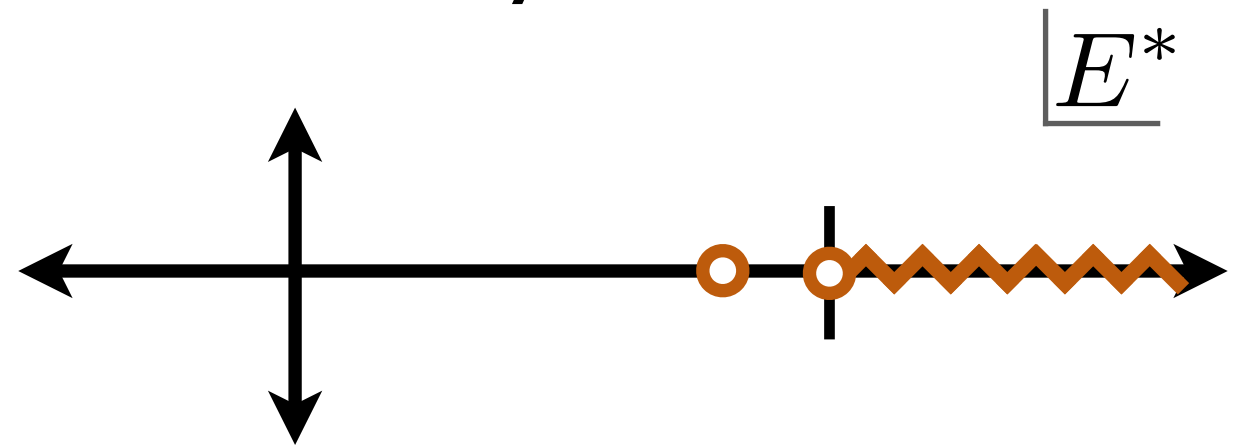
At fixed L , $\mathbf{P}$ poles in C_L give **the finite-volume spectrum**

C_L analytic structure



- ☑ Real function (unitarity hidden)
- ☑ No cuts → only one sheet
- ☑ No resonance poles

C_∞ analytic structure



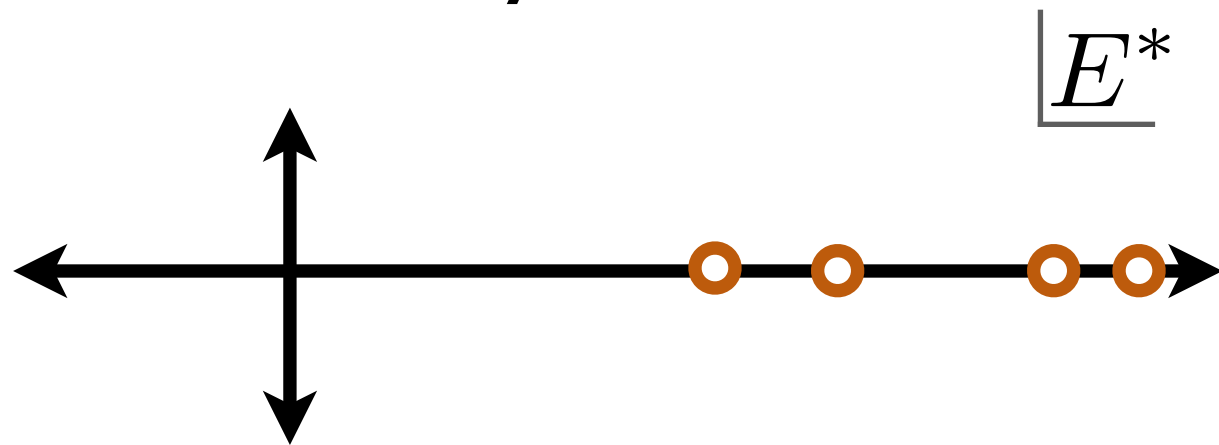
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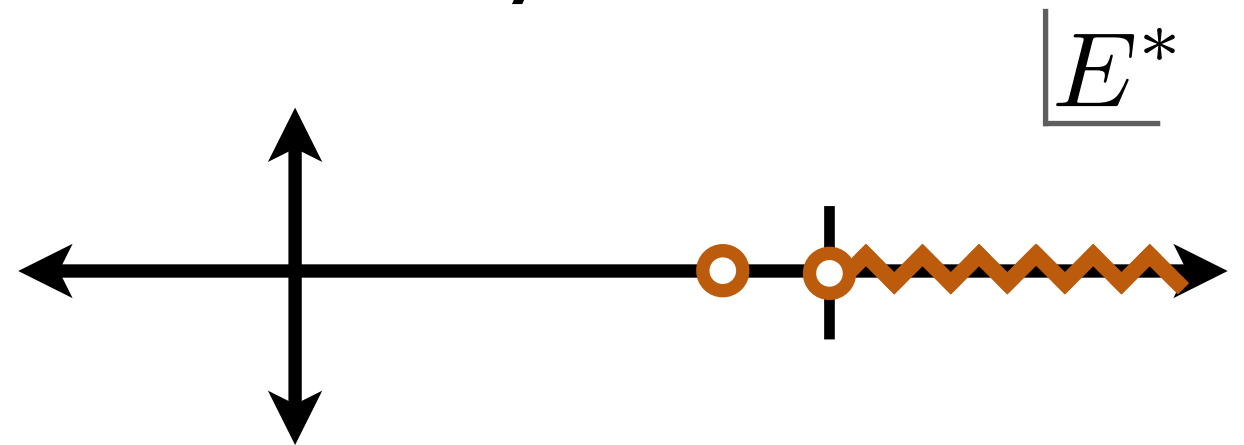
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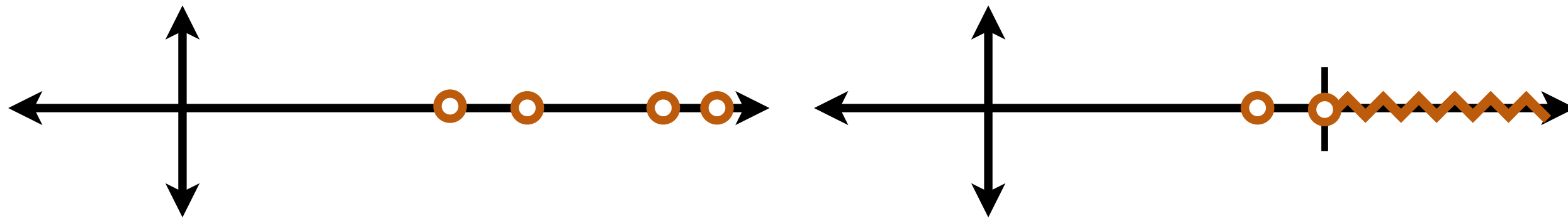


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Want to relate $C_L \longleftrightarrow C_\infty$ ($\mathcal{M}_{n \rightarrow m}$ is just a specific choice of C_∞)

The idea is to “reach in” and correct the singularity structure

From poles to cuts: Toy example

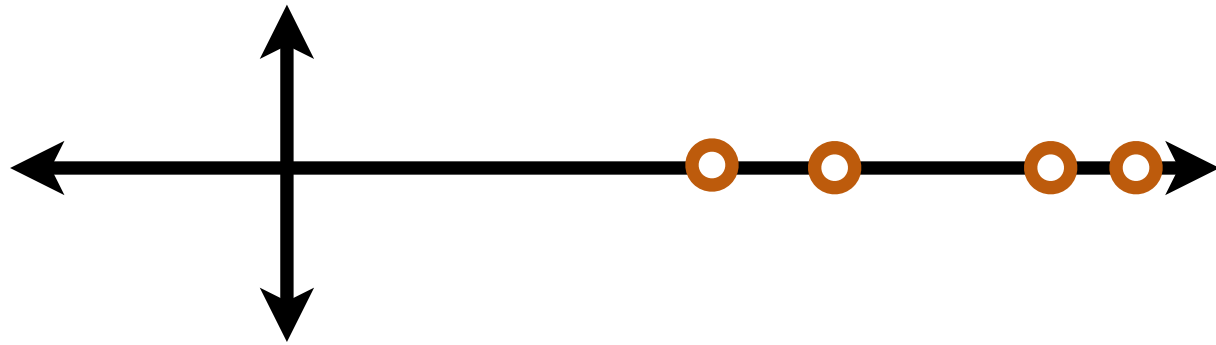


$$f(x, y, \Delta n) \equiv \sum_{n=-10, -10+\Delta n, \dots}^{10} \Delta n \operatorname{Im} \frac{1}{n - x - iy}$$

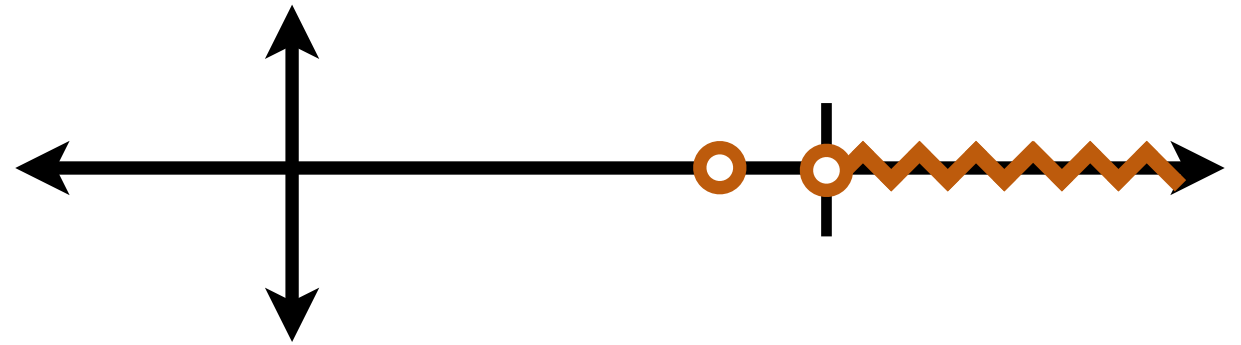
}

$$f_{\infty}(x, y) \equiv \operatorname{Im} \log \left[\frac{10 - x - iy}{-10 - x - iy} \right]$$

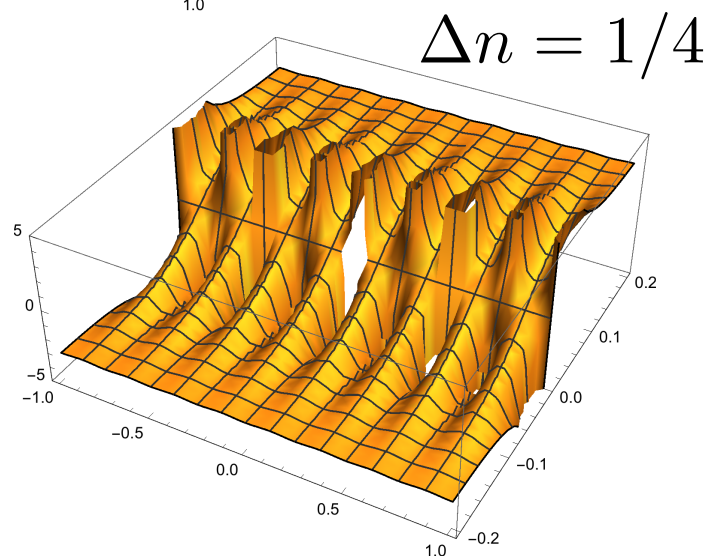
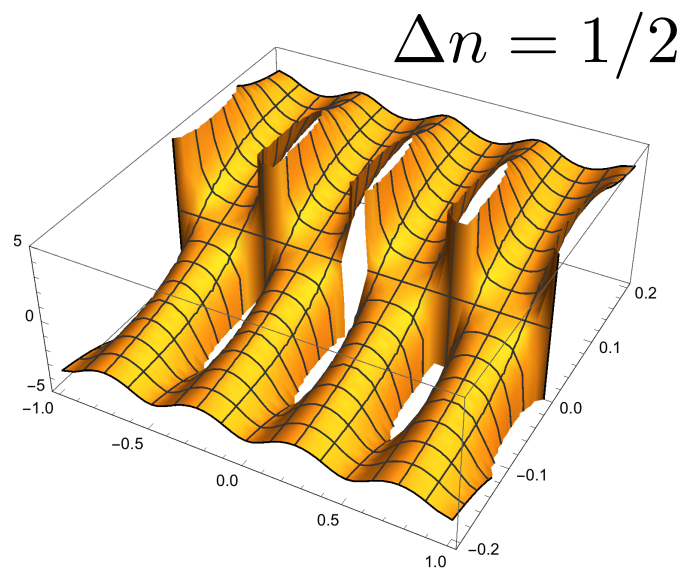
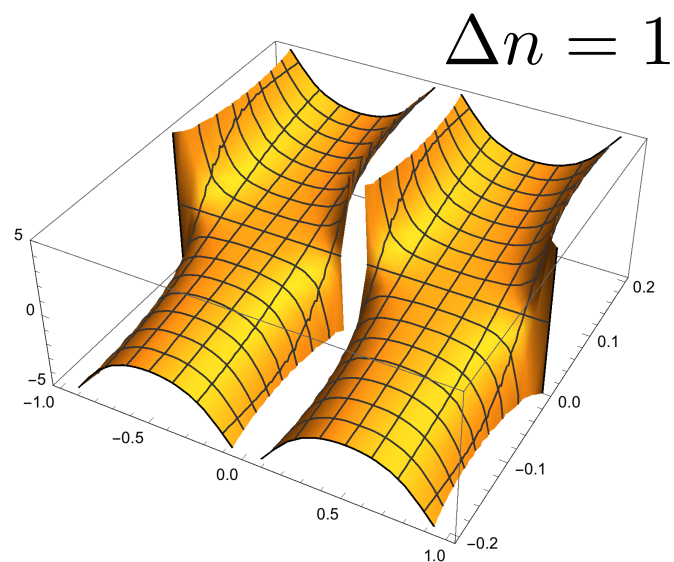
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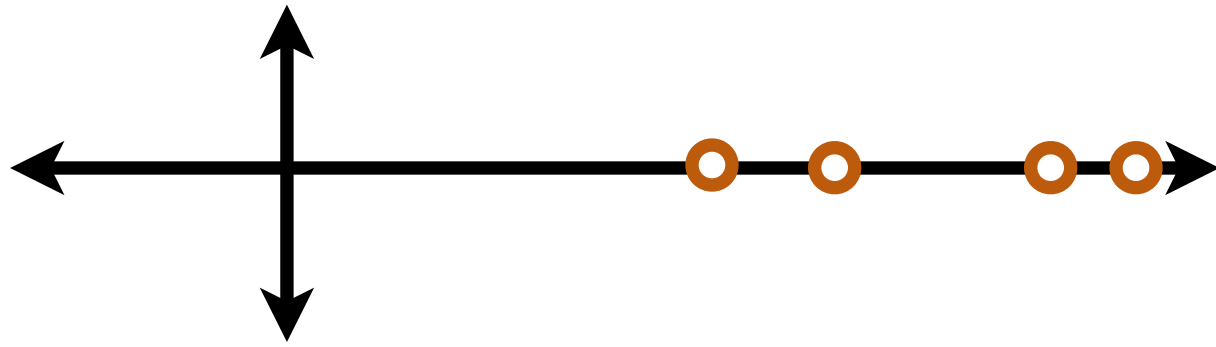
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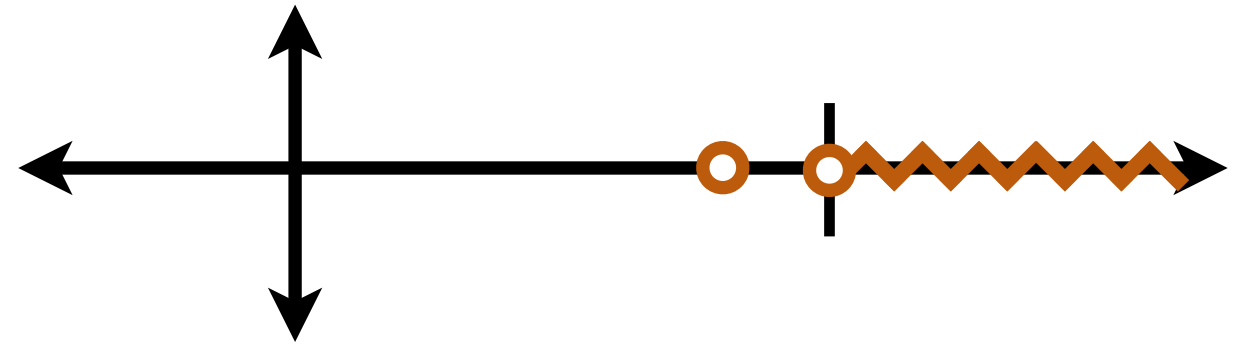
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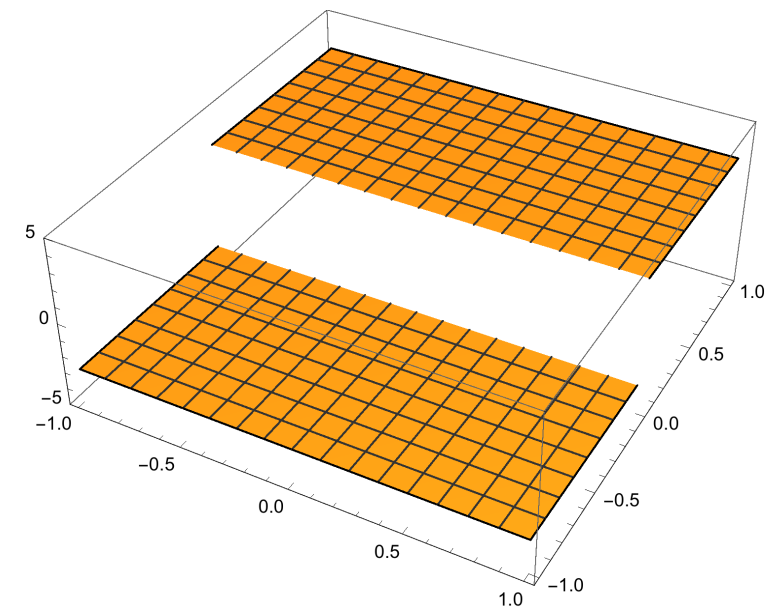
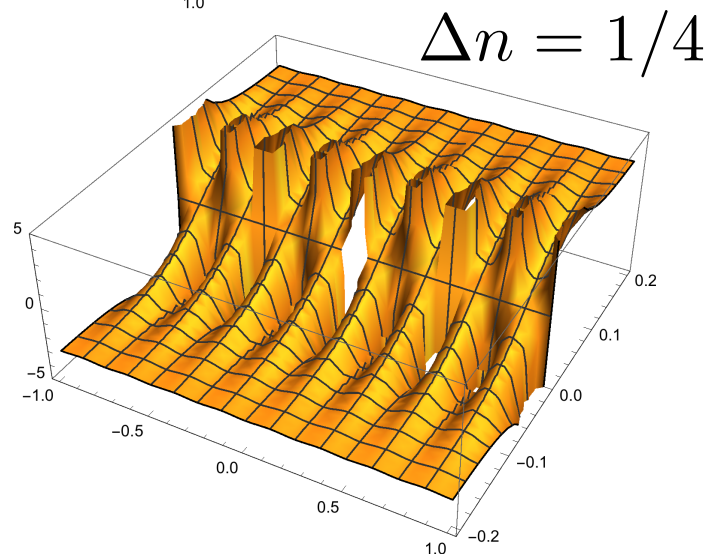
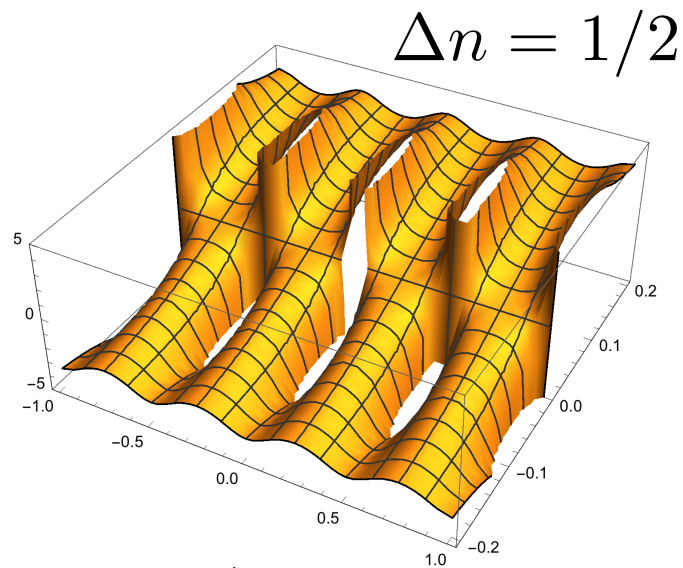
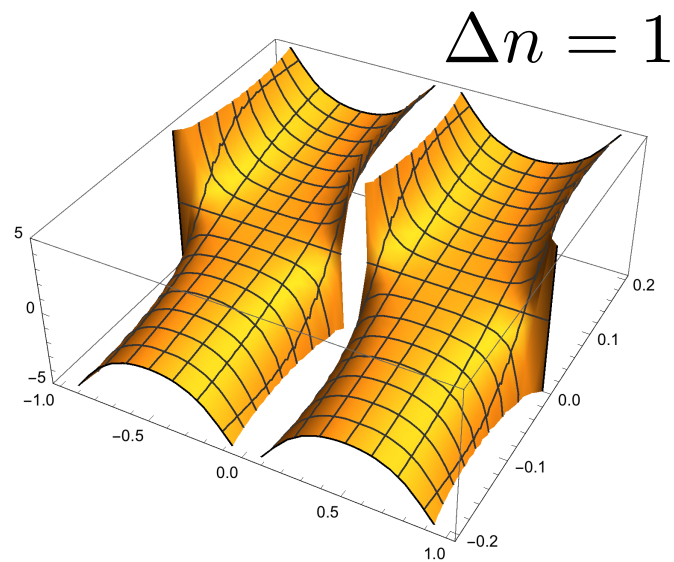
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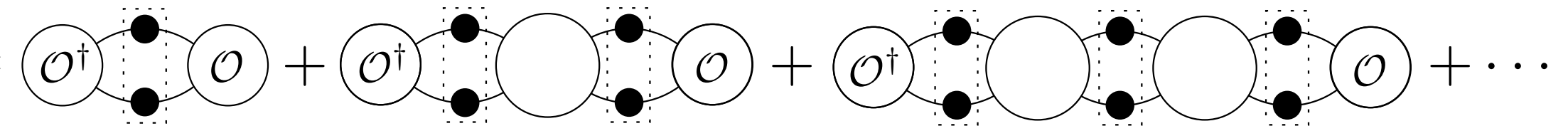
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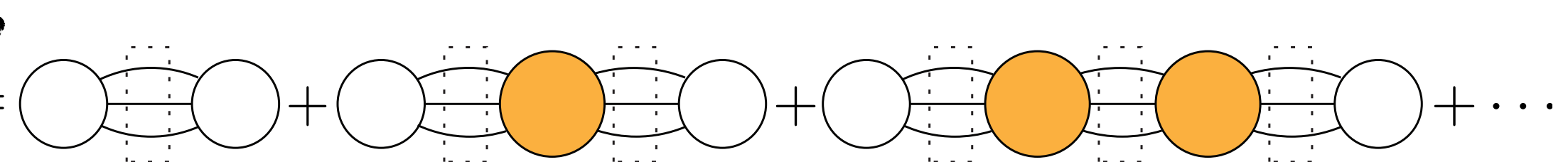
**Only in the infinite-volume limit
can we generate the singularity
that leads to a second sheet**

New skeleton expansion

Recall for two particles we started with a “skeleton expansion”

$$C_L(P) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$
The diagrammatic expansion for two particles shows a sequence of terms. The first term consists of two circles, the left one labeled $\mathcal{O}^\dagger$ and the right one labeled $\mathcal{O}$, connected by two horizontal lines. Each line has a black dot at its midpoint. A dashed rectangle encloses the two dots on the left line. The second term is identical but includes a third, empty circle between the two original ones. The third term includes two empty circles between the original ones. The pattern continues with ellipses.

So now we need the same for three...

$$C_L(E, \vec{P}) \stackrel{?}{=} \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$
The diagrammatic expansion for three particles shows a sequence of terms. The first term consists of three empty circles connected by two horizontal lines. A dashed rectangle encloses the first two circles. The second term is identical but includes a third, orange-filled circle between the first two. The third term includes two orange-filled circles between the first two. The pattern continues with ellipses.

New skeleton expansion

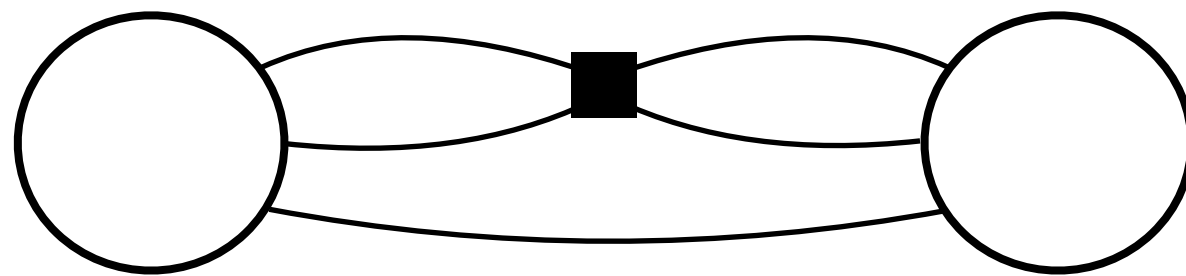
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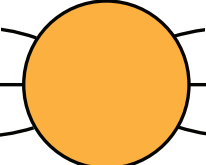
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No!... We must also accommodate diagrams like



Disconnected diagrams in  lead to singularities that invalidate the derivation

Back to three: new skeleton expansion

$$C_L(E, \vec{P}) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

$$+ \text{diagram 4} + \text{diagram 5} + \text{diagram 6} + \dots$$

The diagrams represent terms in a skeleton expansion. The first row shows terms with orange nodes, and the second row shows terms with purple nodes. Each term consists of a sequence of nodes connected by lines, with dashed boxes indicating substructures.

Kernel definitions:

$$\text{purple node} \equiv \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

The purple node is defined as a sum of diagrams. The first diagram is a single vertex. The second diagram is a vertex connected to two vertices by two lines. The third diagram is a vertex connected to two vertices by two lines, with an additional loop on one of the internal vertices.

$$\text{orange node} \equiv \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

The orange node is defined as a sum of diagrams. The first diagram is a single vertex. The second diagram is a vertex connected to two vertices by a single line. The third diagram is a vertex connected to two vertices by two lines, with an additional loop on one of the internal vertices.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots
 \end{aligned}$$

The diagrams represent terms in a skeleton expansion. Each term consists of a sequence of vertices (white circles) connected by lines. Some vertices are replaced by orange or purple circles, which represent kernels. Dashed boxes indicate the skeleton structure.

Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The purple circle kernel is defined as the sum of diagrams showing a vertex with two external lines, a loop, and a tadpole.

$$\text{Orange circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The orange circle kernel is defined as the sum of diagrams showing a vertex with two external lines, a line segment, and a loop.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
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 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams represent terms in the skeleton expansion of the longitudinal correlation function $C_L(E, \vec{P})$. They are organized into rows, with each row starting with a plus sign. The diagrams consist of white circles (external lines) connected by horizontal lines (propagators). Some internal lines are highlighted with dashed boxes, indicating summations over finite-volume momenta. The colors of the internal lines (purple and orange) correspond to the kernel definitions provided below.

Kernel definitions:

$$\text{Purple Kernel} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The purple kernel is defined as a sum of diagrams. The first diagram is a purple circle with four external lines. The second diagram is a white circle with four external lines and two internal lines forming a loop. The third diagram is a white circle with four external lines and two internal lines forming a loop, with a dashed box around the loop. The fourth diagram is a white circle with four external lines and two internal lines forming a loop, with a dashed box around the loop and a dashed box around the entire diagram.

$$\text{Orange Kernel} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The orange kernel is defined as a sum of diagrams. The first diagram is an orange circle with four external lines. The second diagram is a white circle with four external lines and two internal lines forming a loop. The third diagram is a white circle with four external lines and two internal lines forming a loop, with a dashed box around the loop. The fourth diagram is a white circle with four external lines and two internal lines forming a loop, with a dashed box around the loop and a dashed box around the entire diagram.

- All lines are fully dressed propagators
- Boxes represent sums over finite-volume momenta
- Kernels may contain fixed poles

Basic approach

1. Work out the three particle skeleton expansion

$$C_L(E, \vec{P}) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

The expansion is represented by three rows of diagrams, each enclosed in a dashed box and followed by an ellipsis. The first row contains three diagrams: the first has two white circles connected by two horizontal lines; the second has a white circle, a dashed box containing an orange circle, and another white circle; the third has two orange circles in a dashed box between two white circles. The second row contains three diagrams: each has a white circle, a dashed box containing one, two, or three purple circles, and another white circle. The third row contains three diagrams: each has a white circle, a dashed box containing two, three, or four purple circles, and another white circle.

2. Break diagrams into finite- and infinite-volume parts

3. Organize and sum terms to identify
infinite-volume observables

Basic approach

1. Work out the three particle skeleton expansion

$$C_L(E, \vec{P}) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

The diagram shows the skeleton expansion of the three-particle correlation function $C_L(E, \vec{P})$. It consists of three rows of diagrams, each representing a different type of skeleton. The first row shows diagrams with orange circles (representing two-particle irreducible diagrams) connected by lines. The second row shows diagrams with purple circles (representing three-particle irreducible diagrams) connected by lines. The third row shows diagrams with purple circles connected by lines, representing higher-order irreducible diagrams. Each diagram is enclosed in a dashed box, and the expansion is shown as a sum of these diagrams.

2. Break diagrams into finite- and infinite-volume parts

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infinite-volume observables

Result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) - A' F_3 \frac{1}{1 + \mathcal{K}_{\text{df},3} F_3} A$$

- 🔍 Looks similar to the two-particle case
- 🔍 All quantities defined with PV-pole prescription
- 🔍 F_3 depends on finite-volume and two-to-two scattering

Quantization condition

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

$$\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$$

$F_3 \equiv$ matrix that depends on geometric functions and $\mathcal{M}_{2 \rightarrow 2}$.

MTH and Sharpe (2014)

Quantization condition

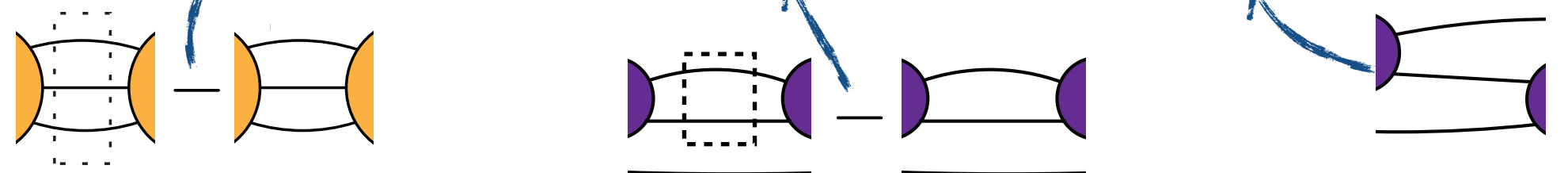
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MTH and Sharpe (2014)

All of the complication is buried inside F_3

$$F_3 = \frac{F}{6\omega L^3} - \frac{F}{2\omega L^3} \frac{1}{1 + \mathcal{M}_{2,L} G} \mathcal{M}_{2,L} F$$


These are all matrices with indices

momentum of
one particle

$$\vec{k} = \frac{2\pi\vec{n}}{L}$$

angular momentum
of the other two



ℓ, m

F and **G** are geometric functions

$\mathcal{M}_{2,L}$ depends on **F** and $\mathcal{M}_2$

Determining $\mathcal{K}_{\text{df},3}$

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MTH and Sharpe (2014)

(1). Use two-particle q.c. to constrain $\mathcal{M}_2$ and determine $F_3(E, \vec{P}, L)$.

$$\det[\mathcal{M}_2^{-1} + F_2] = 0 \longrightarrow \mathcal{M}_2 \longrightarrow F_3(E, \vec{P}, L)$$

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$$\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \mathcal{K}_{\text{df},3}[\alpha_1, \dots, \alpha_N] \longleftarrow \text{Recall, this is a real, smooth function}$$

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(3). Use three-particle q.c. with finite-volume energies to determine $\mathcal{K}_{\text{df},3}(E^*)$.

$$\det[\mathcal{K}_{\text{df},3}^{-1} + F_3] = 0 \longrightarrow \mathcal{K}_{\text{df},3}(E^*) \checkmark$$

Relating $\mathcal{K}_{\text{df},3}$ to $\mathcal{M}_3$

First we relate $\mathcal{K}_{\text{df},3}$ to a special choice of finite-volume correlator

$$i\mathcal{M}_{L,3\rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{l} \text{Orange circle} + \text{Orange circle} \text{---} \text{Orange circle} + \dots \\ + \text{Purple circle} \text{---} \text{Purple circle} + \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} + \dots \\ + \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} + \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} + \dots \\ + \dots \\ + \text{Purple circle} \text{---} \text{Orange circle} + \text{Orange circle} \text{---} \text{Purple circle} + \dots \end{array} \right\}$$

$$\mathcal{M}_{L,3} = \mathcal{S} \left[\mathcal{D}_L + \mathcal{L}_L \frac{1}{\mathcal{K}_{\text{df},3}^{-1} + F_3} \mathcal{R}_L \right]$$

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With this **analytic relation** in hand we can...

(a) Set $E \rightarrow E + i\epsilon$, (b) Send $L \rightarrow \infty$, (c) Send $\epsilon \rightarrow 0^+$.

Leads to an integral equation for the scattering amplitude

$$\mathcal{M}_3(E^*) = \mathcal{I}[\mathcal{K}_{\text{df},3}(E^*), \mathcal{M}_2, \mathcal{M}_3(E^*)]$$

Current status

Model- & EFT-independent relation between
finite-volume energies and relativistic two-and-three particle scattering

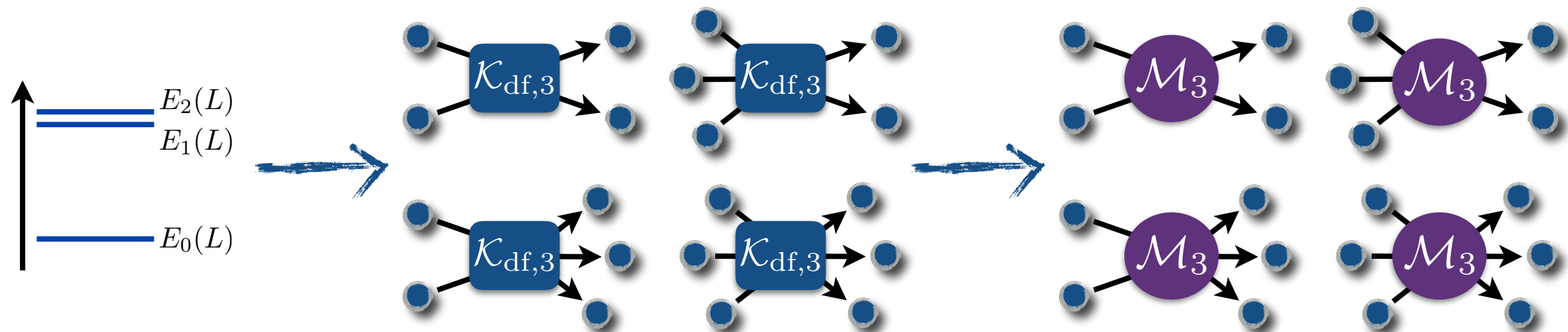
- ☑ Requires energy is below four- or five-particle production threshold
- ☑ Derivation ignores (drops) suppressed volume effects ($e^{-M_\pi L}$)
- ☑ Only useful if one truncates angular momentum space

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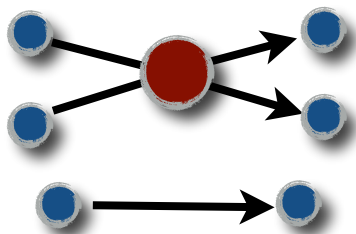
Model- & EFT-independent relation between
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- ✓ Requires energy is below four- or five-particle production threshold
- ✓ Derivation ignores (drops) suppressed volume effects ($e^{-M_\pi L}$)
- ✓ Only useful if one truncates angular momentum space

- ✓ Result relies on an intermediate three-particle quantity



- Assumes no sub-channel two-particle resonances



Extension to remove this restriction is underway

- Derivation assumes identical scalar particles

Usability?

☑ How do we make the two-particle formalism usable?

Truncate partial waves

$$\mathcal{M}_2(E_2^*, \theta^*) \approx \sum_{\ell=0}^N P_\ell(\cos \theta^*) \mathcal{M}_{2,\ell}(E_2^*) \longrightarrow$$

Single partial wave

$$\mathcal{M}_2(E_2^*, \theta^*) \approx \mathcal{M}_{2,s}(E_2^*) \propto \frac{1}{p^* \cot \delta_0(p^*) - ip^*}$$

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☑ Is there a three-particle analog?

$$\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \sum_{n=0}^N \mathcal{P}_n(\Omega'_3, \Omega_3) \mathcal{K}_{\text{df},3,n}(E^*) \longrightarrow$$

$$\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \mathcal{K}_{\text{df},3}^{\text{iso}}(E^*) \in \mathbb{R}$$

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At fixed energy $\frac{\mathcal{M}_2(E_2^*, \theta^*)}{\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3)}$ is a smooth function on a compact space.

Further investigation is needed to understand suppression of higher $\mathcal{K}_{\text{df},3,n}(E^*)$

Numerics (keeping only s-wave and $\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \mathcal{K}_{\text{df},3}^{\text{iso}}(E^*)$)

$$1/\mathcal{K}_{\text{df},3}^{\text{iso}}(E^*) = -F_3^{\text{iso}}[E, \vec{P}, L, \mathcal{M}_2^s] \quad \mathcal{M}_3(E^*, \Omega'_3, \Omega_3) = \mathcal{S} \left[\mathcal{D} + \mathcal{L} \frac{1}{1/\mathcal{K}_{\text{df},3}^{\text{iso}} + F_{3,\infty}^{\text{iso}}} \mathcal{R} \right]$$

For the numerical approach we restrict attention to... $p^* \cot \delta_0(p^*) = -\frac{1}{a}$, $\vec{P} = 0$

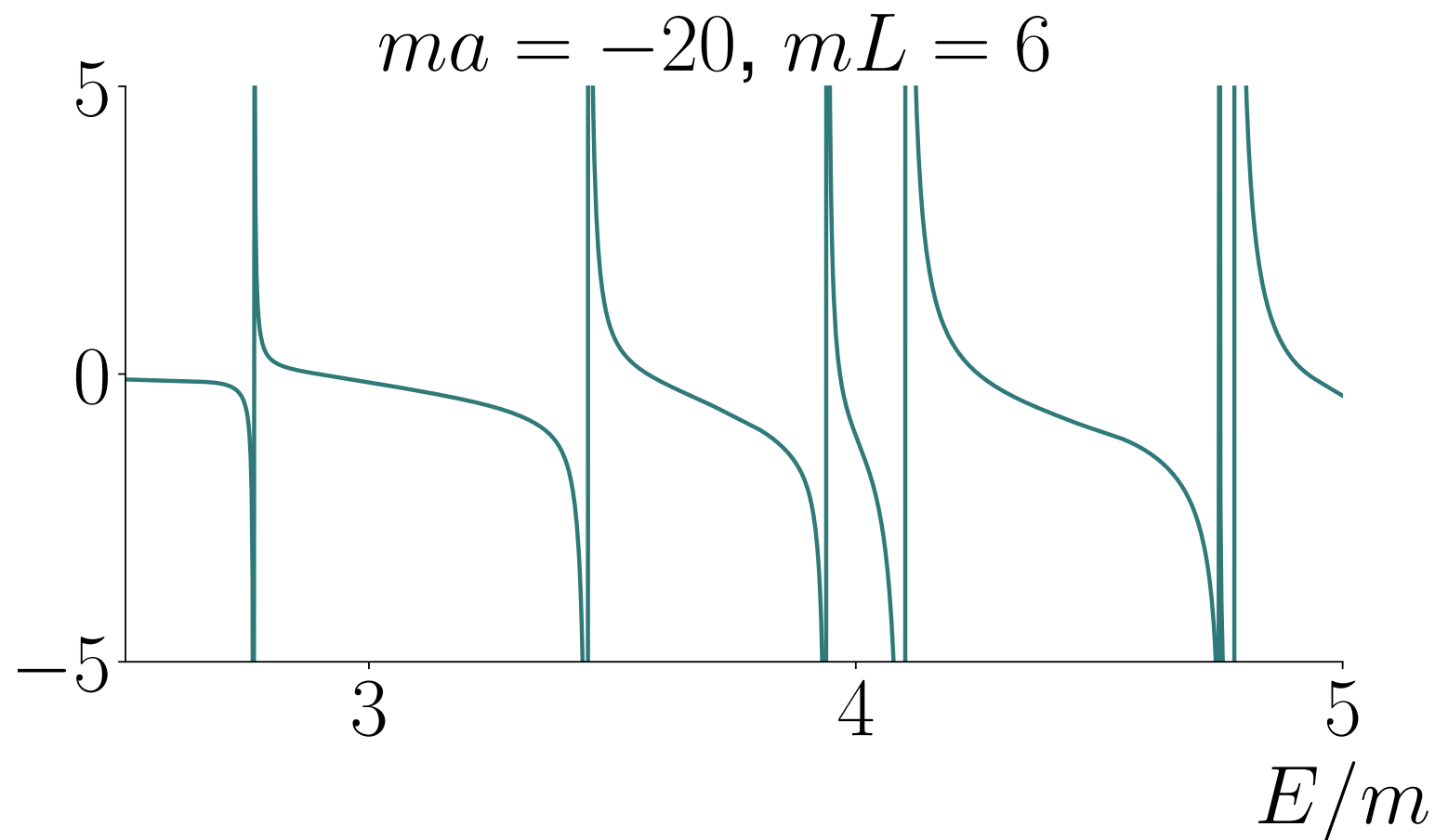
Then the quantization condition is based on $F_3^{\text{iso}}(E, L, a)$

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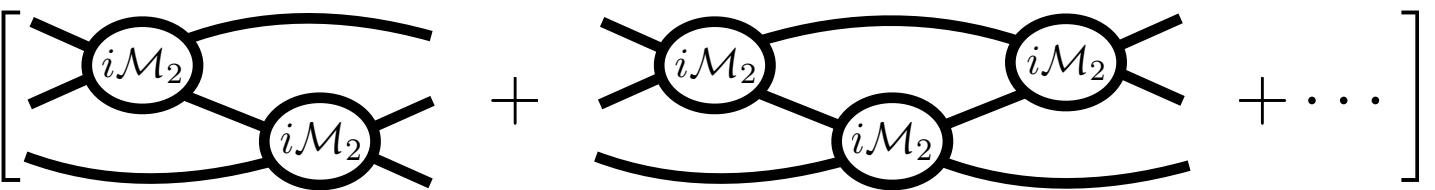
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Finite-volume energies wherever these curves intersect $-1/\mathcal{K}_{\text{df},3}^{\text{iso}}(E)$

$\mathcal{K}_{\text{df},3}^{\text{iso}}(E) = 0$ solutions

- Provides a useful benchmark: Deviations measure three-particle physics
- Meaning for three-to-three scattering is clear

$$i\mathcal{M}_3 = \mathcal{S} \left[\text{Diagram 1} + \text{Diagram 2} + \dots \right]$$


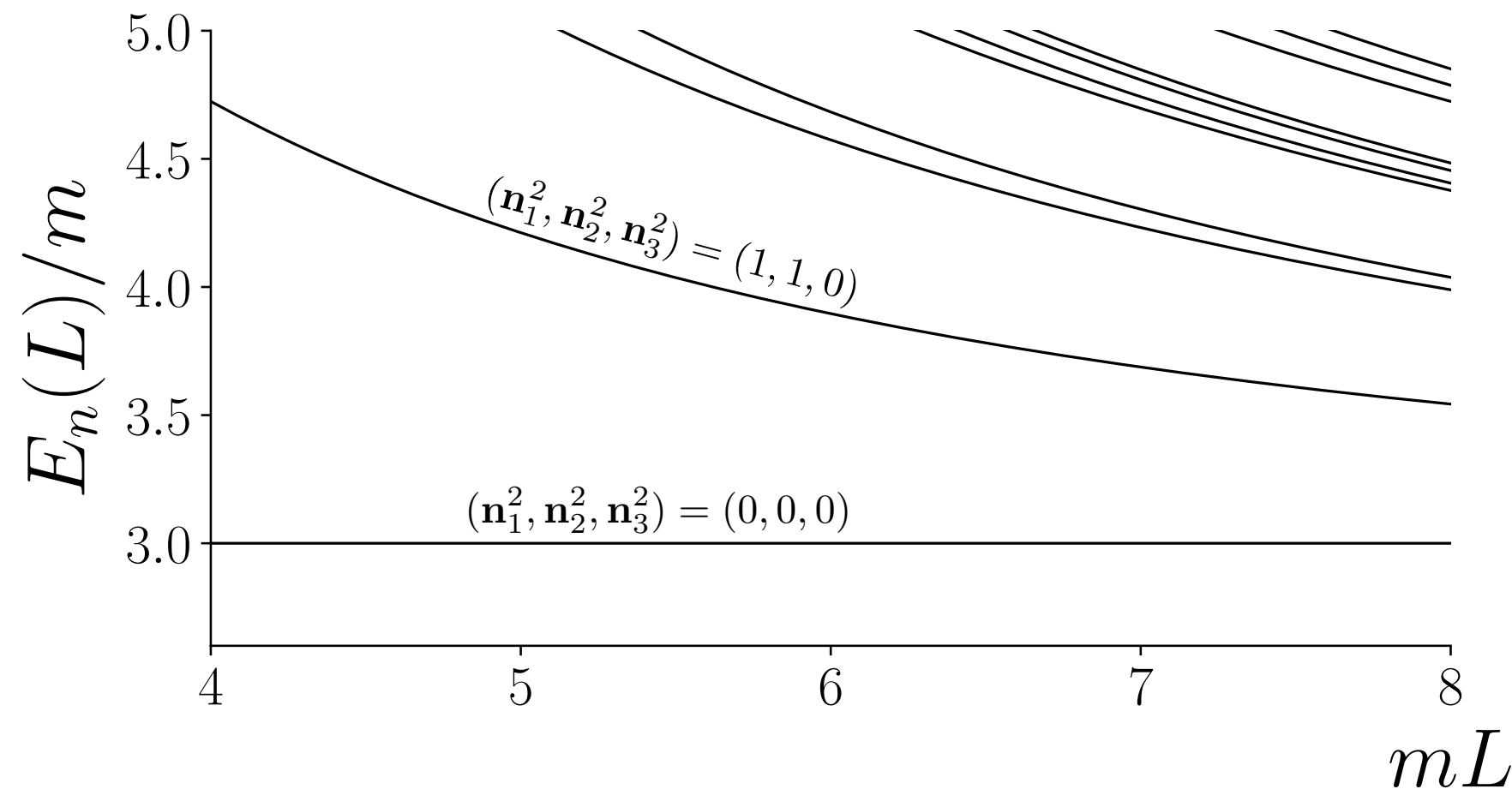
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$$i\mathcal{M}_3 = \mathcal{S} \left[\text{diagram 1} + \text{diagram 2} + \dots \right]$$

The diagrams represent Feynman-like diagrams for three-particle scattering. The first diagram shows two internal circles labeled $i\mathcal{M}_2$ connected by a horizontal line, with external lines entering and exiting. The second diagram shows a similar structure but with an additional circle labeled $i\mathcal{M}_2$ on the right side, connected to the first circle by a horizontal line.

First: **Non-interacting states** ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)



Three non-interacting particles

$$E_n(L) = \omega_1 + \omega_2 + \omega_3$$

$$\omega_i = \sqrt{m^2 + (2\pi/L)^2 \mathbf{n}_i^2} \quad \sum_i \mathbf{n}_i = 0$$

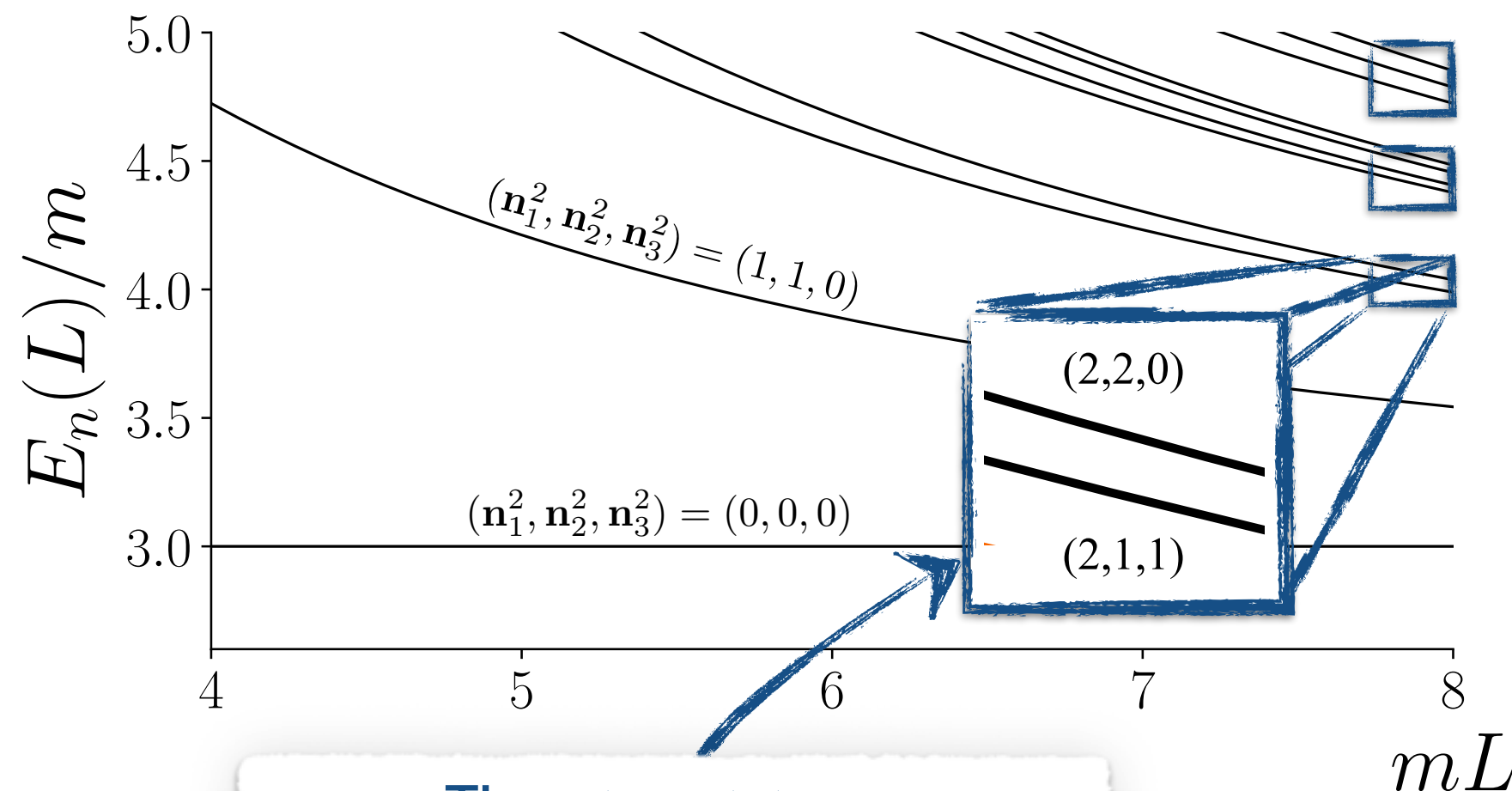
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The diagrams represent Feynman-like diagrams for three-particle scattering. The first diagram shows two internal vertices labeled $i\mathcal{M}_2$ connected by a line, with three external lines. The second diagram shows a more complex interaction with three internal vertices labeled $i\mathcal{M}_2$ and three external lines.

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These two states are degenerate in the NR theory

$$E_n^{\text{NR}}(L) = 3m + \frac{2\pi^2}{L^2} (\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2)$$

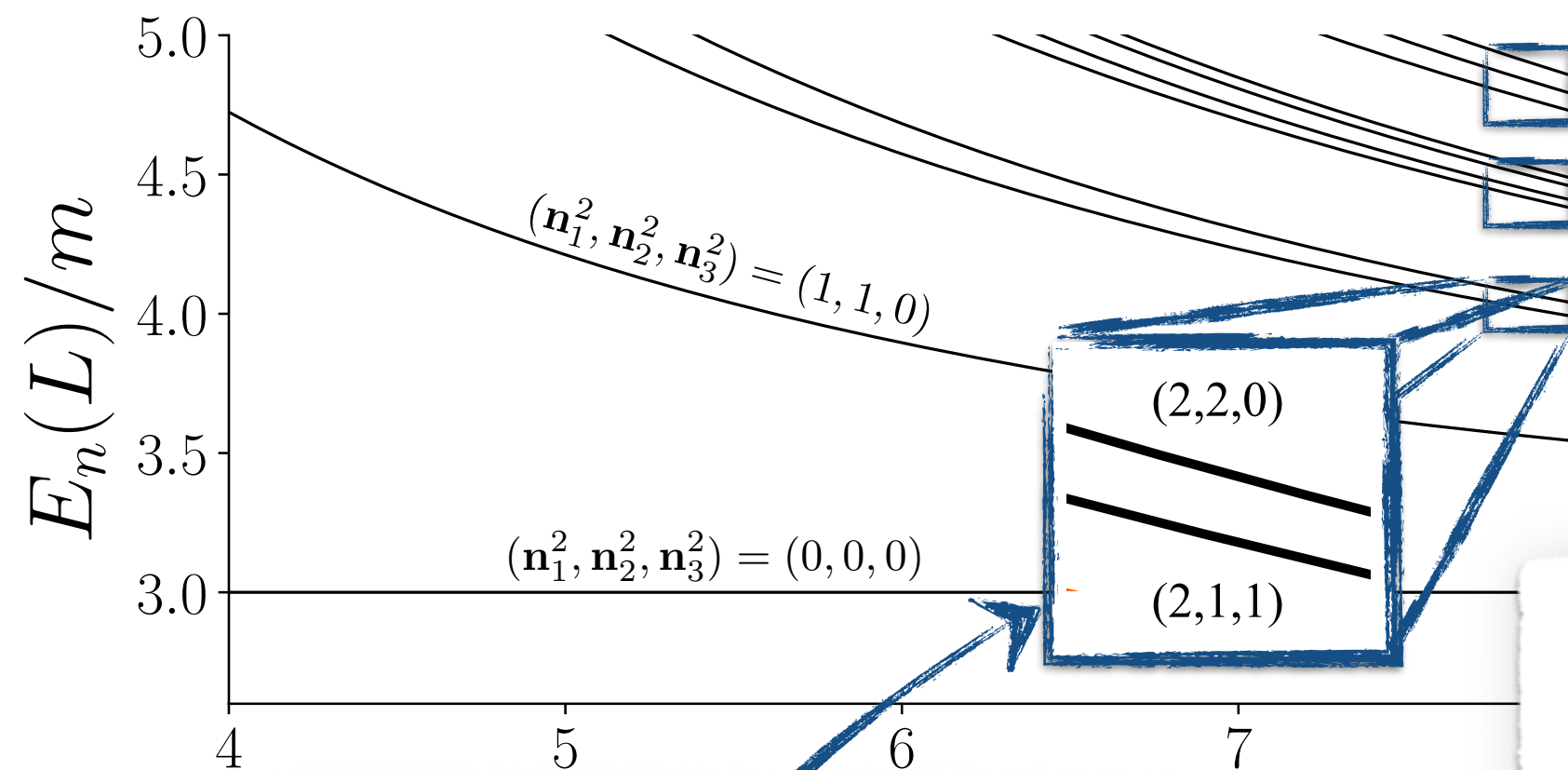
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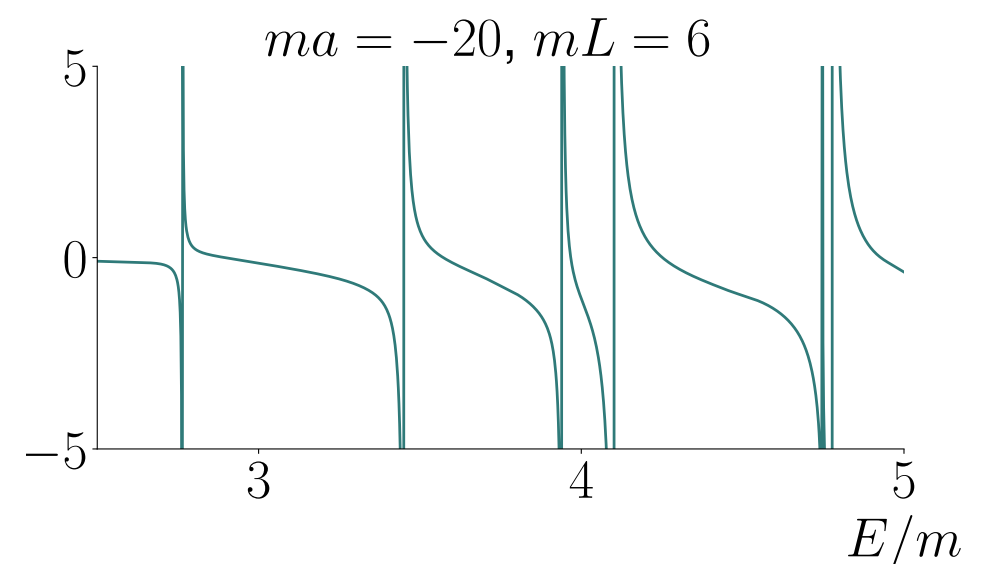
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We have already seen these clusters

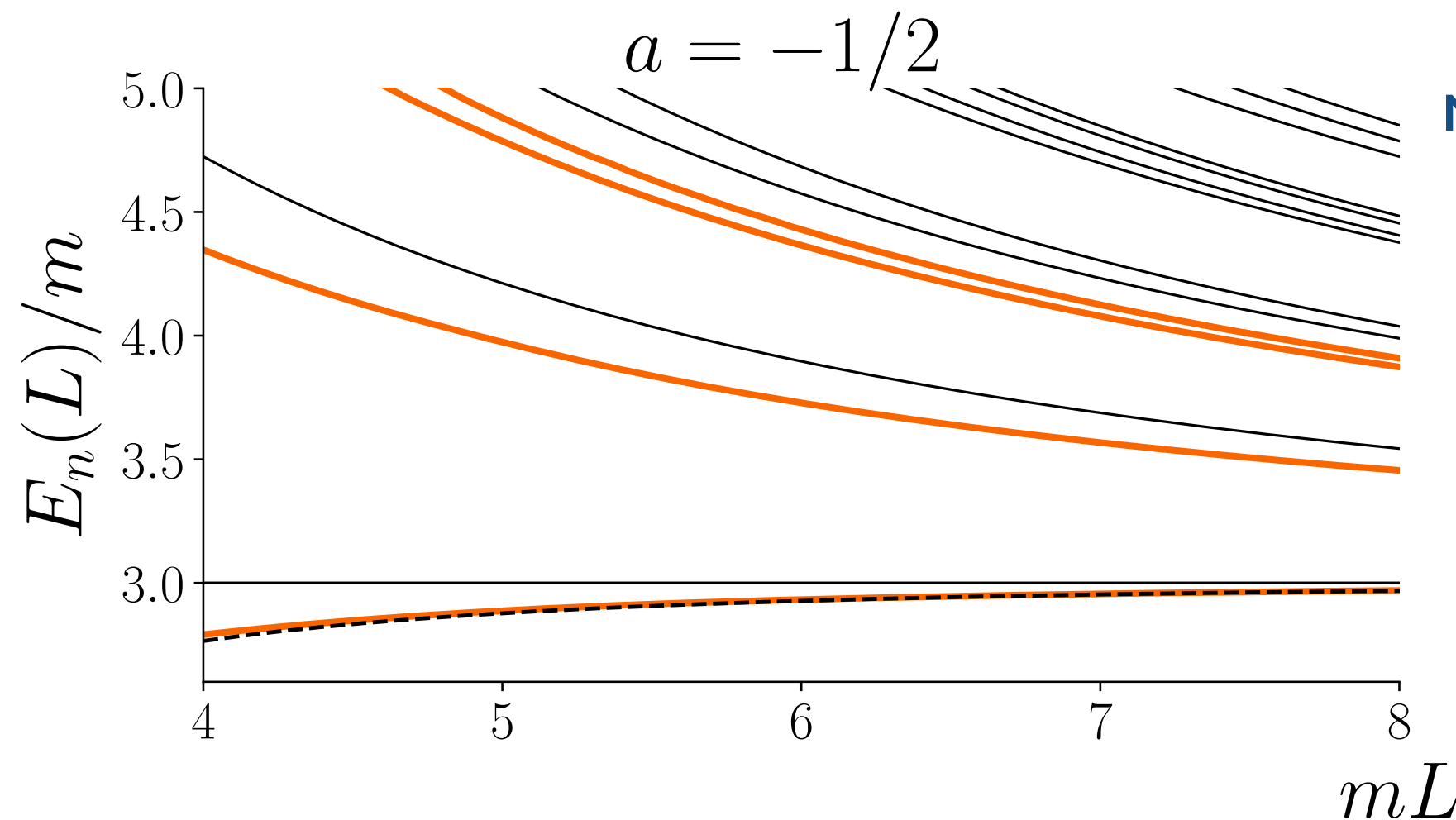


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Now we turn on the interactions

$$1/F_3^{\text{iso}}(E, L, a) = 0$$

$$F_3^{\text{iso}} = \frac{1}{L^3} \sum_{\vec{k}, \vec{p}} \left[\frac{\tilde{F}_s}{3} - \tilde{F}_s \frac{1}{\mathcal{H}} \tilde{F}_s \right]_{kp}$$

$$\mathcal{H} = 1/(2\omega\mathcal{K}_2) + \tilde{F}_s + \tilde{G}_s$$

known functions

----- $1/L$ expansion

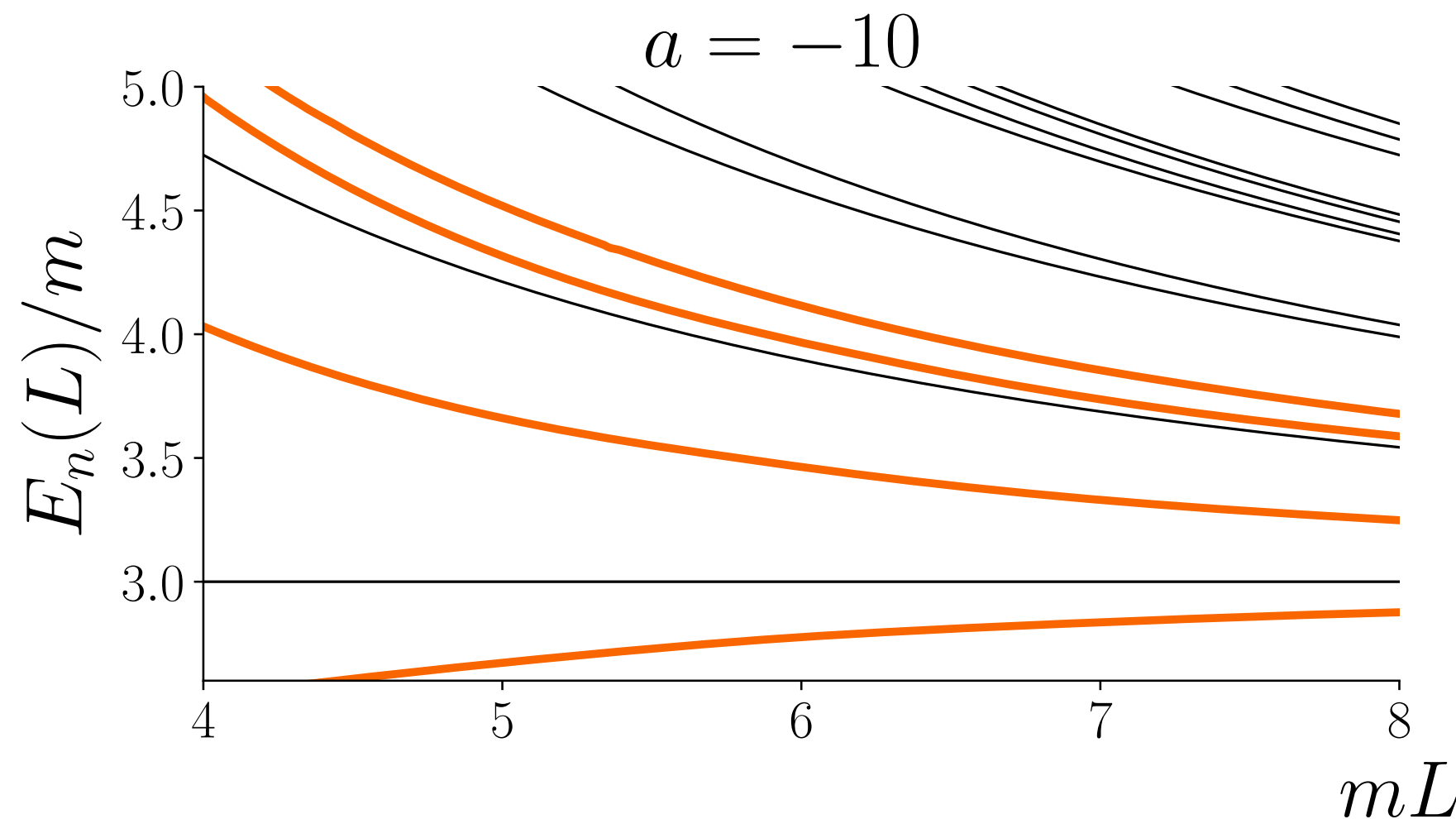
Briceño, Hansen and Sharpe (2018)

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Can also accommodate large a

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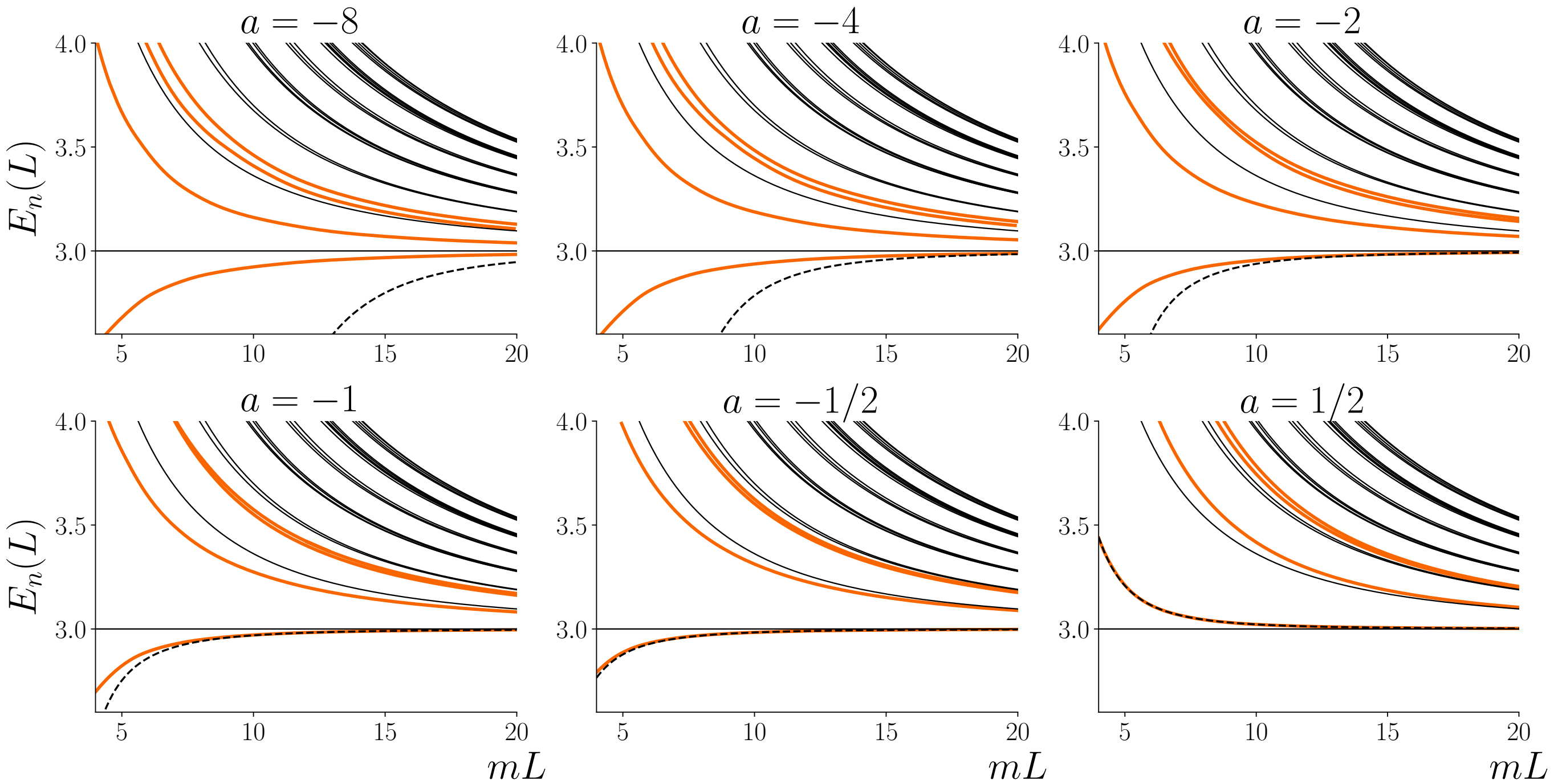
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Briceño, Hansen and Sharpe (2018)

$$\mathcal{K}_{\text{df},3}^{\text{iso}}(E) = 0 \text{ solutions}$$

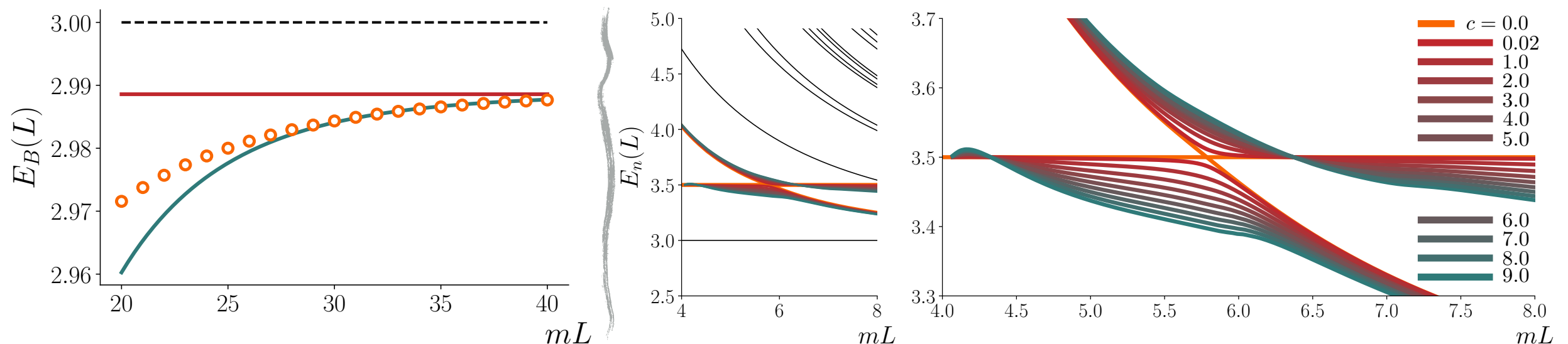
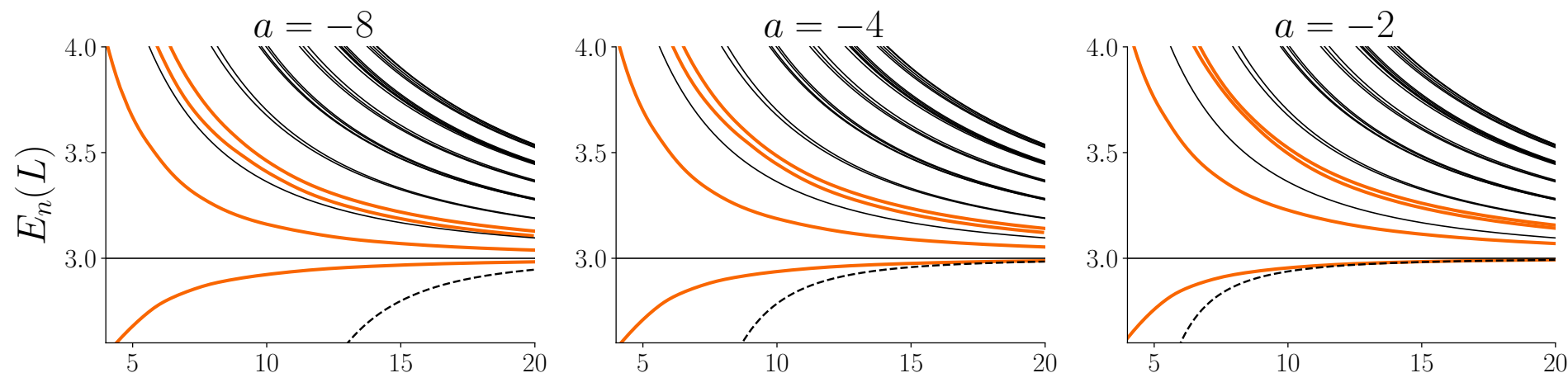
Straightforward to vary a and to study large volumes



Still lots to do

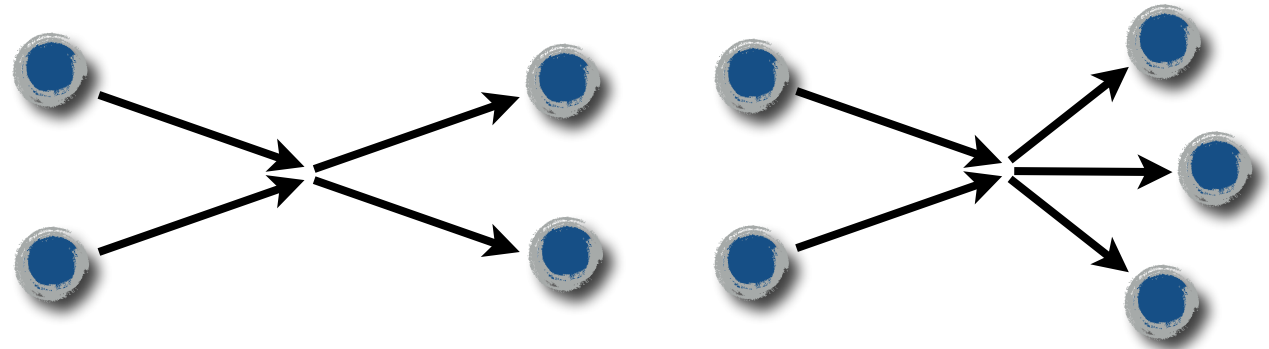
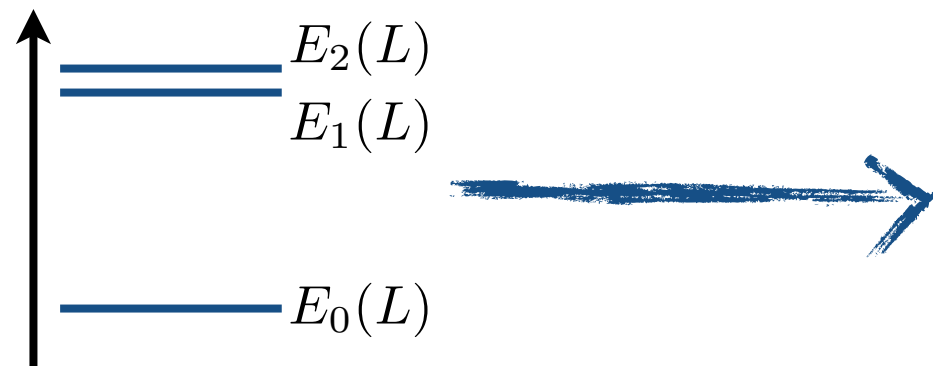
- ☐ Finish result with intermediate two-particle resonances
- ☐ Extend to non-identical, non-degenerate, multiple channels, spin
- ☐ Study subduction to finite-volume irreps
- ☐ Understand rigorous parametrizations for the infinite-volume observables
- ☐ Convince practitioners that the formalism is mature
- ☐ Reliably measure finite-volume spectra
- ☐ Extract three-particle scattering from LQCD

Big picture: making progress, but not quite there yet

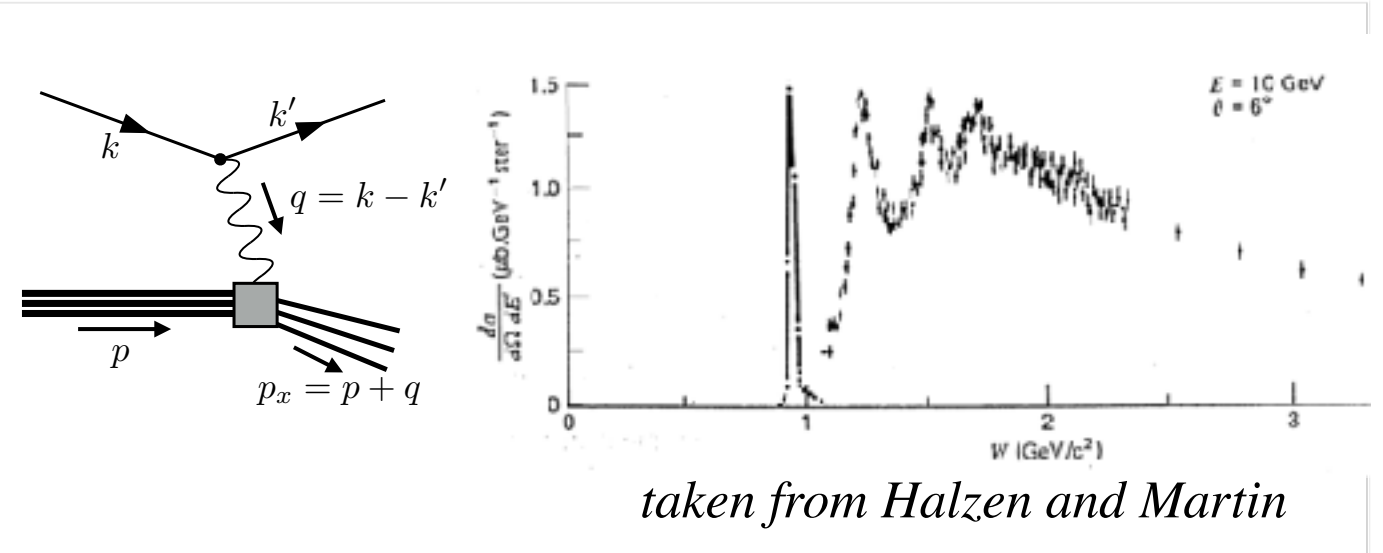
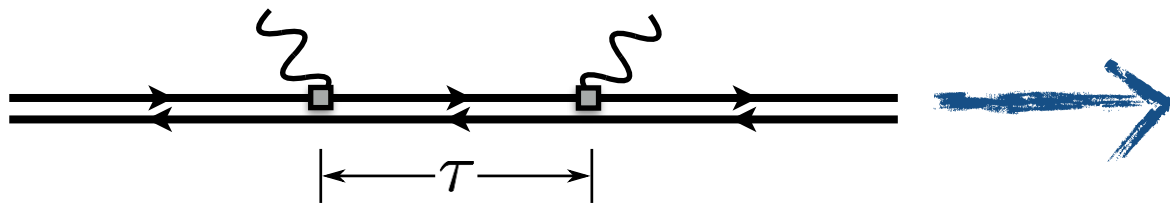


Outline

☑ Three-particle scattering

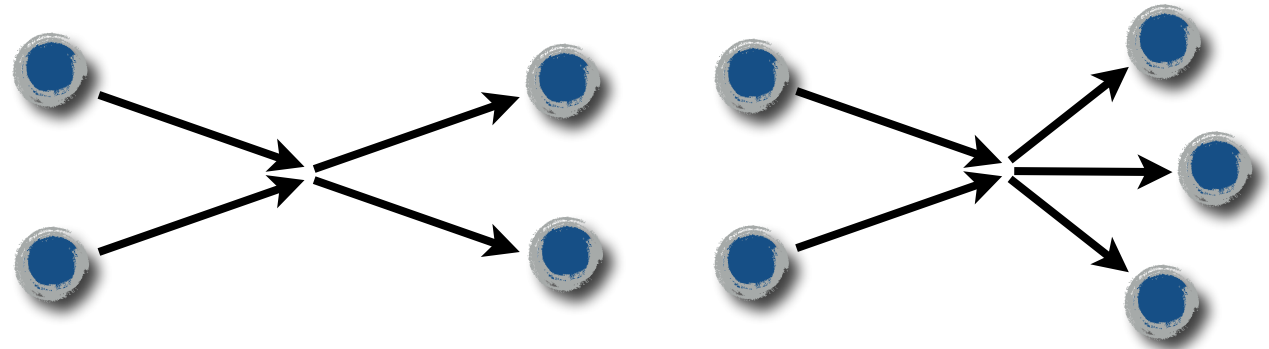
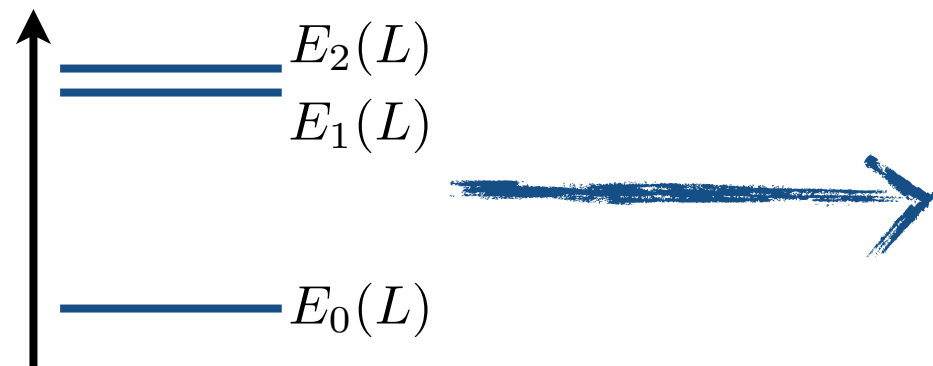


☐ n-particle inclusive rates

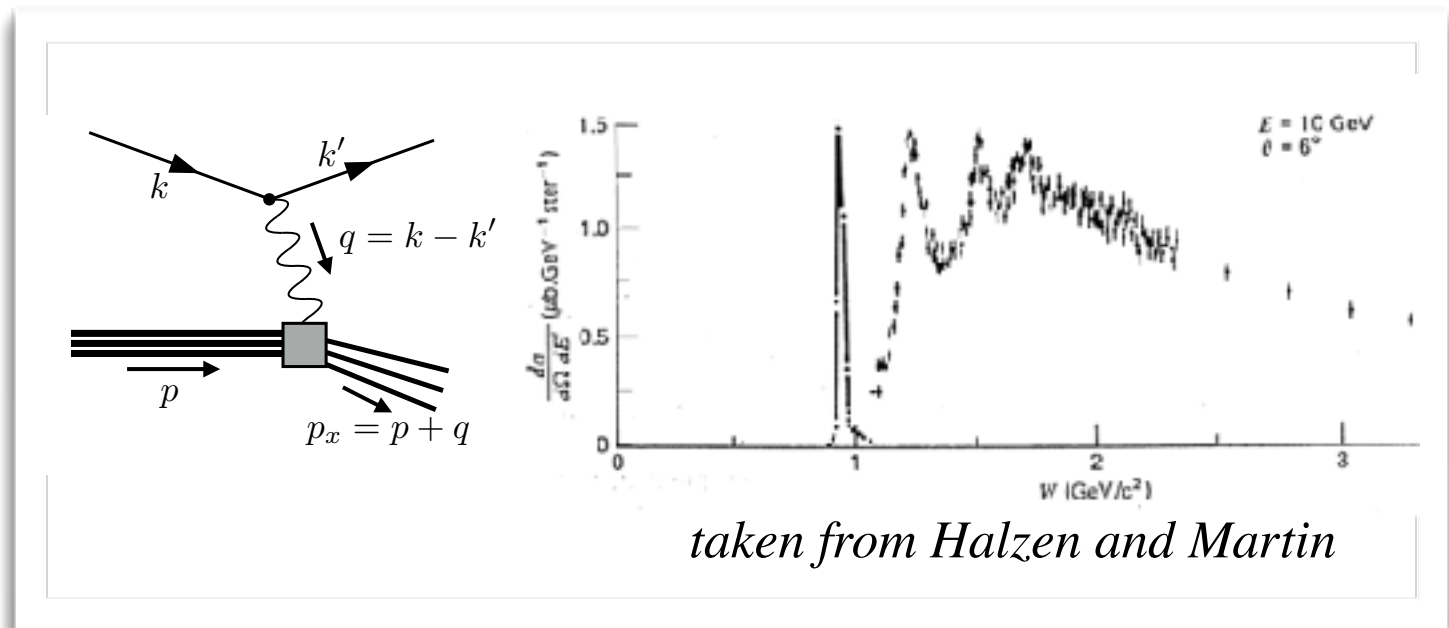
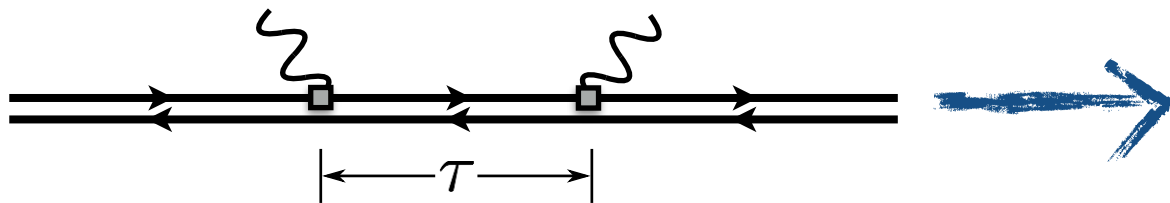


Outline

☑ Three-particle scattering

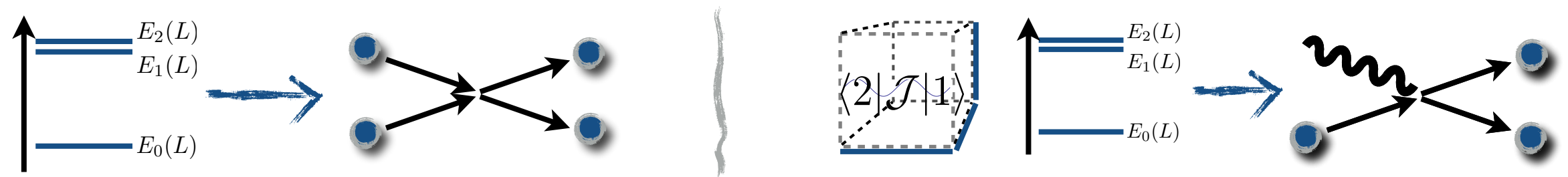


☐ n-particle inclusive rates



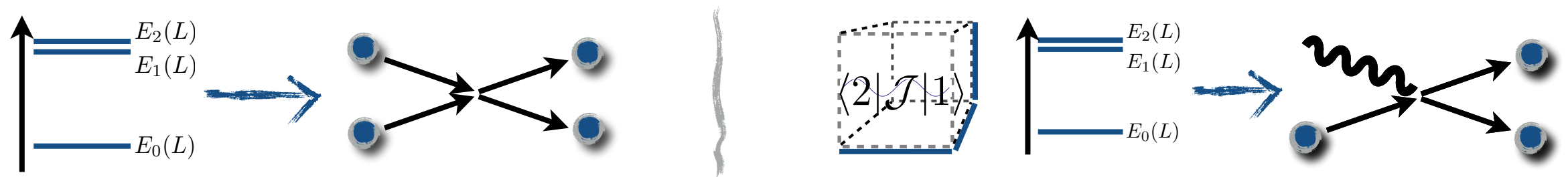
Based on MTH, H. Meyer & D. Robaina, (1704.08993)

Rates instead of amplitudes...



Finite-volume quantities are used to determine scattering and transition **amplitudes**

Rates instead of amplitudes...

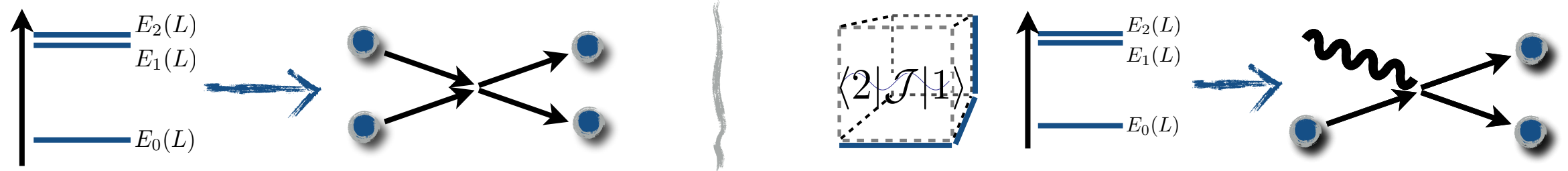


Finite-volume quantities are used to determine scattering and transition **amplitudes**

As the energy increases and more channels open,
this becomes increasingly challenging

$E_n(L) \leftarrow$ Contains information about
all open channels with given QCD quantum numbers $\pi\pi, \pi\pi\pi\pi, K\bar{K}, \dots$

Rates instead of amplitudes...



Finite-volume quantities are used to determine scattering and transition **amplitudes**

As the energy increases and more channels open,
this becomes increasingly challenging

$E_n(L)$ ← Contains information about all open channels with given QCD quantum numbers $\pi\pi, \pi\pi\pi\pi, K\bar{K}, \dots$

Suppose we attempt an “easier” task: **Total production rates**

$$\sigma_{\pi\gamma^* \rightarrow X} \equiv \int d\Phi \left| \begin{array}{c} \text{wavy line} \\ \nearrow \searrow \\ \bullet \quad \bullet \end{array} \right|^2 + \int d\Phi \left| \begin{array}{c} \text{wavy line} \\ \nearrow \searrow \nearrow \searrow \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \right|^2 + \int d\Phi \left| \begin{array}{c} \text{wavy line} \\ \nearrow \searrow \\ \bullet \quad \bullet \end{array} \right|^2 + \dots$$

Sum over all open channels with given QCD quantum numbers

The hadronic tensor

$$\sigma_{\pi\gamma^* \rightarrow X} \equiv \int d\Phi \left| \begin{array}{c} \text{wavy line} \\ \nearrow \searrow \\ \bullet \quad \bullet \end{array} \right|^2 + \int d\Phi \left| \begin{array}{c} \text{wavy line} \\ \nearrow \searrow \nearrow \searrow \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \right|^2 + \int d\Phi \left| \begin{array}{c} \text{wavy line} \\ \nearrow \searrow \\ \bullet \quad \bullet \end{array} \right|^2 + \dots$$

This rate is related to the matrix element of the current...

$$\sigma_{\pi\gamma^* \rightarrow X} \propto \sum_{\alpha} \int d\Phi_{\alpha} |\langle E, \mathbf{p}, \alpha, \text{out} | \mathcal{J}(0) | \pi \rangle|^2$$

$\nearrow$ proportional because of missing kinematic factors
 $\nearrow$ n-particle phase space integral
 $\nearrow$ $\pi\pi, \pi\pi\pi\pi, K\bar{K}, \dots$

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integrate over all energies
but include a delta

The hadronic tensor

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the delta is rewritten using

$$\int dx e^{ipx} = 2\pi \delta(p)$$

The hadronic tensor

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the sum over states is just the identity, I

the current is translated

The hadronic tensor

$$\sigma_{\pi\gamma^* \rightarrow X} \equiv \int d\Phi \left| \begin{array}{c} \text{wavy line} \rightarrow \text{two blue dots} \\ \text{blue dot} \rightarrow \text{two blue dots} \end{array} \right|^2 + \int d\Phi \left| \begin{array}{c} \text{wavy line} \rightarrow \text{four blue dots} \\ \text{blue dot} \rightarrow \text{four blue dots} \end{array} \right|^2 + \int d\Phi \left| \begin{array}{c} \text{wavy line} \rightarrow \text{two red dots} \\ \text{blue dot} \rightarrow \text{two red dots} \end{array} \right|^2 + \dots$$

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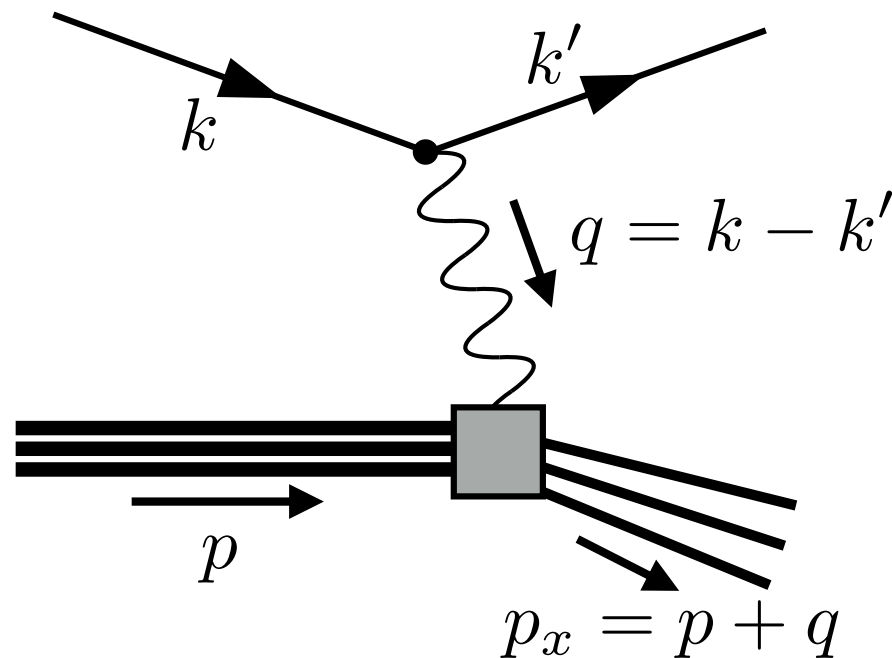
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This motivates the hadronic tensor...

$$W_{\mu\nu}(p, q) \equiv \int d^4x e^{iqx} \langle \pi, p | \mathcal{J}_{\mu}^{\dagger}(x) \mathcal{J}_{\nu}(0) | \pi, p \rangle$$

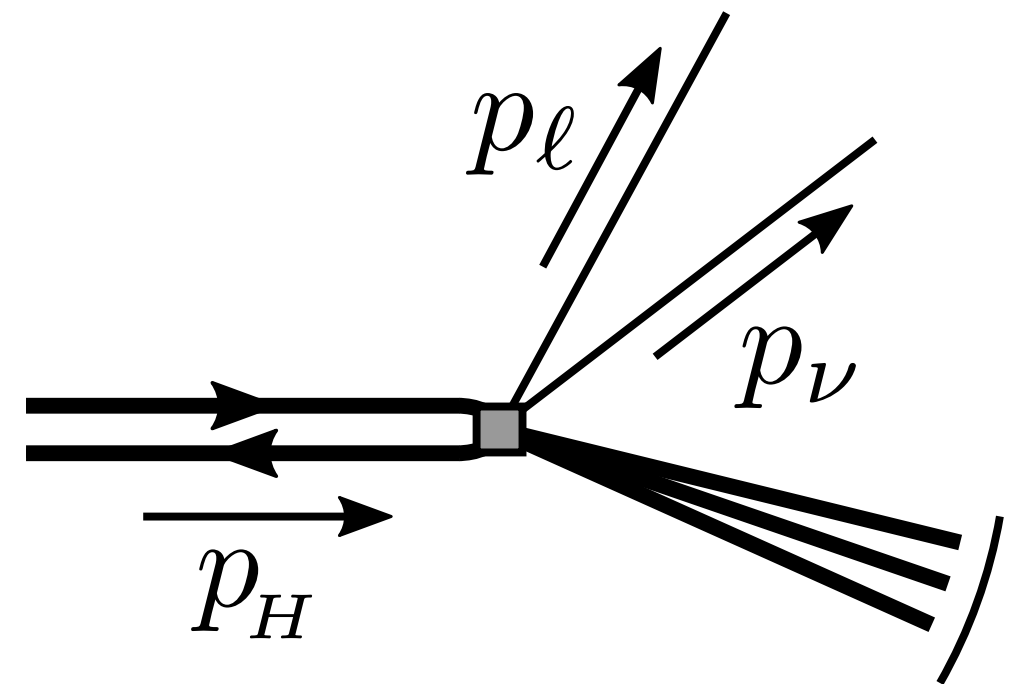
Potential applications

Onset of deep-inelastic scattering



$$\langle N | J_\mu(q) J_\nu(0) | N \rangle$$

Heavy-flavor decays



$$\langle D | J_\mu(q) J_\nu(0) | D \rangle$$

The idea is to describe systems where **many hadrons are produced** and they are **not individually detected**

$$\sigma_{\pi\gamma^* \rightarrow X} \equiv \int d\Phi \left| \text{Diagram 1} \right|^2 + \int d\Phi \left| \text{Diagram 2} \right|^2 + \int d\Phi \left| \text{Diagram 3} \right|^2 + \dots$$

The equation shows the cross-section $\sigma_{\pi\gamma^* \rightarrow X}$ as a sum of squared amplitudes for different final states X . The diagrams represent different ways a virtual photon can decay into multiple hadrons (represented by blue and red dots). The first diagram shows a photon decaying into two blue dots. The second shows a photon decaying into three blue dots. The third shows a photon decaying into two red dots. The sum is over all possible final states.

See also... Liu, Dong (hep-ph/9306299), Liu (arXiv:1703.04690), Hashimoto (arXiv:1703.01881), Chambers *et al.* (arXiv:1703.01153)

Can we calculate this using LQCD?

$$W_{\mu\nu}(p, q) \equiv \int d^4x \, e^{iqx} \langle \pi, p | \mathcal{J}_\mu^\dagger(x) \mathcal{J}_\nu(0) | \pi, p \rangle$$

Can we calculate this using LQCD?

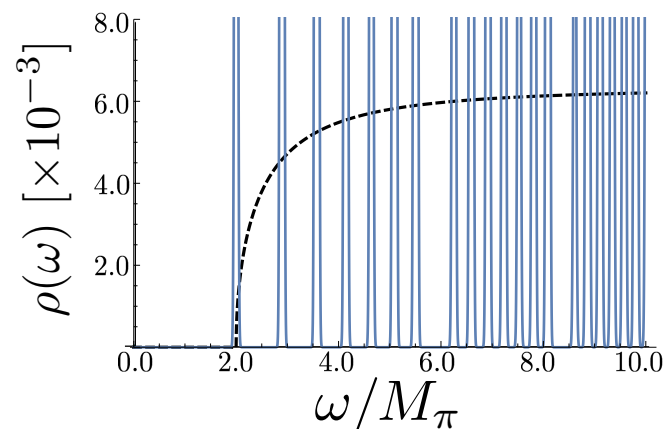
$$W_{\mu\nu}(p, q) \equiv \int d^4x \, e^{iqx} \langle \pi, p | \mathcal{J}_\mu^\dagger(x) \mathcal{J}_\nu(0) | \pi, p \rangle$$

The main obstacles are **finite volume** and **Euclidean time**

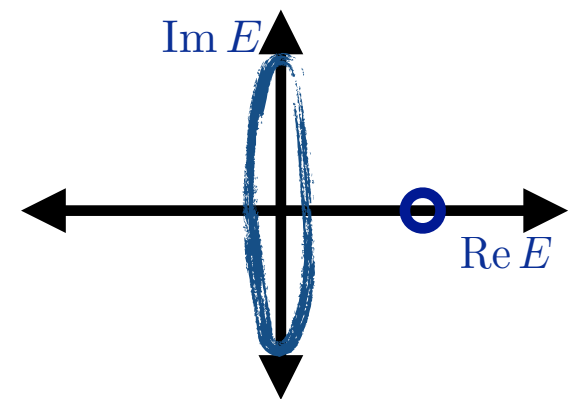
$$\sum_{\alpha} \int d\Phi_{\alpha} \, |p_x, \alpha, \text{out}\rangle \langle p_x, \alpha, \text{out}| \neq \sum_n |E_n, \mathbf{p}_x, L\rangle \langle E_n, \mathbf{p}_x, L|$$

$$e^{i\omega t} \neq e^{i\omega\tau}$$

Finite volume



Euclidean time



Can we calculate this using LQCD?

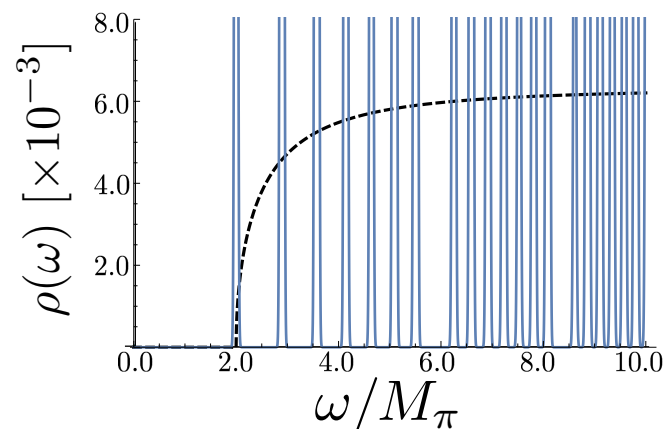
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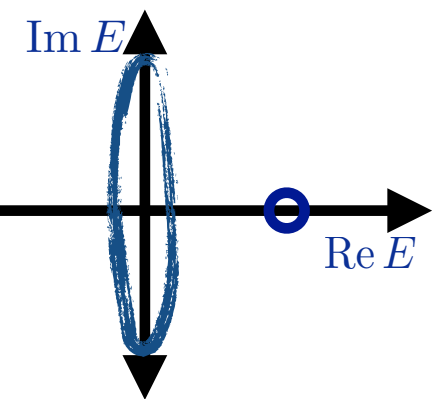
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Finite volume

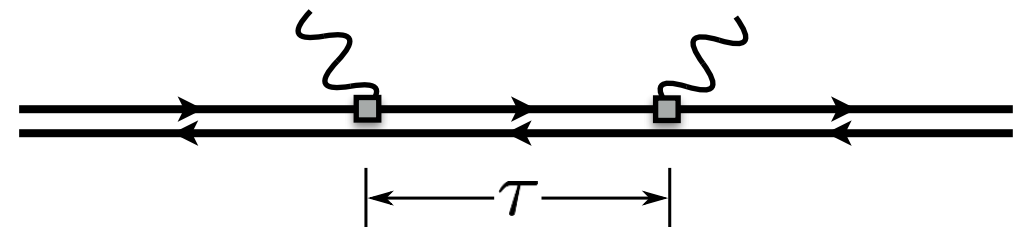


Euclidean time



What is the closest lattice quantity?

$$G_{\mu\nu}(\tau, \mathbf{q}, L) \propto \frac{\langle \pi_{\mathbf{p}}(\tau_f) \mathcal{J}_\mu^\dagger(\tau, \mathbf{q}) \mathcal{J}_\nu(0) \pi_{\mathbf{p}}(\tau_i) \rangle_{\text{conn}}}{\langle \pi_{\mathbf{p}}(\tau_f) \pi(\tau_i) \rangle}$$



Can we calculate this using LQCD?

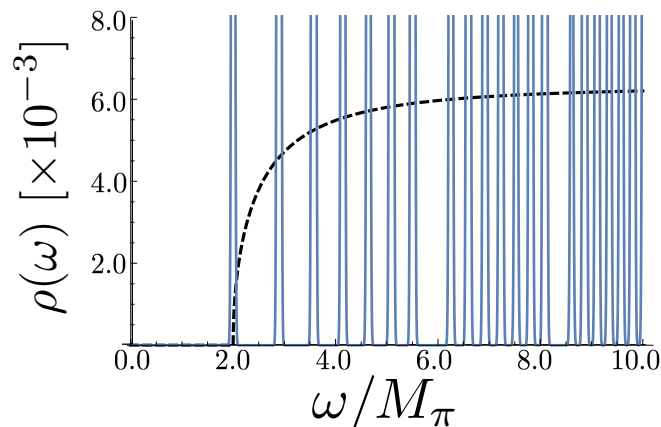
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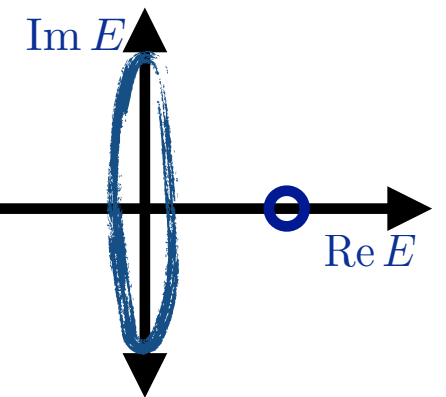
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Finite volume

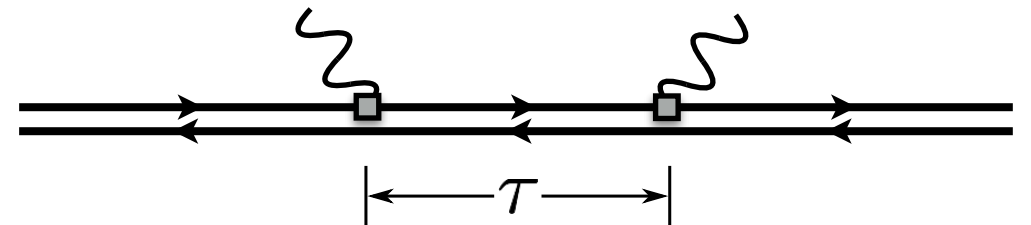


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$$G_{\mu\nu}(\tau, \mathbf{q}, L) = e^{-\omega_{\mathbf{p}}\tau} \langle \pi, \mathbf{p} | \mathcal{J}_\mu(\tau, \mathbf{q}) \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle_L + \dots$$

Our goal...

(a) This is what we have...

$$G_{\nu\nu}(\tau, \mathbf{q}, L) = \sum_n e^{-E_n(L)\tau} |\langle E_n(L), \mathbf{p}_x | \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle_L|^2$$

(b) This is what we want...

$$W_{\nu\nu}(p, q) = \int_{\text{all states}, E'} (2\pi) \delta(E - E') |\langle E', \mathbf{p}_x, \text{out} | \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle|^2$$

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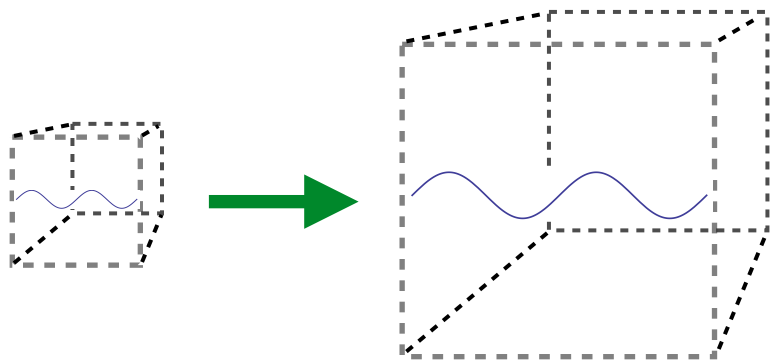
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To go from (a) to (b) one must first take the infinite-volume limit



$$G_{\nu\nu}(\tau, \mathbf{q}) = \int_{\text{all states}, E'} e^{-E'\tau} |\langle E', \mathbf{p}_x, \text{out} | \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle|^2$$

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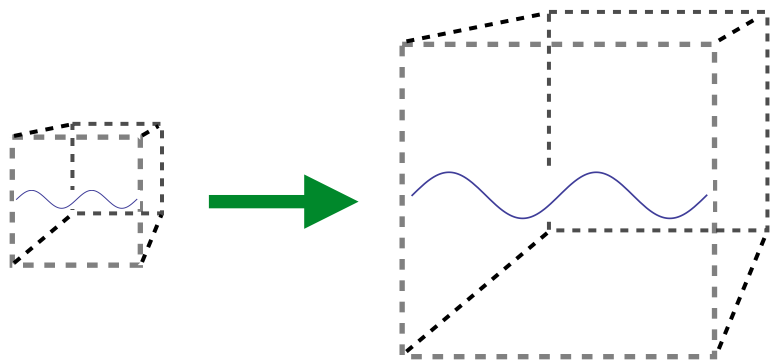
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(b) This is what we want...

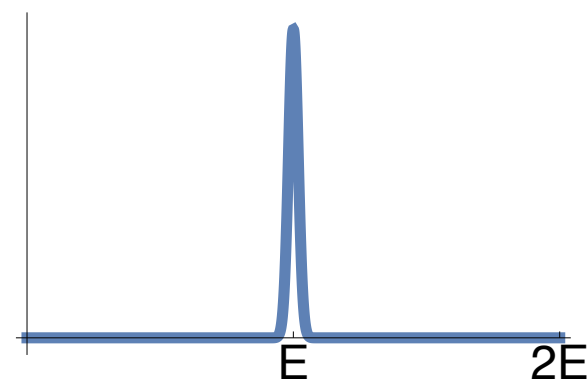
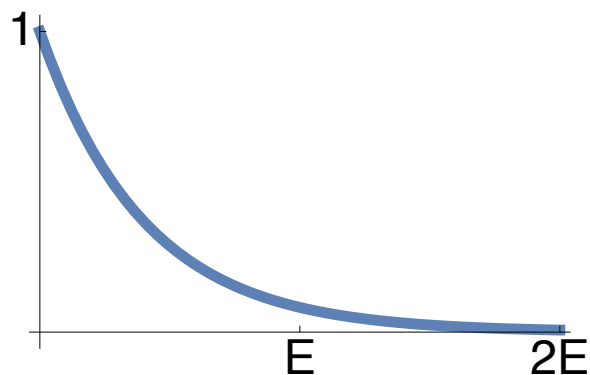
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To go from (a) to (b) one must first take the infinite-volume limit



$$G_{\nu\nu}(\tau, \mathbf{q}) = \int_{\text{all states}, E'} e^{-E'\tau} |\langle E', \mathbf{p}_x, \text{out} | \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle|^2$$

Then replace $e^{-E'\tau}$ with $(2\pi) \delta(E - E')$

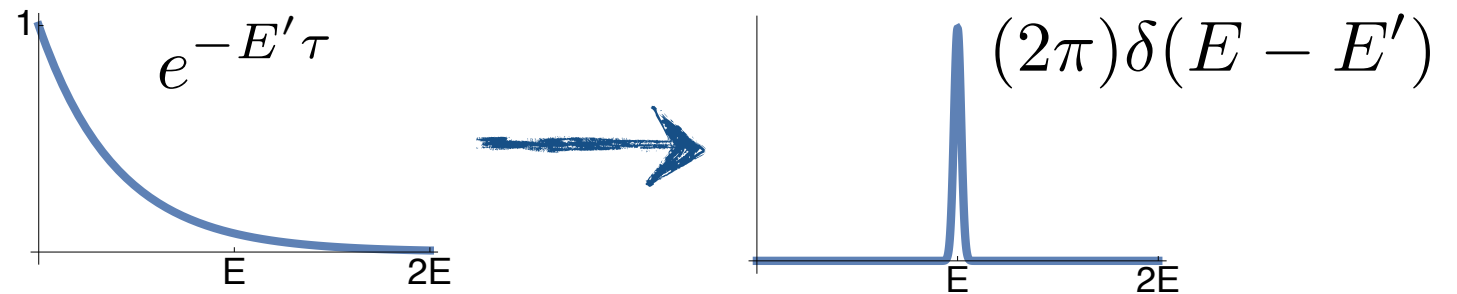


The importance of infinite-volume

Suppose we take some (large) finite volume...

$$G_{\nu\nu}(\tau, \mathbf{q}, L) = \sum_n e^{-E_n(L)\tau} |\langle E_n(L), \mathbf{p}_x | \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle_L|^2$$

and make the replacement

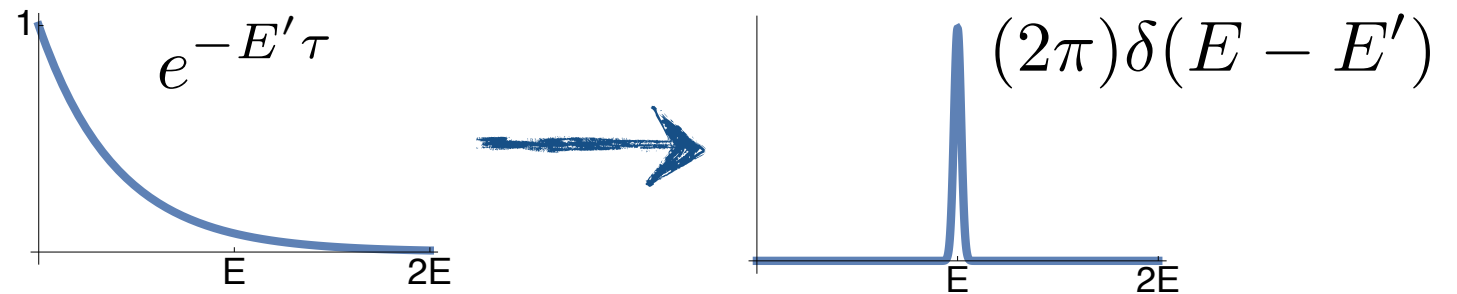


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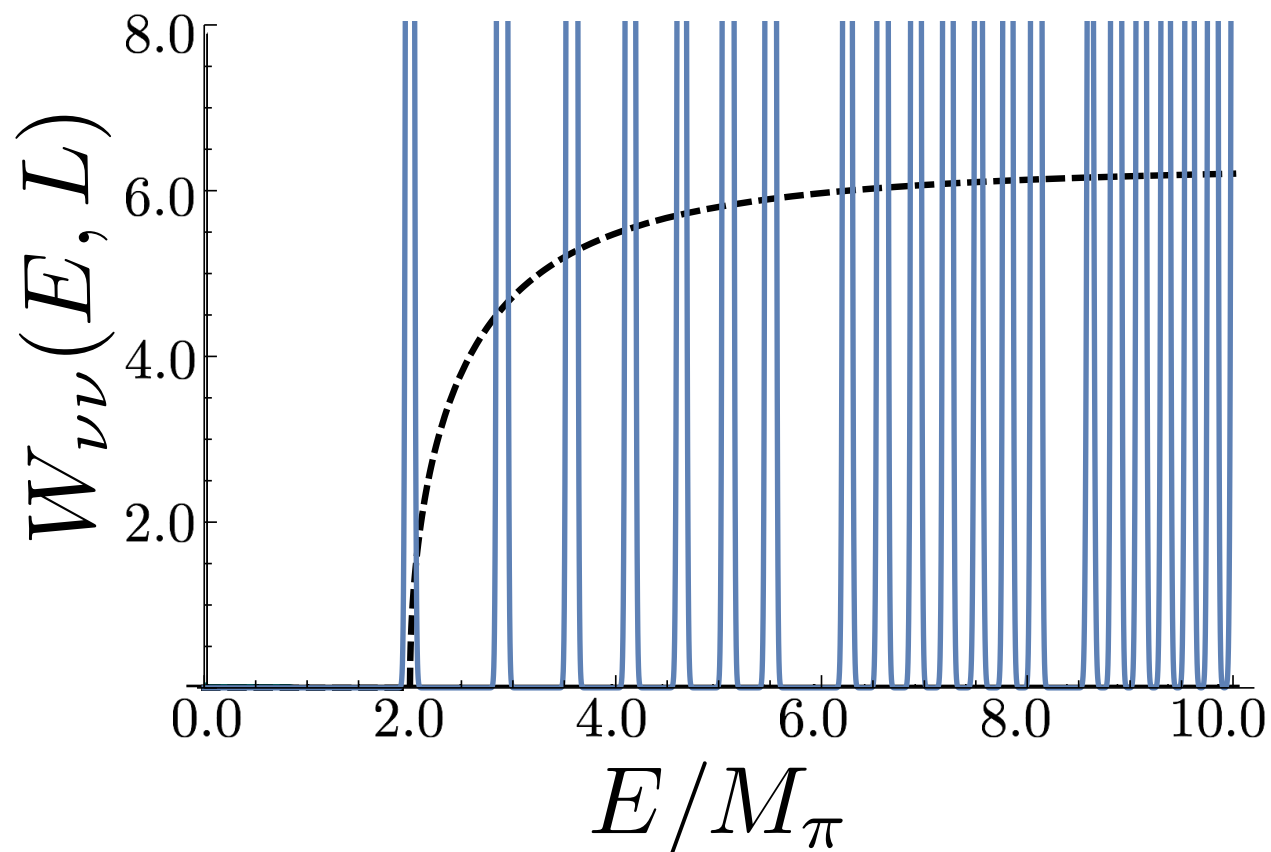
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and make the replacement



This does not give a useful estimate



For a **fixed volume** the sum over **delta functions** does **not** directly give a useful estimate of the **infinite-volume target quantity**

Our goal (cont.)...

Suppose we could send $L \rightarrow \infty$

$$\lim_{L \rightarrow \infty} G_{\nu\nu}(\tau, \mathbf{q}, L) = \int_{\text{all states}, E'} e^{-E'\tau} |\langle E', \mathbf{p}_x, \text{out} | \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle|^2$$

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Then the remaining step is an inverse Laplace transform

$$G(\tau) = \int_0^\infty dE' e^{-E'\tau} \rho(E') \quad \xrightarrow{?} \quad \rho(E)$$



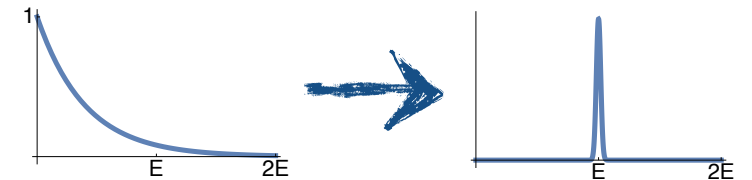
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This amounts to forming clever linear combinations...

$$q_1(E) e^{-E' a_\tau} + q_2(E) e^{-E' 2a_\tau} + q_3(E) e^{-E' 3a_\tau} + \dots \approx \delta(E - E')$$

Our goal (cont.)...

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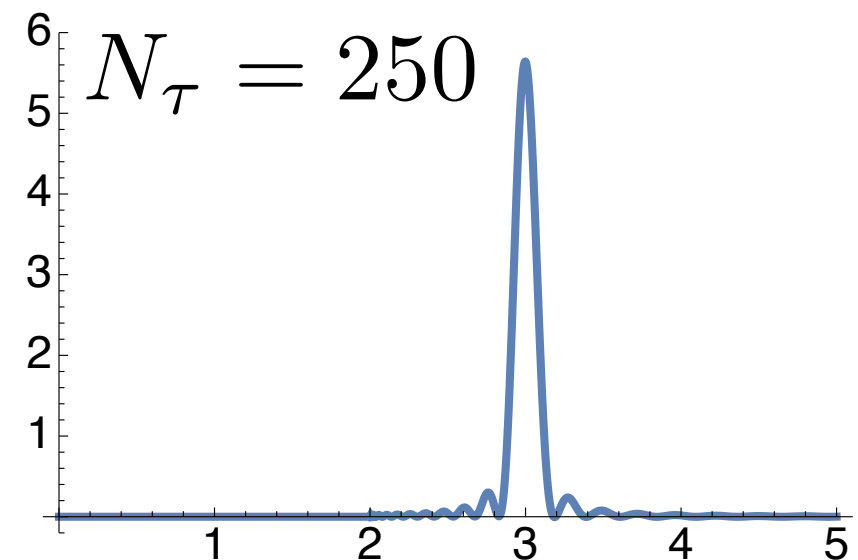
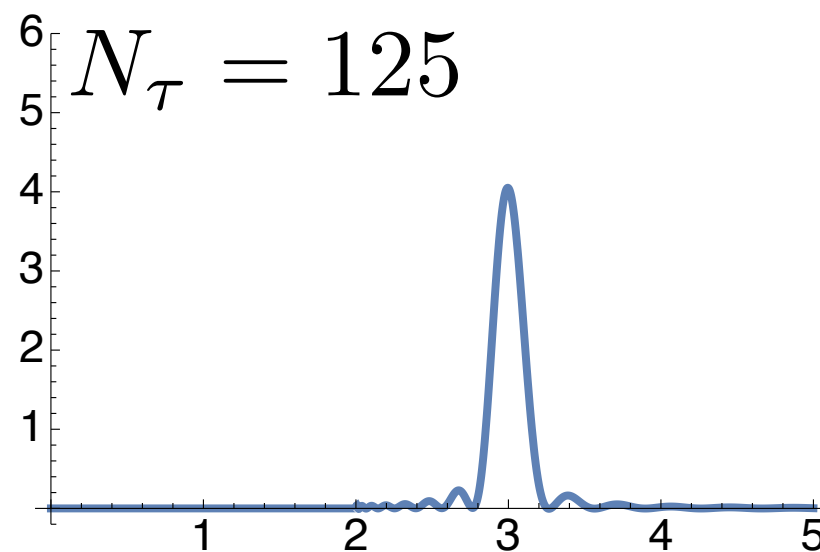
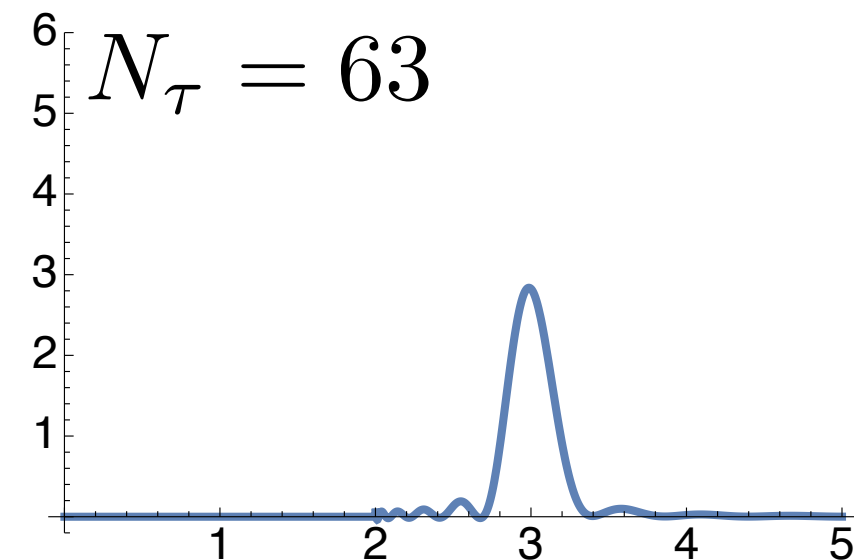
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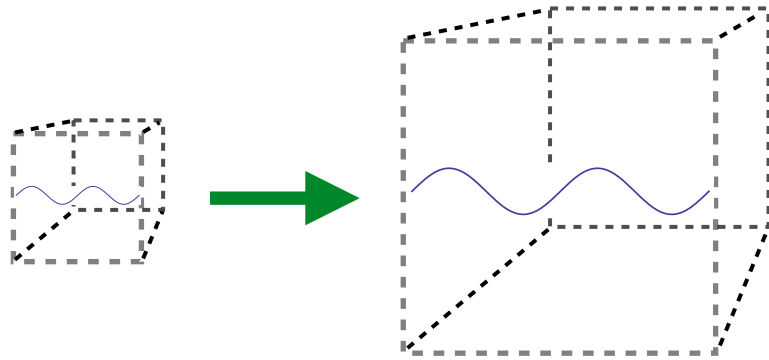
$$q_1(E) e^{-E' a_\tau} + q_2(E) e^{-E' 2a_\tau} + q_3(E) e^{-E' 3a_\tau} + \dots \approx \delta(E - E')$$

In fact this is possible...



Practical limitations

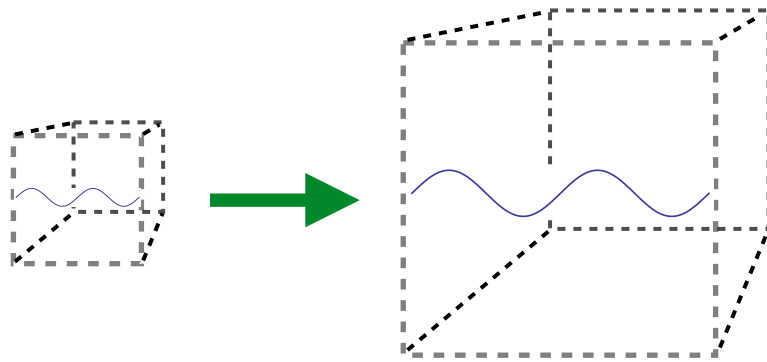
One cannot strictly take the infinite-volume limit...



$$\lim_{L \rightarrow \infty} G_{\nu\nu}(\tau, \mathbf{q}, L) = \int_{\text{all states}, E'} e^{-E' \tau} |\langle E', \mathbf{p}_x, \text{out} | \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle|^2$$

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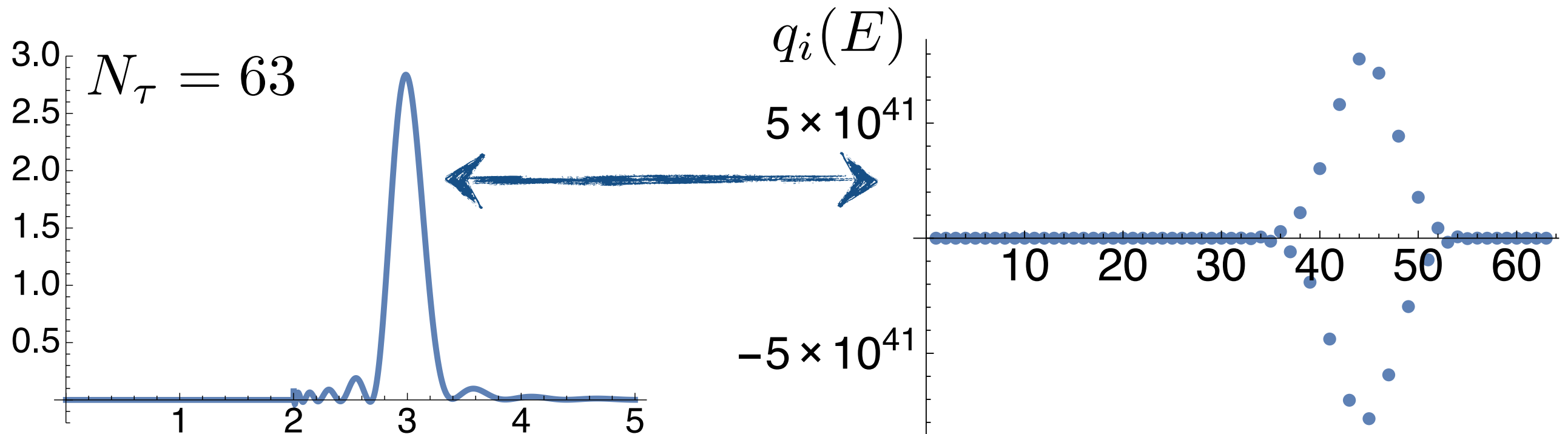
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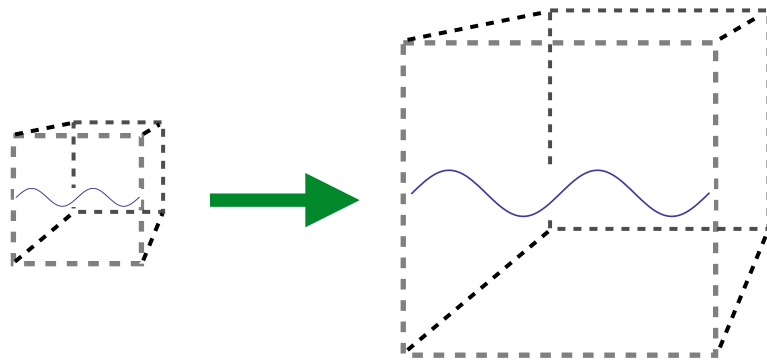
and the inverse problem is not so easily defeated!

$$G(\tau) = \int_0^\infty dE' e^{-E' \tau} \rho(E') \quad \xrightarrow{?} \quad \rho(E)$$



Practical limitations

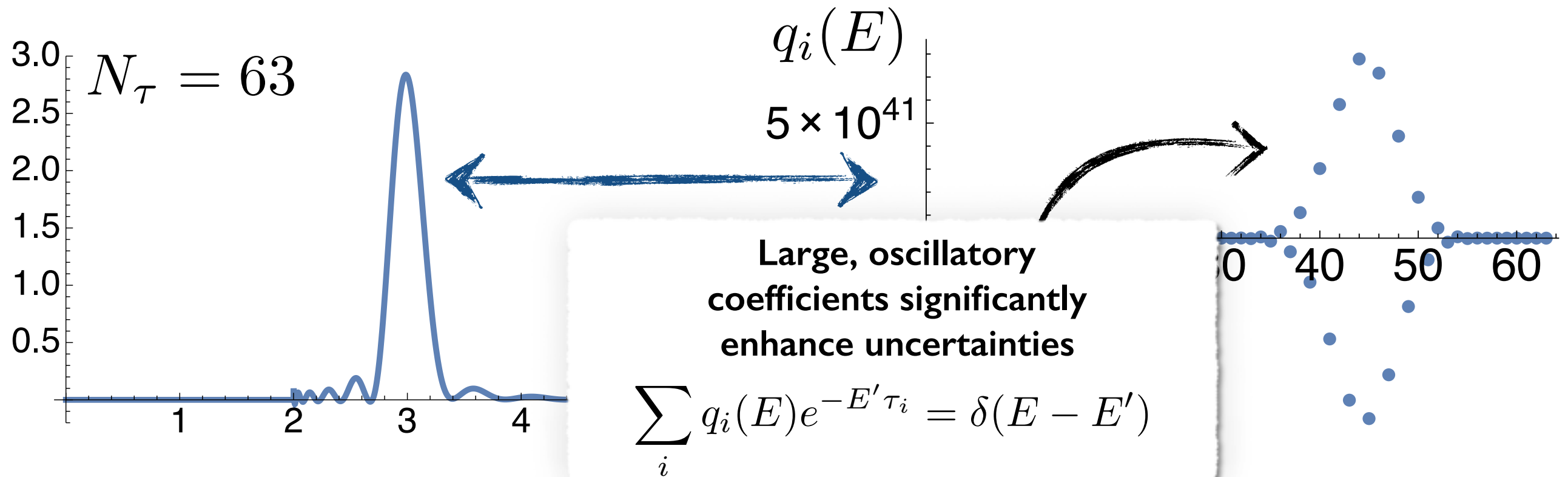
One cannot strictly take the infinite-volume limit...



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- The quality of the data determines the **resolution**

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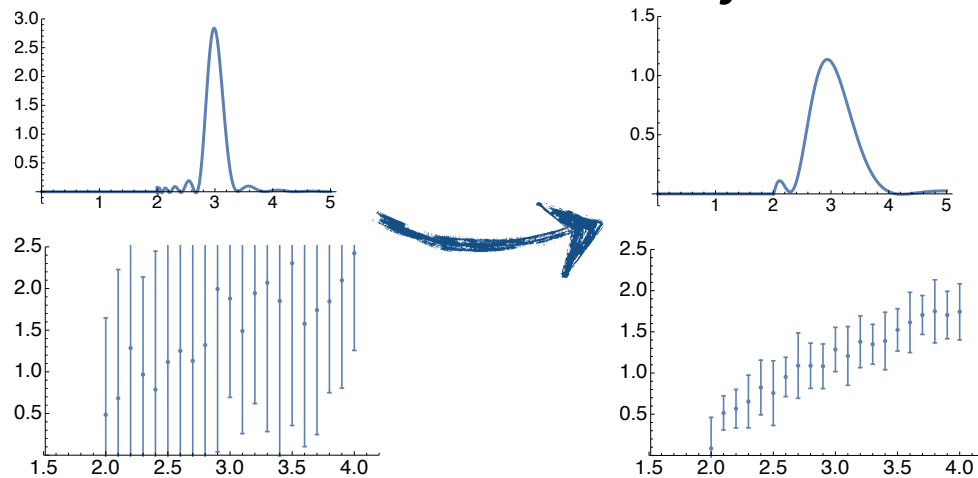
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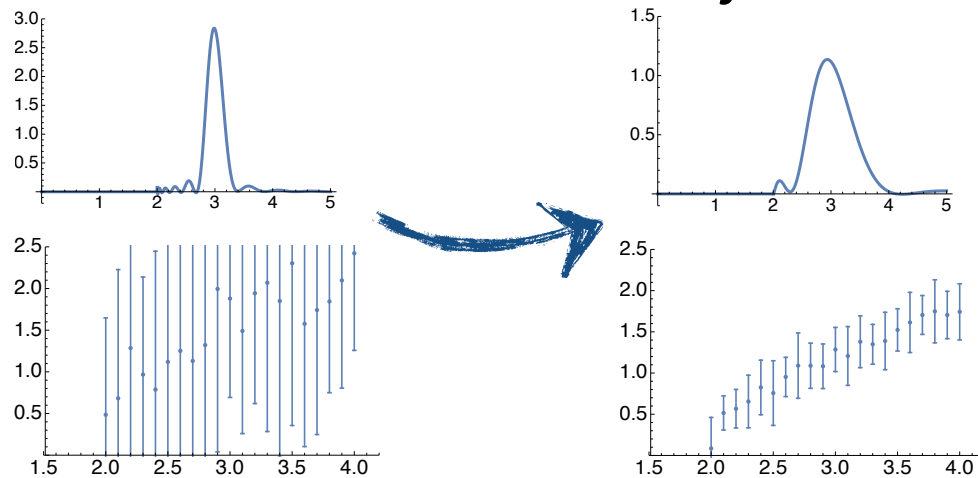
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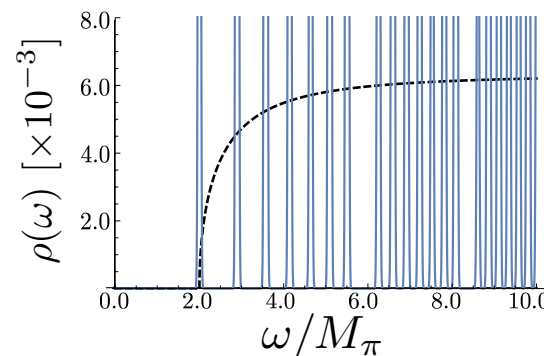
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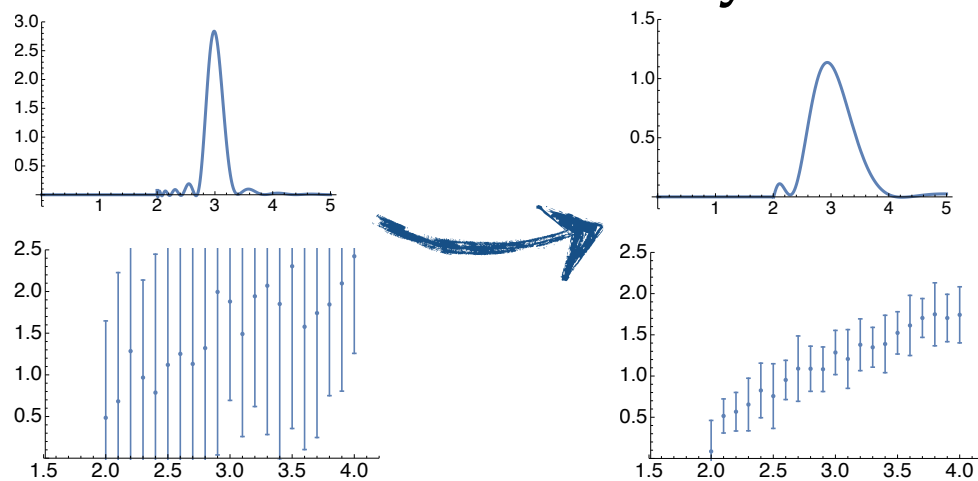
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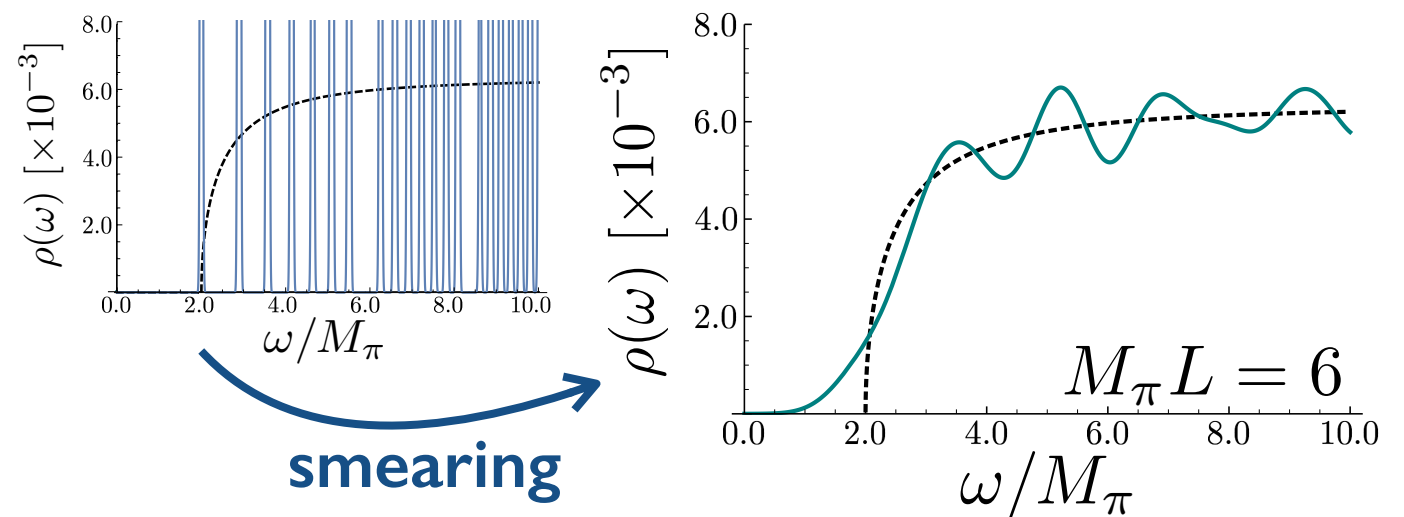
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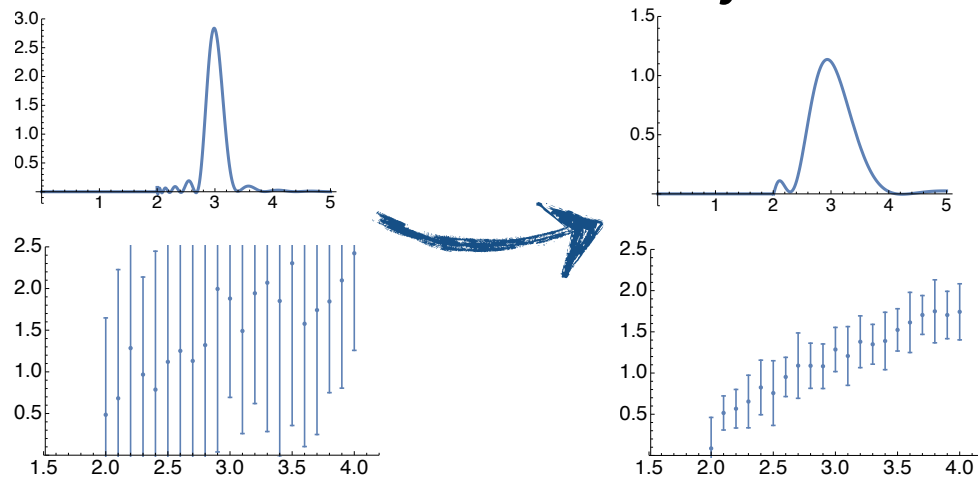
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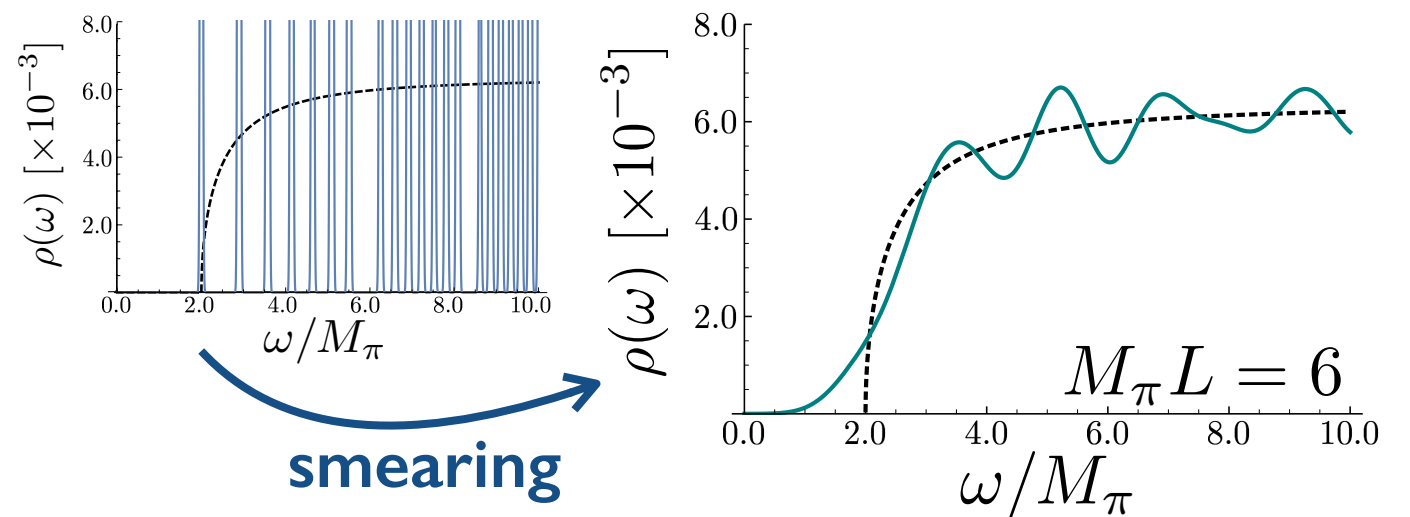
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Target quantity is formally recovered in an **ordered** double limit

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The approach in full...

normalization factors

$$R_i \equiv \int_0^\infty dE' e^{-E' \tau_i}$$

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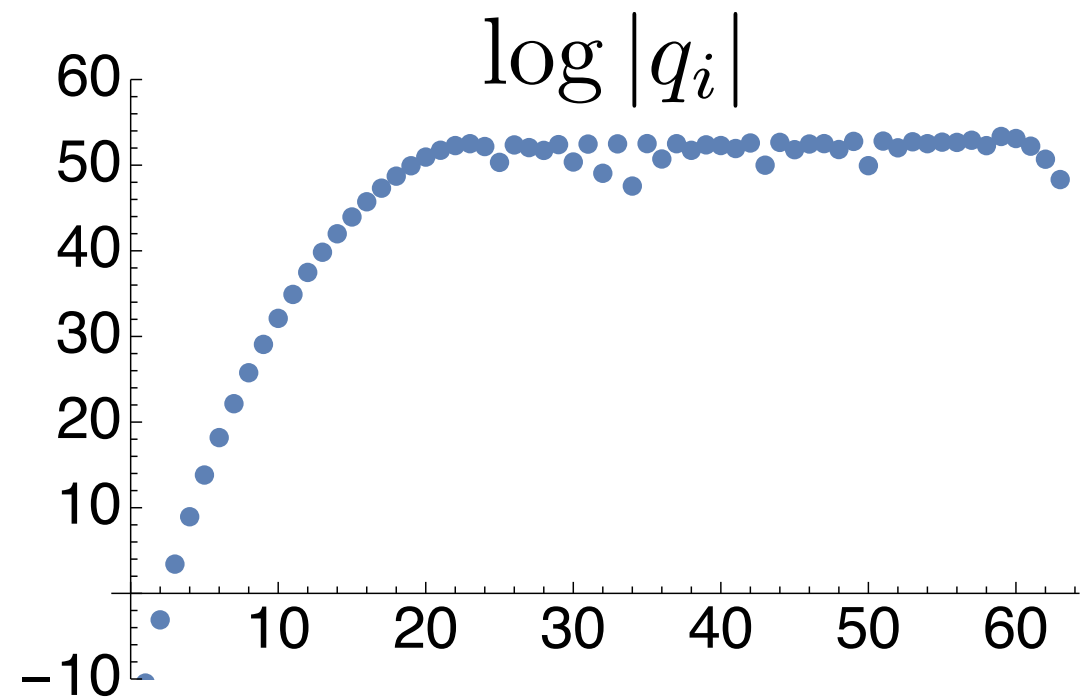
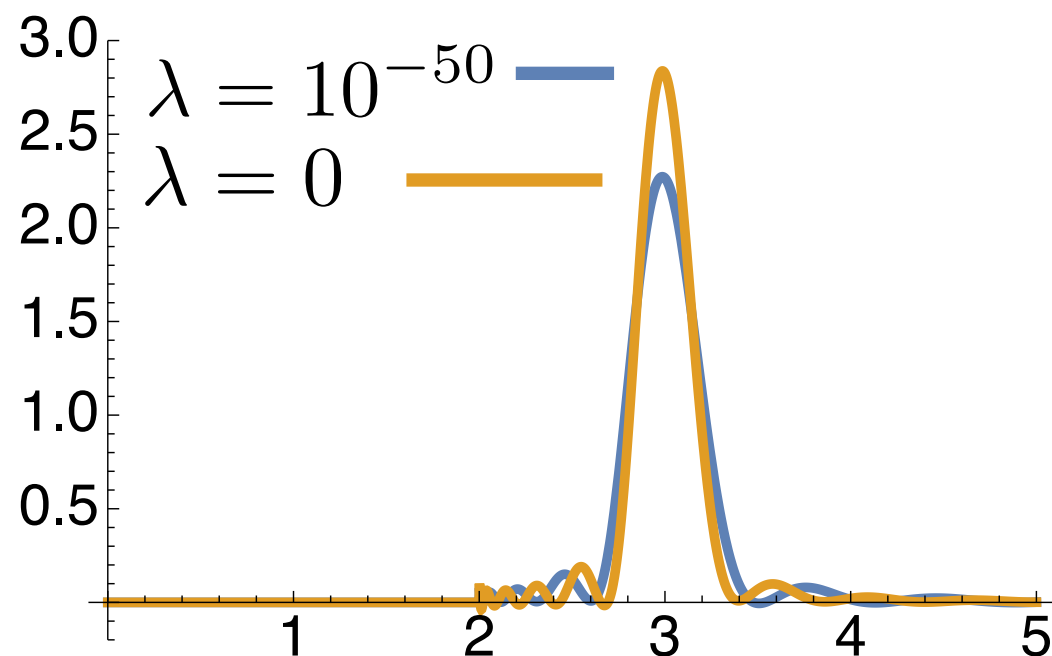
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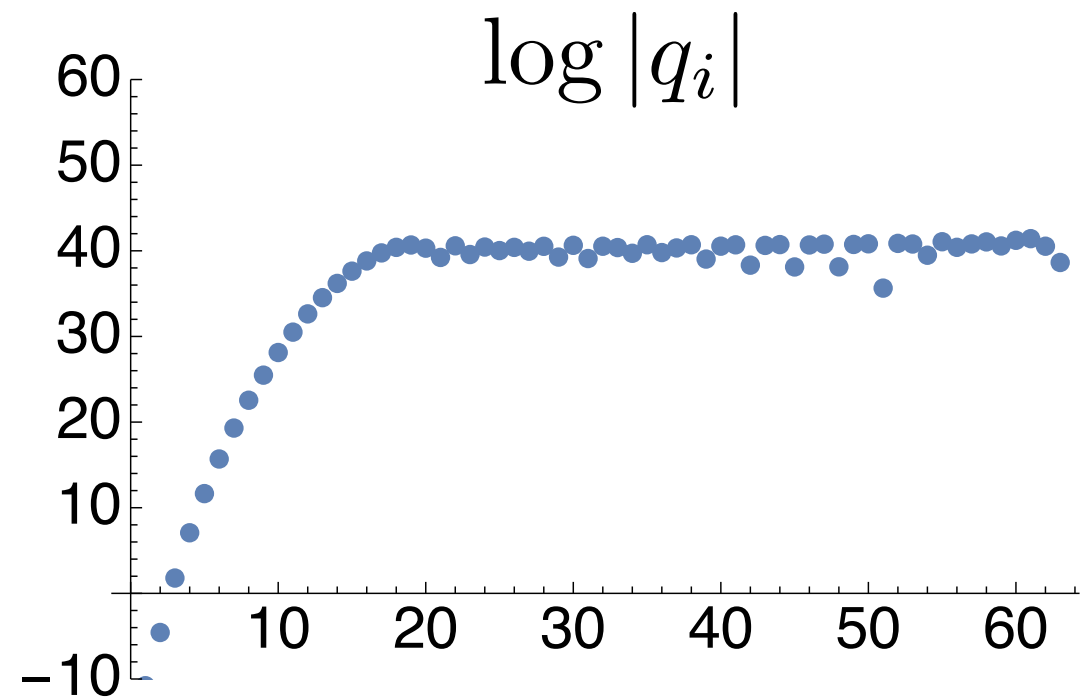
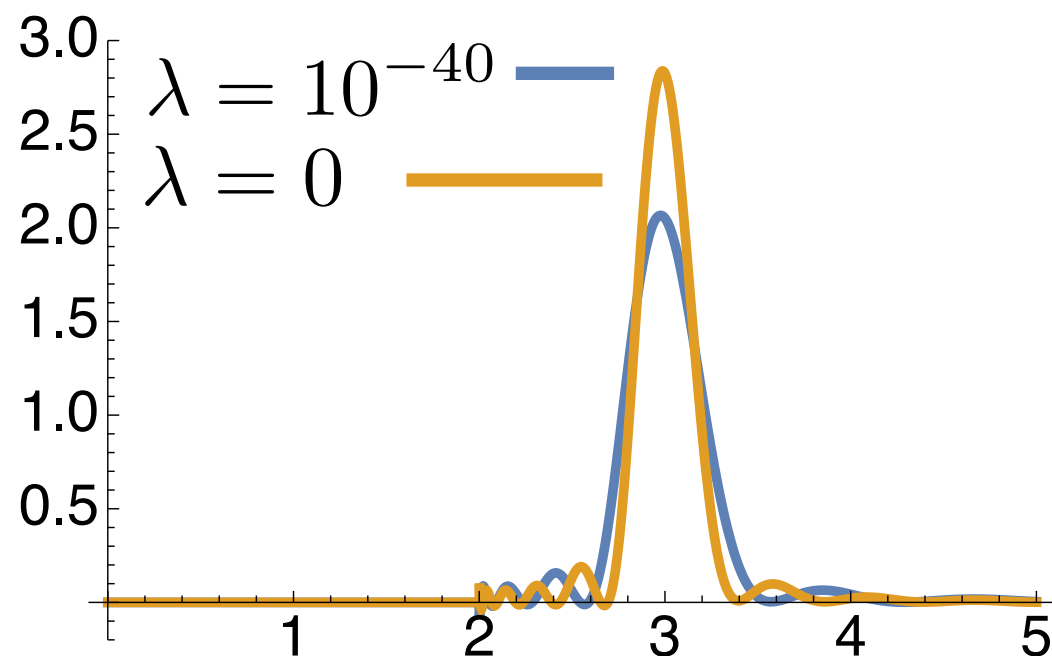
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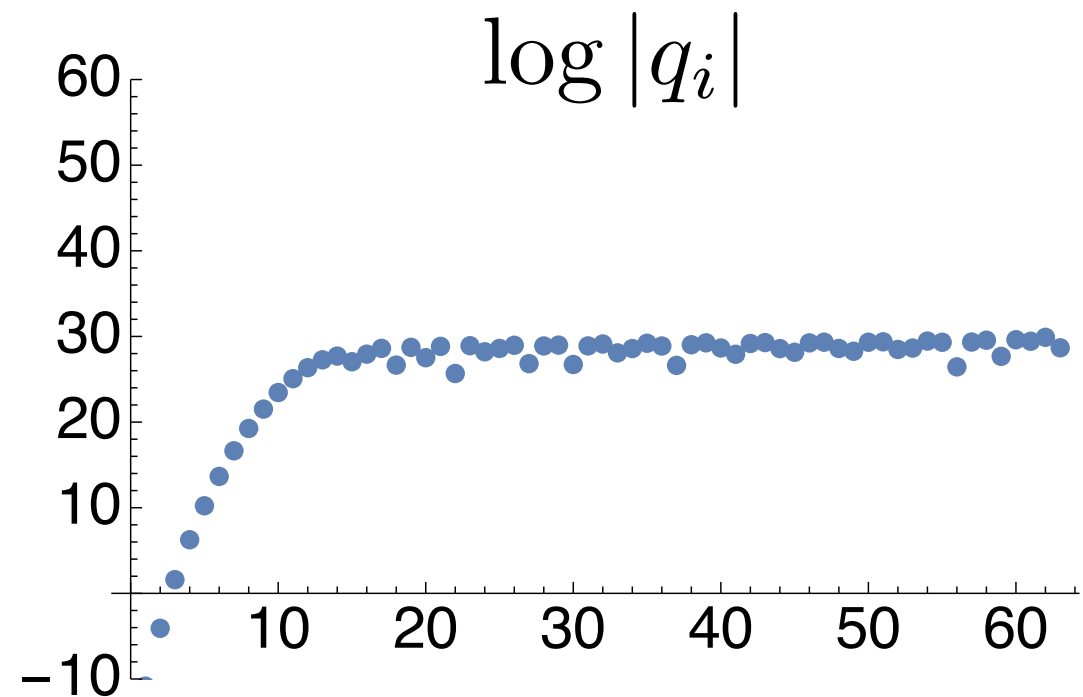
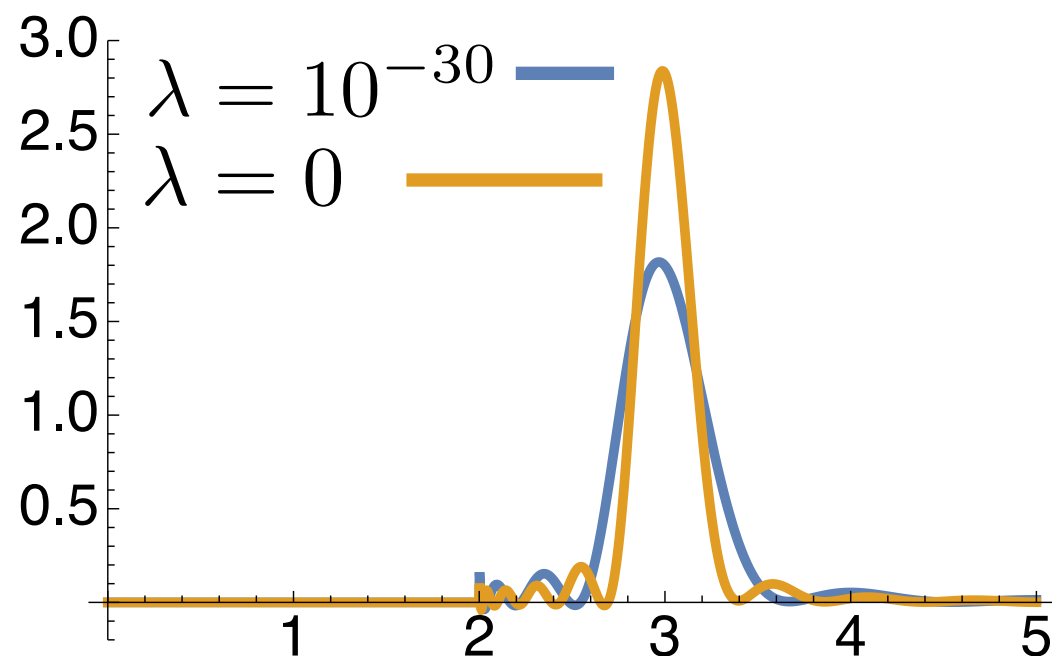
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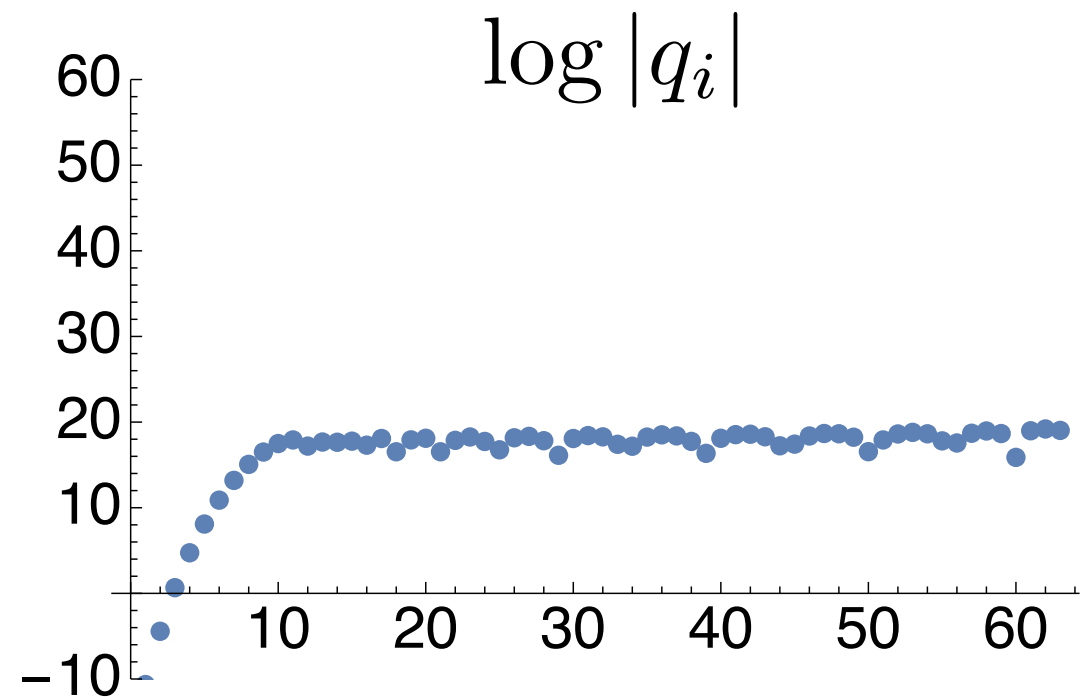
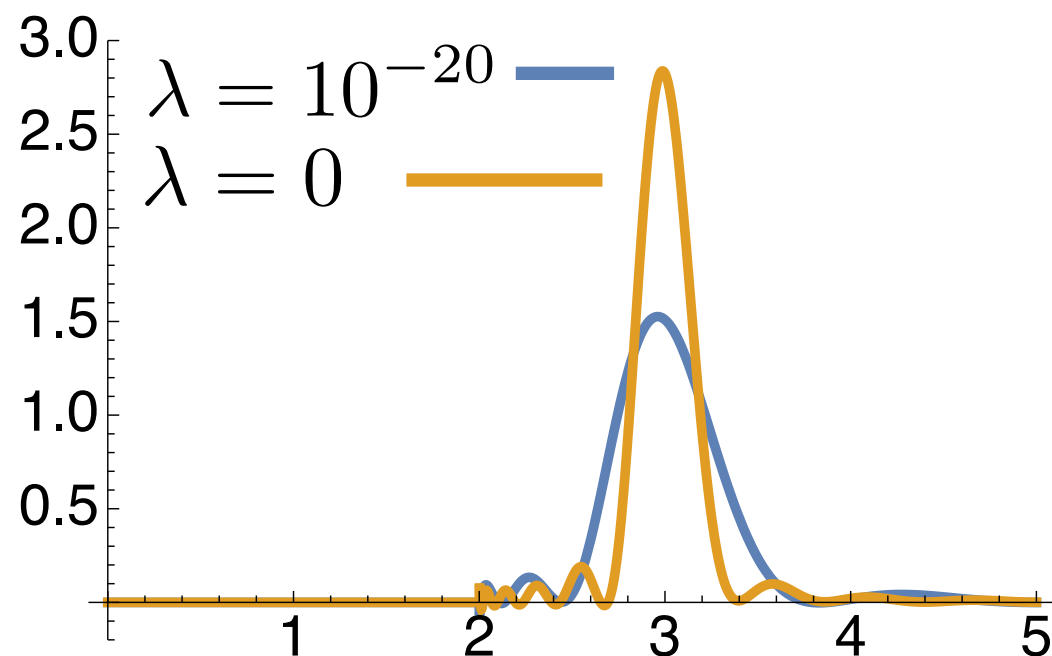
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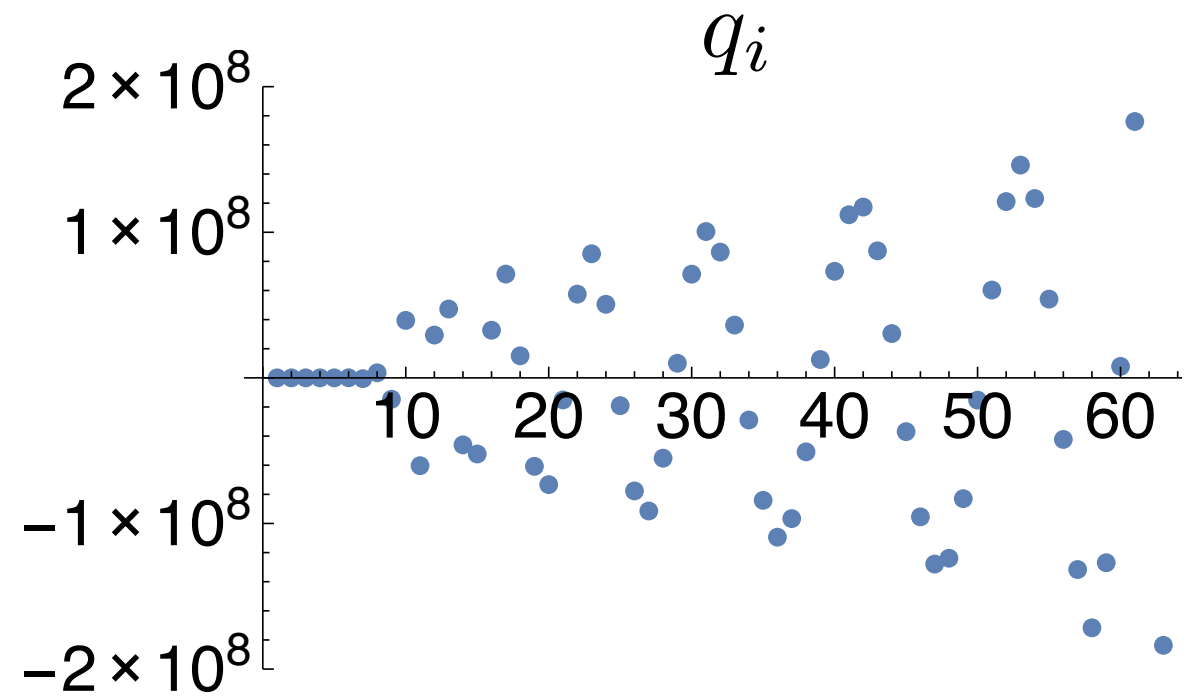
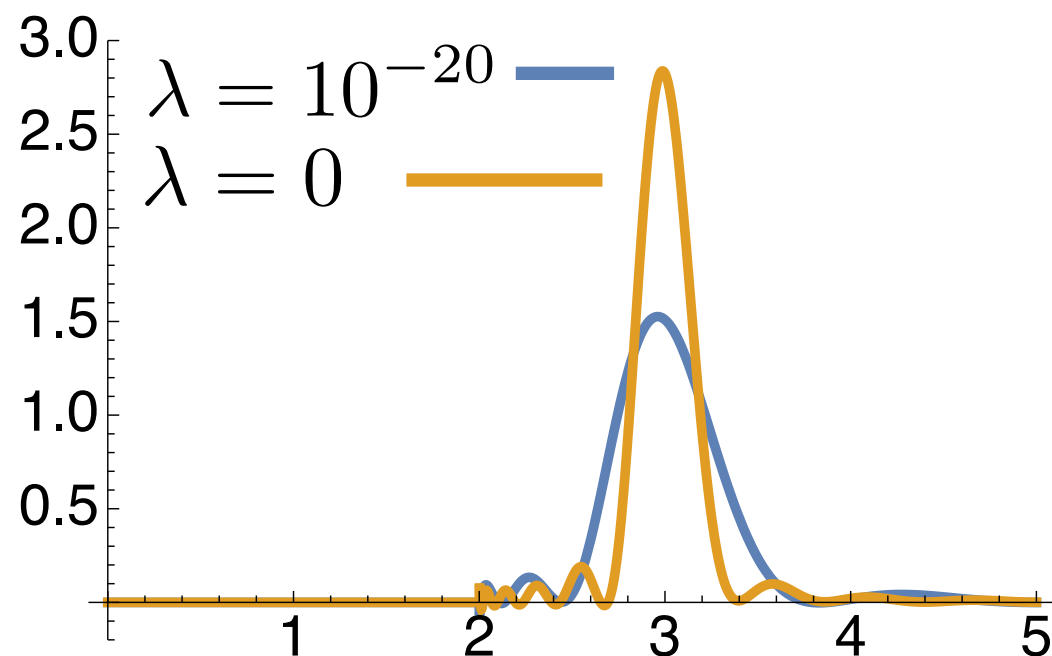
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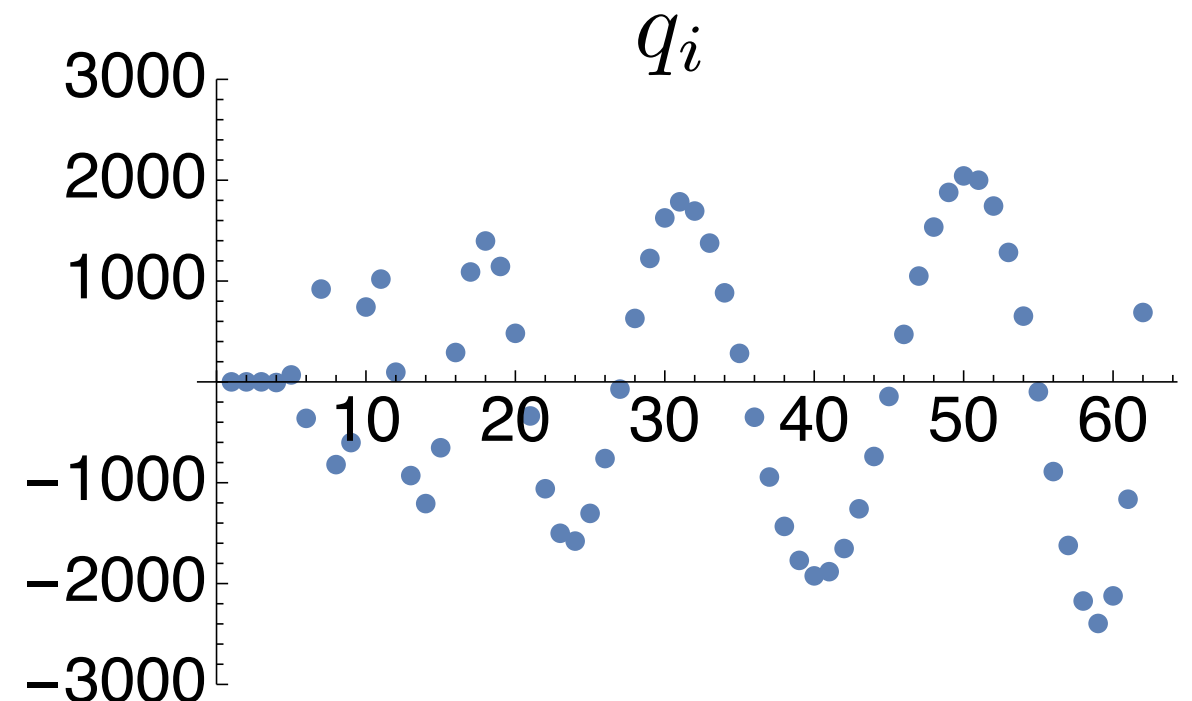
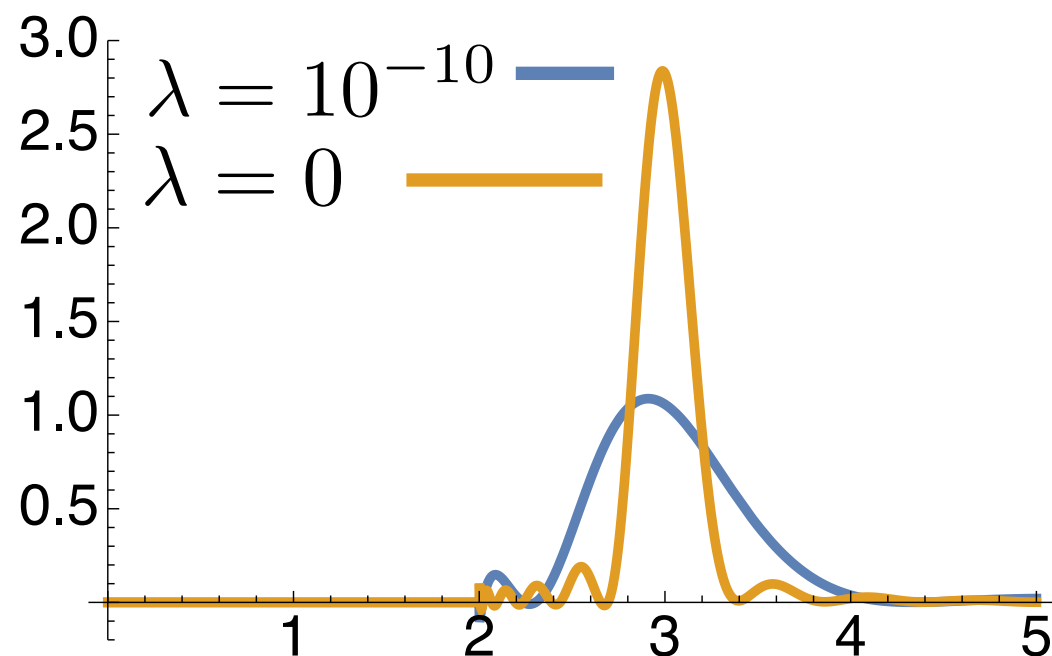
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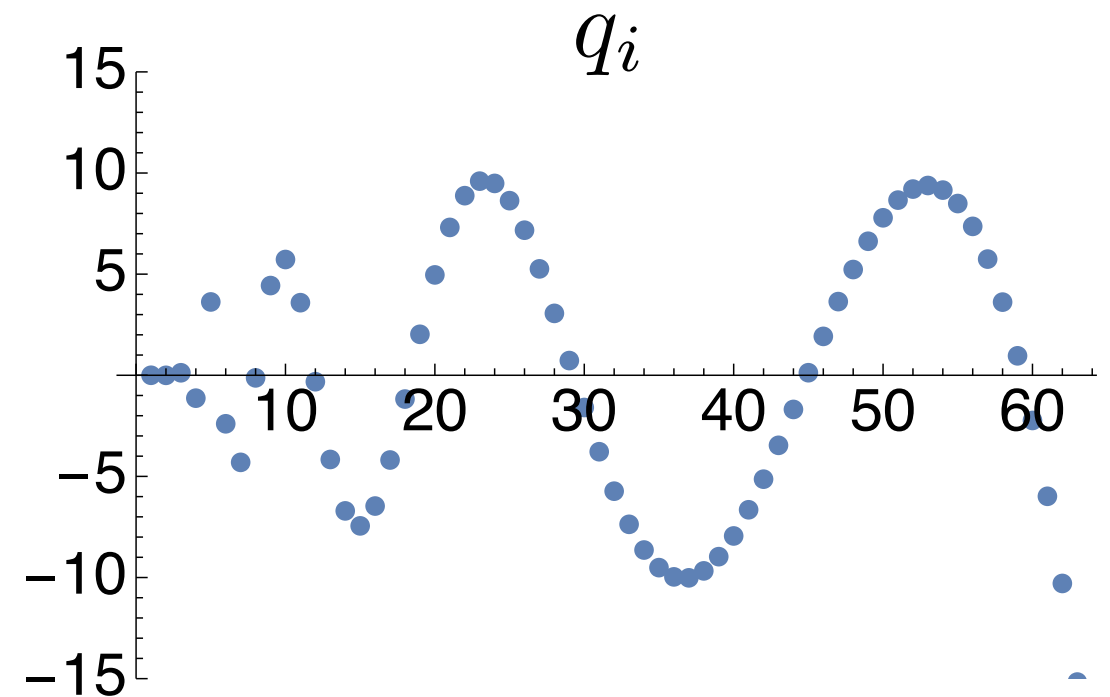
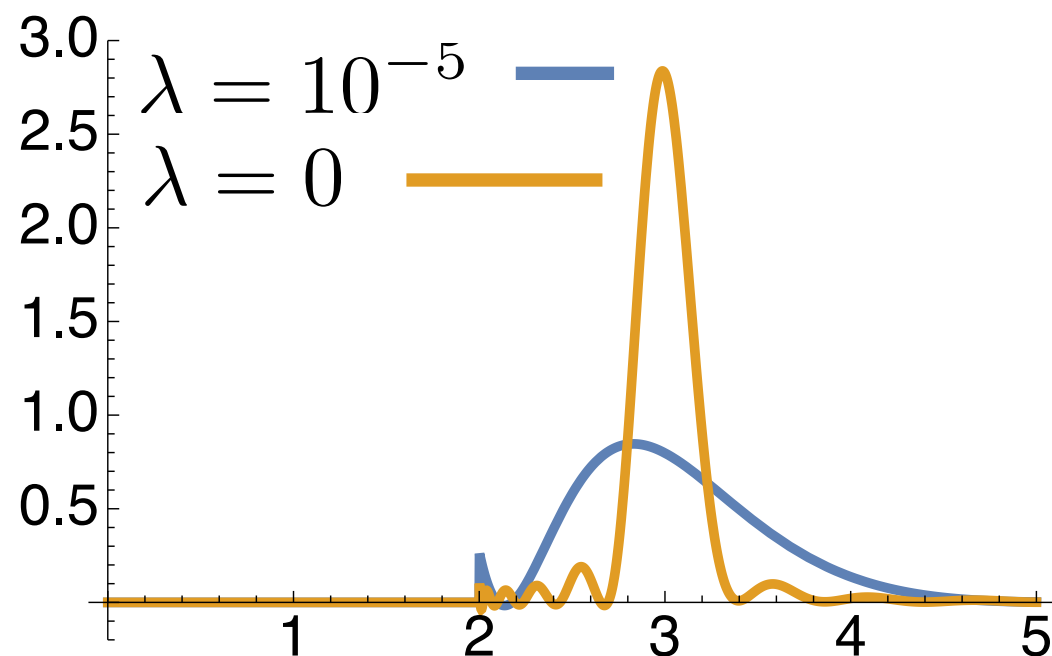
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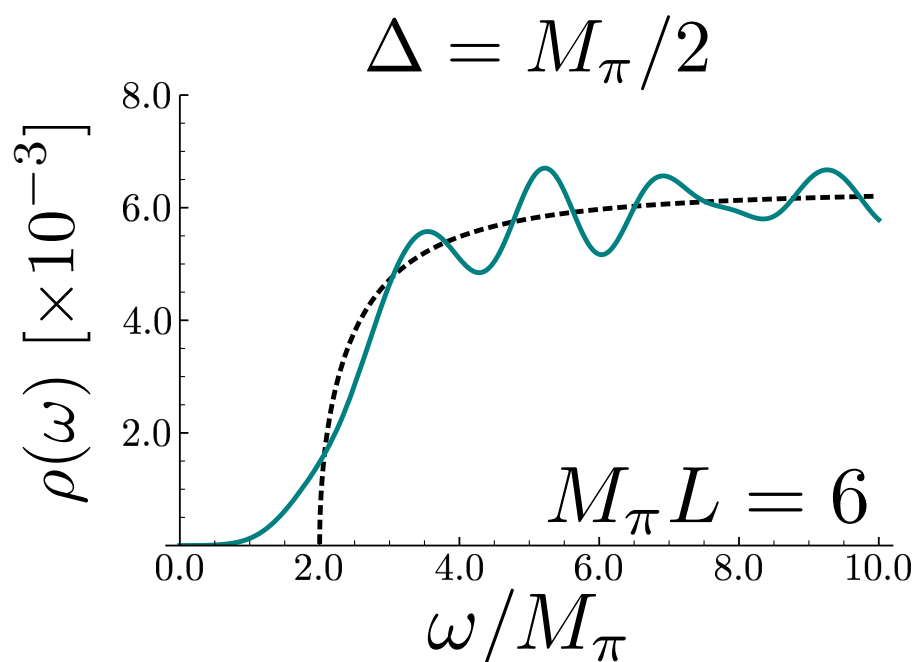
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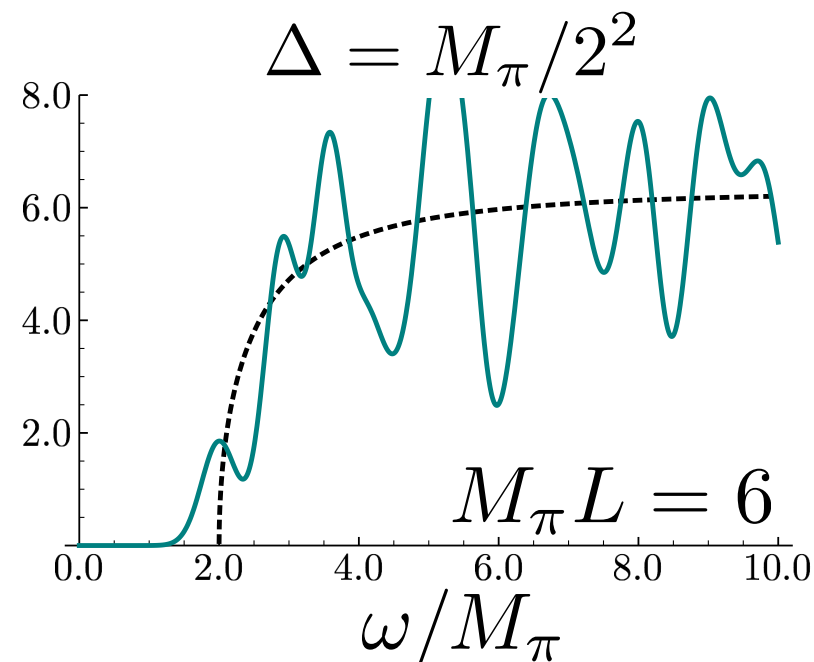
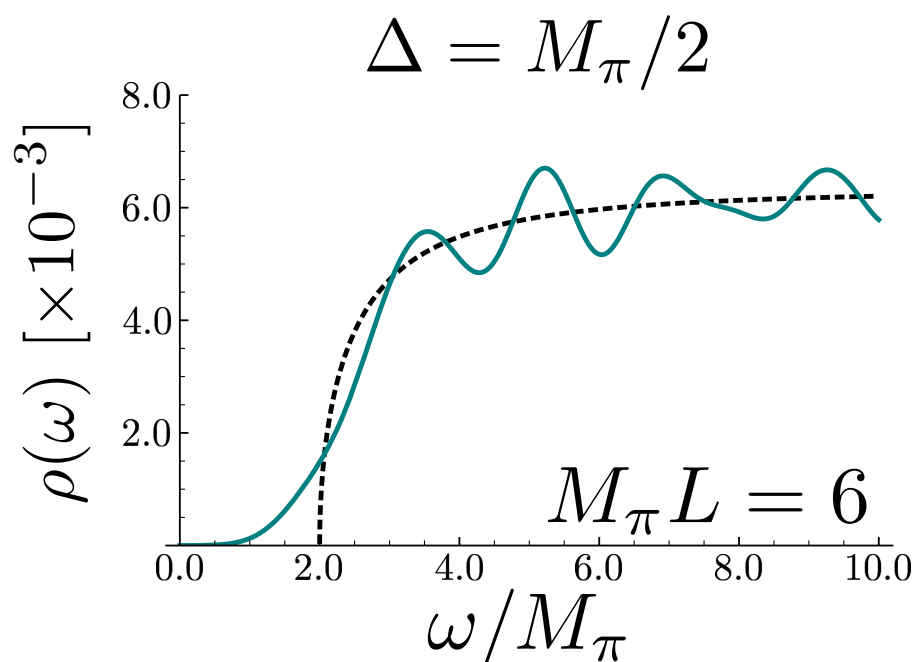
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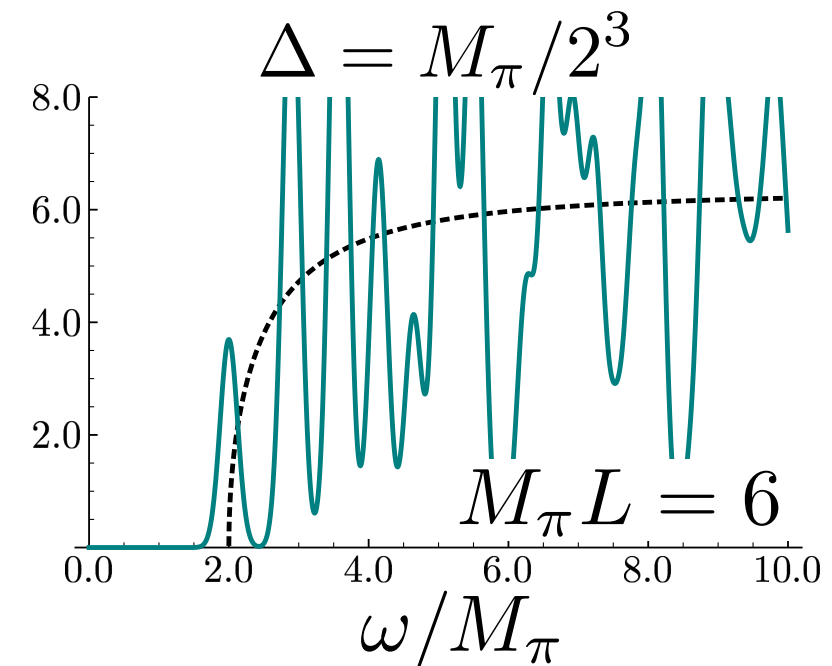
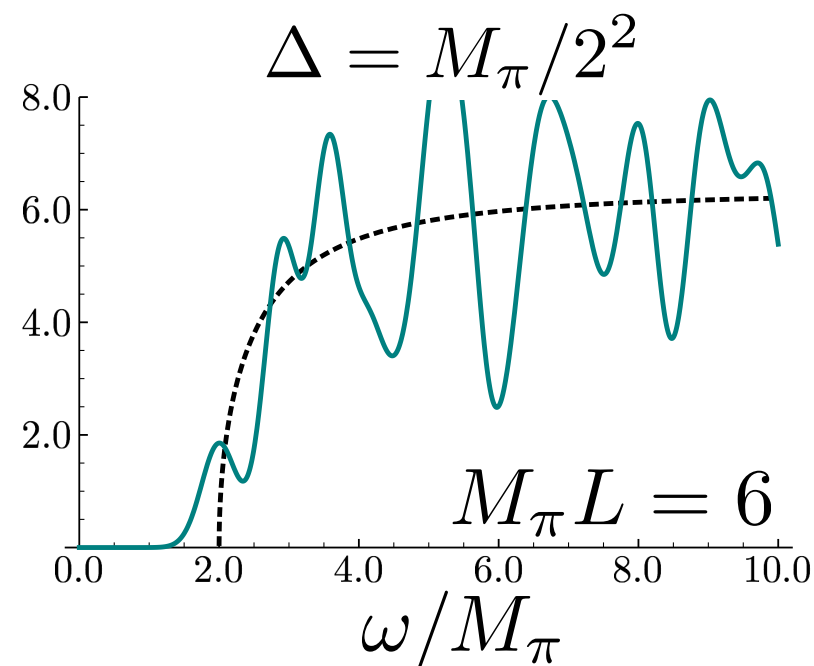
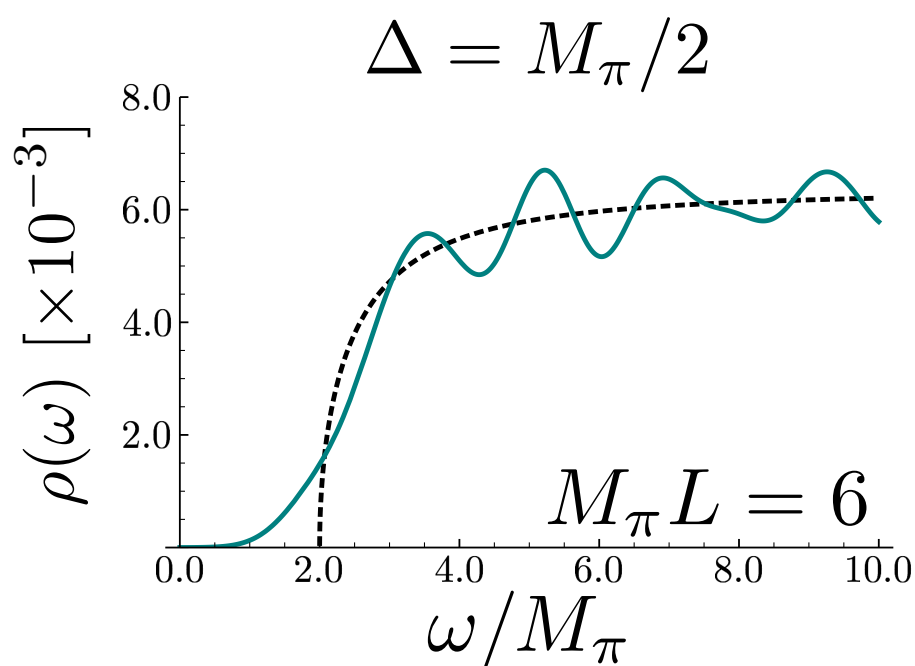
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**free, two-particle
spectral function**

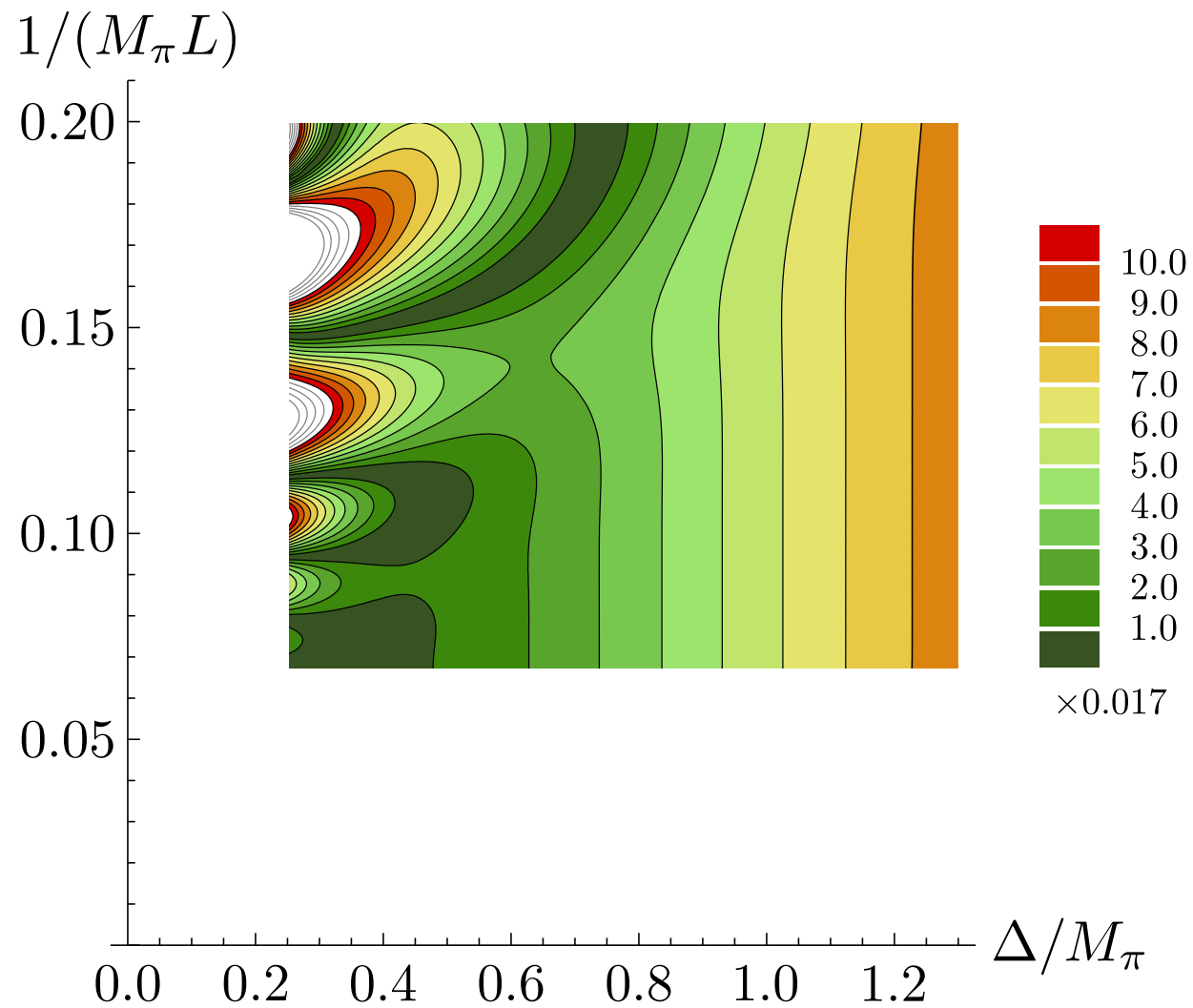
**Gaussian resolution
function**

Δ to 0 first gives a useless spectral function...



Ordered double limit: $\rho(\omega) = \lim_{\Delta \rightarrow 0} \lim_{L \rightarrow \infty} \hat{\rho}(\omega, L, \Delta)$

$$\frac{|\rho(M_K) - \hat{\rho}(M_K, L, \Delta)|}{\rho(M_K)}$$

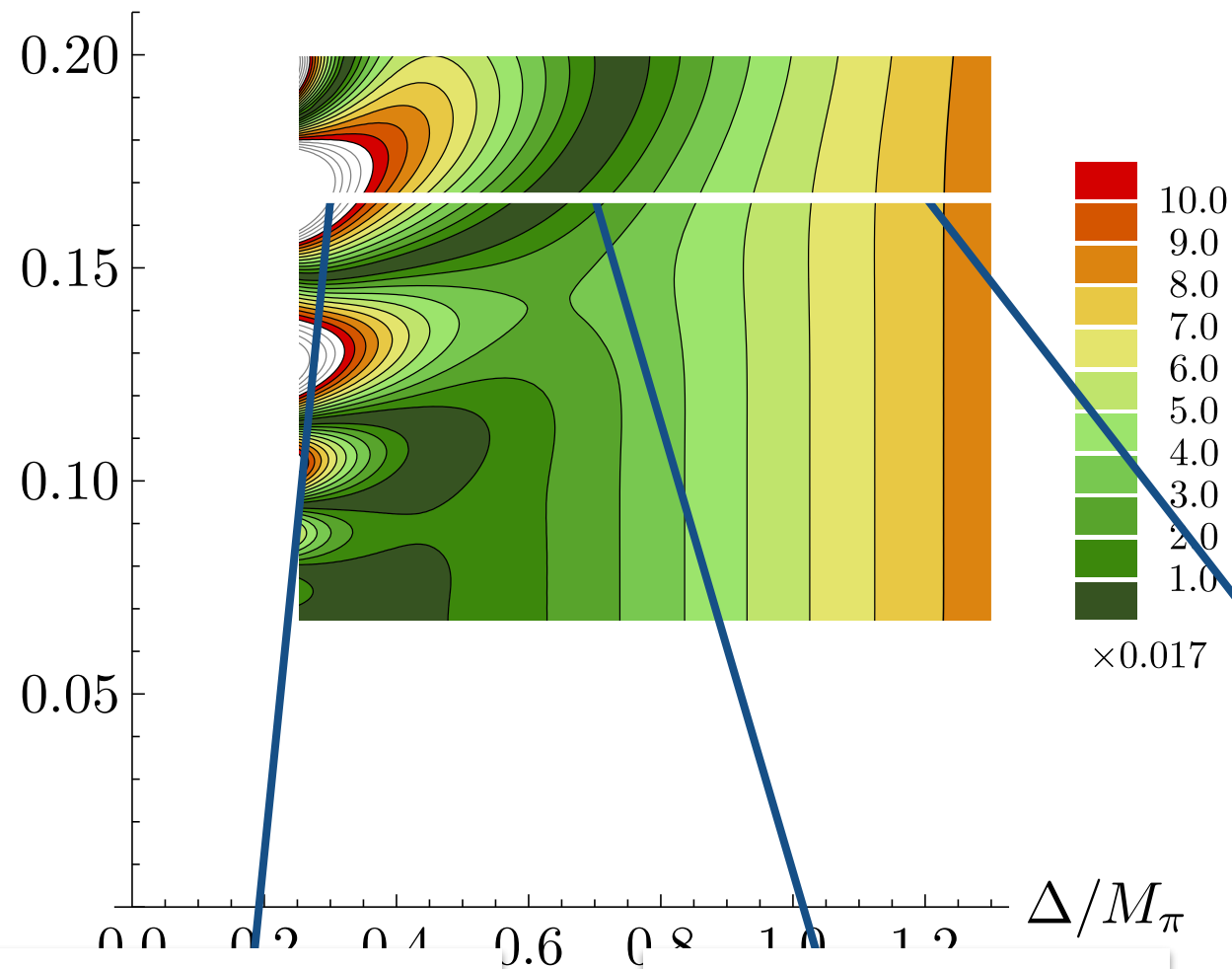


The goal is to access an optimal trajectory in the $(\Delta, 1/L)$ plane

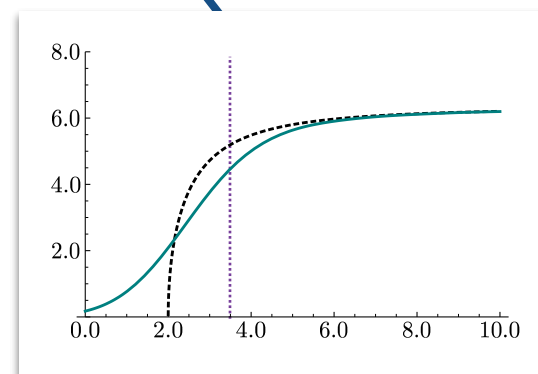
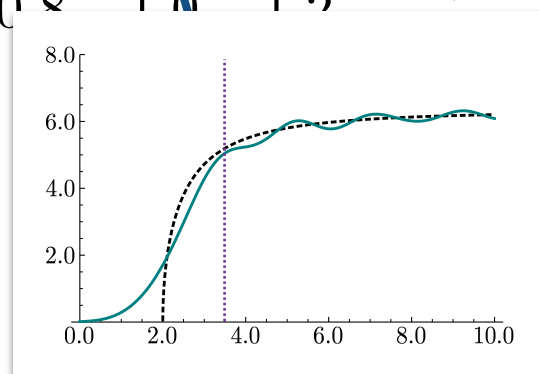
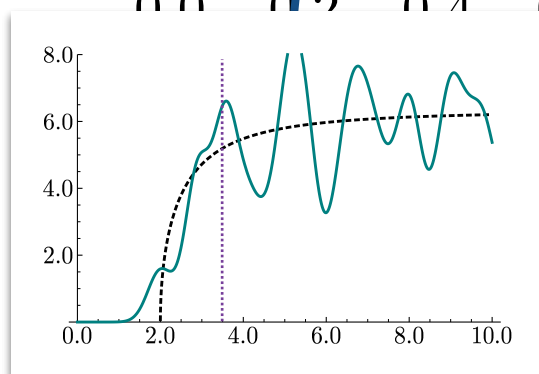
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$1/(M_\pi L)$



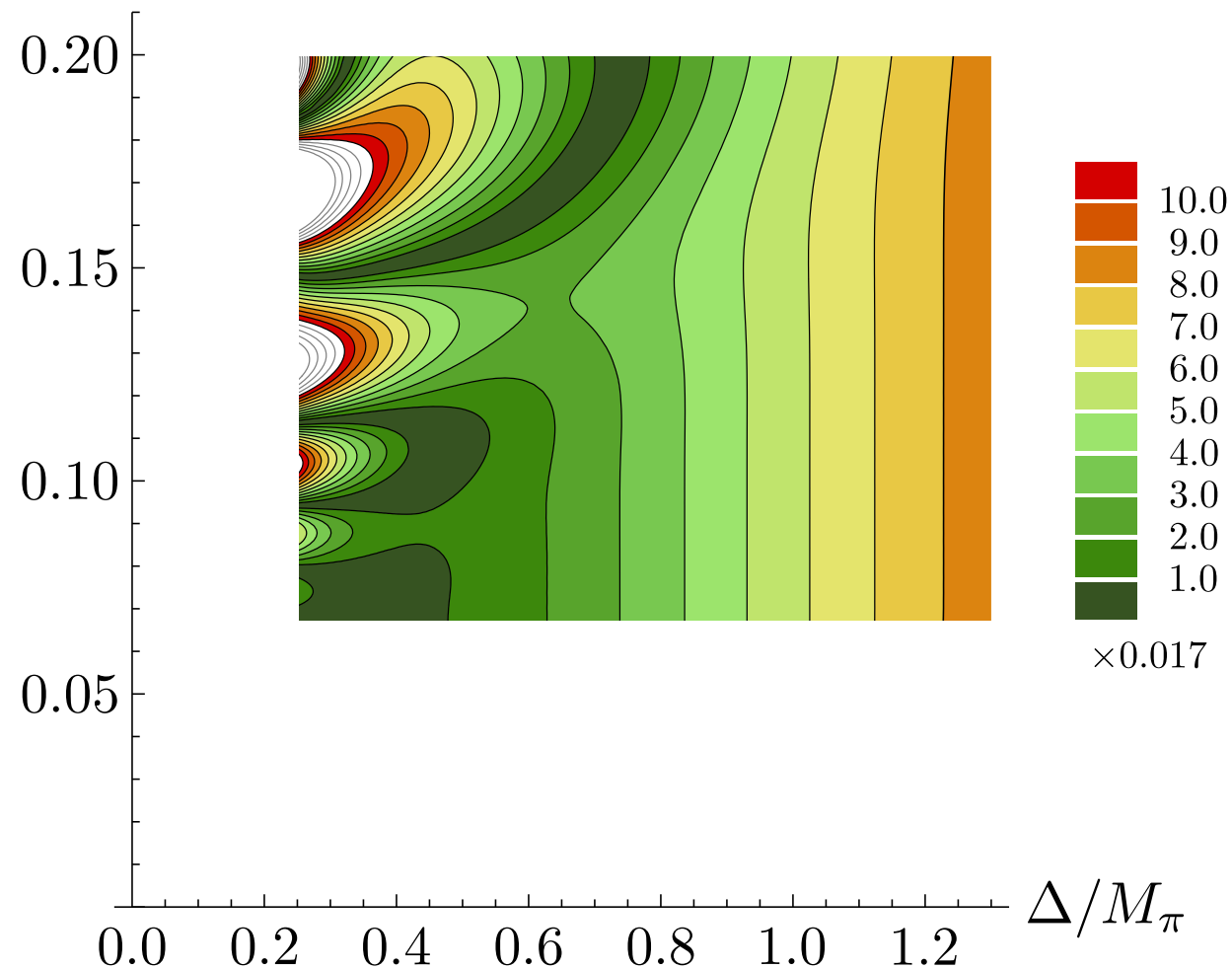
The goal is to access an **optimal trajectory** in the $(\Delta, 1/L)$ plane



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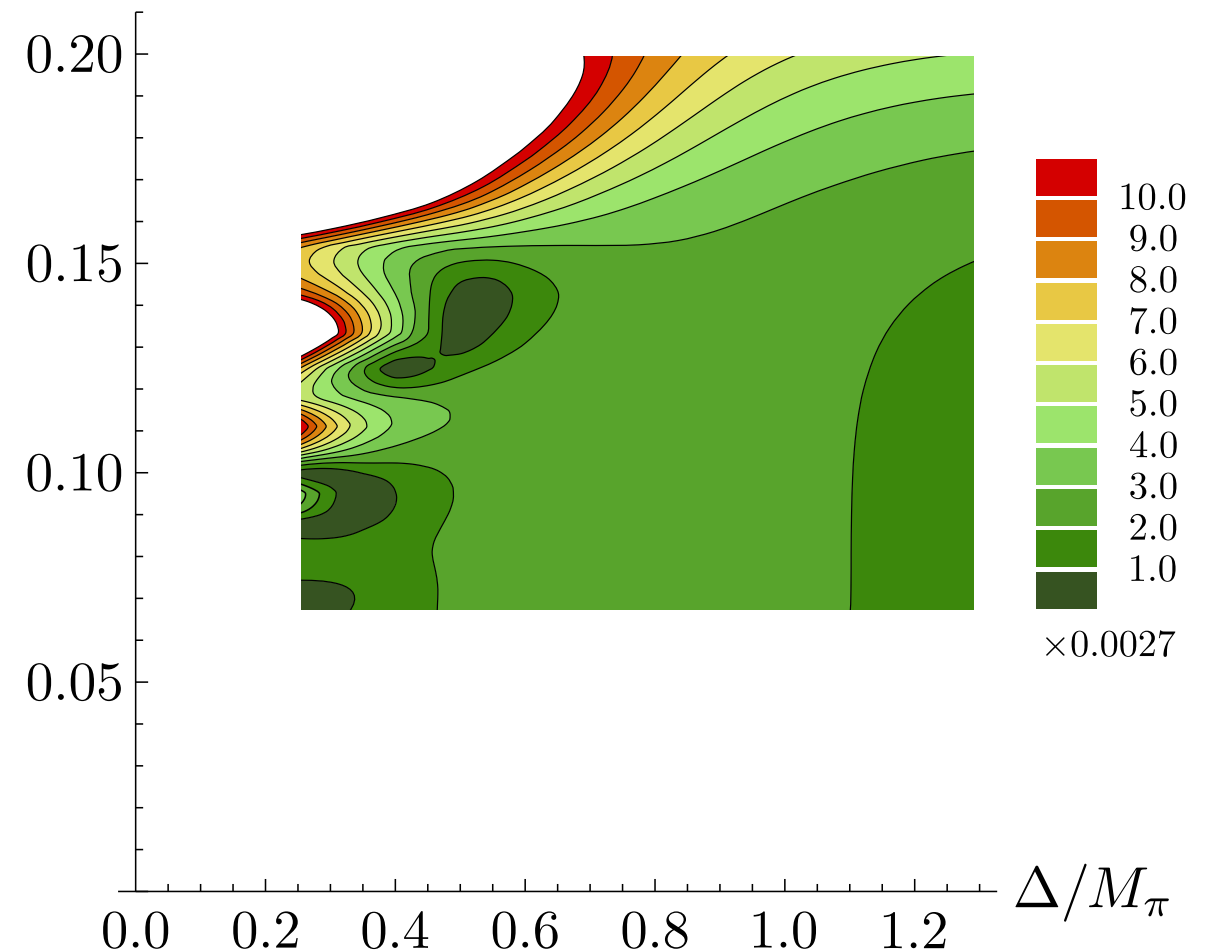
$$\frac{|\rho(M_K) - \hat{\rho}(M_K, L, \Delta)|}{\rho(M_K)}$$

$1/(M_\pi L)$



$$\frac{\left| \int^{4M_\pi} d\omega \rho(\omega) - \int^{4M_\pi} d\omega \hat{\rho}(\omega, L, \Delta) \right|}{\int^{4M_\pi} d\omega \rho(\omega)}$$

$1/(M_\pi L)$



Integrated spectral functions are generally more precise and less dependent on the choice of $(\Delta, 1/L)$

Testing the method...

To test the idea we studied a **toy system** with two open channels

$$\phi \rightarrow \pi\pi\pi, \quad K\bar{K} \quad 3M_\pi < 2M_K \quad \longrightarrow \quad \Gamma_\phi \propto \langle \phi | \mathcal{H}(q) \mathcal{H}(0) | \phi \rangle$$


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spectral function perturbatively

(2) Determine finite-volume
lattice correlator perturbatively

(3) Applied Backus Gilbert
using a correlation matrix
from lattice data

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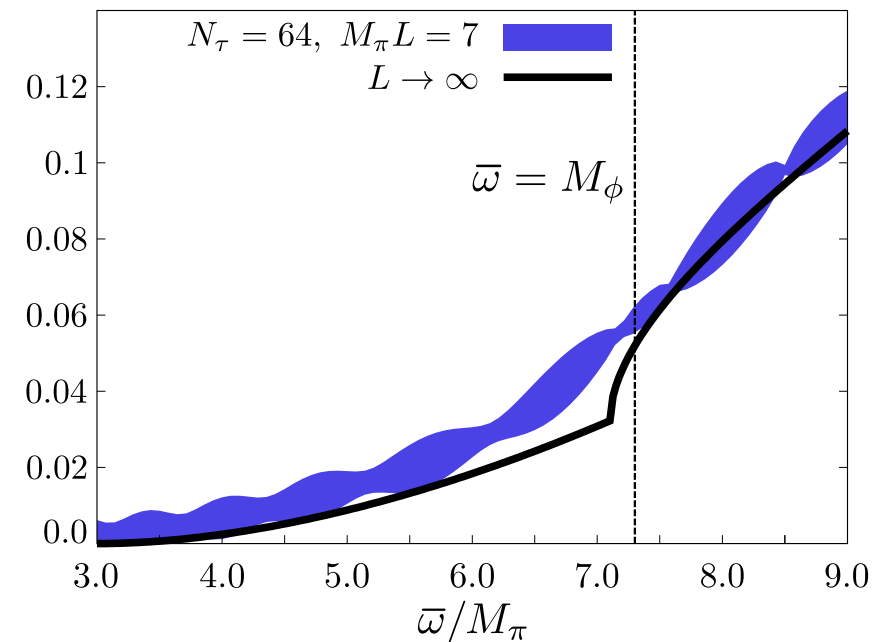
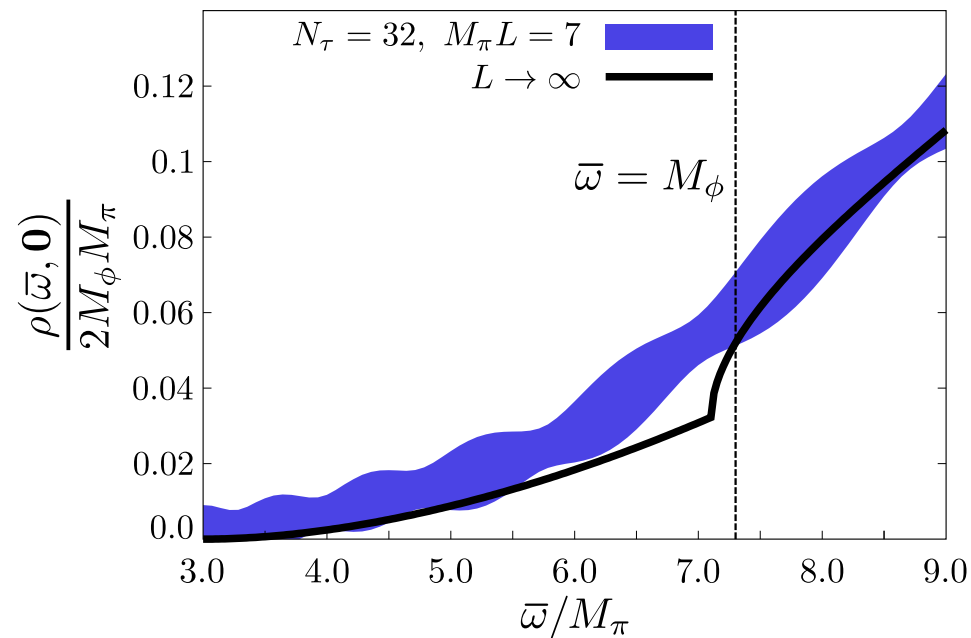
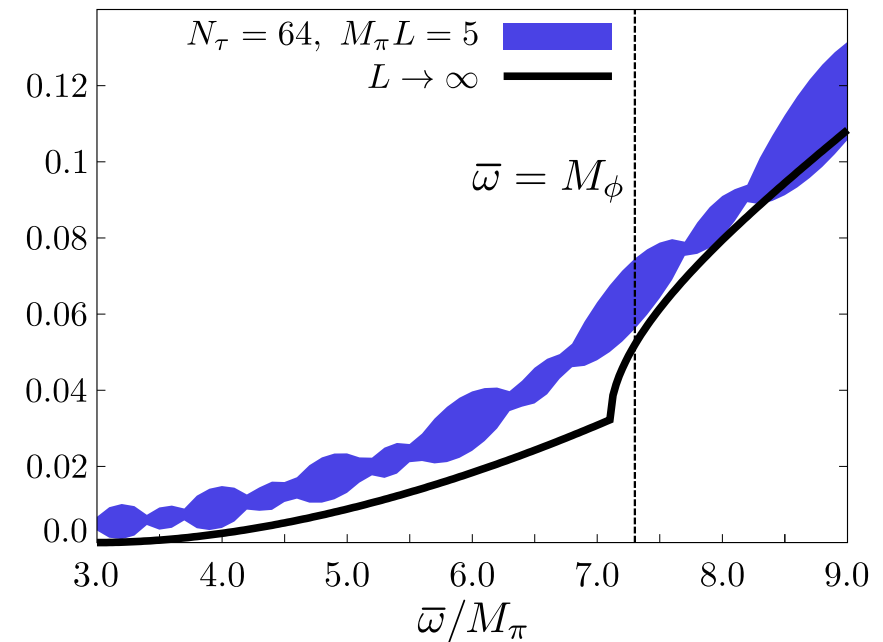
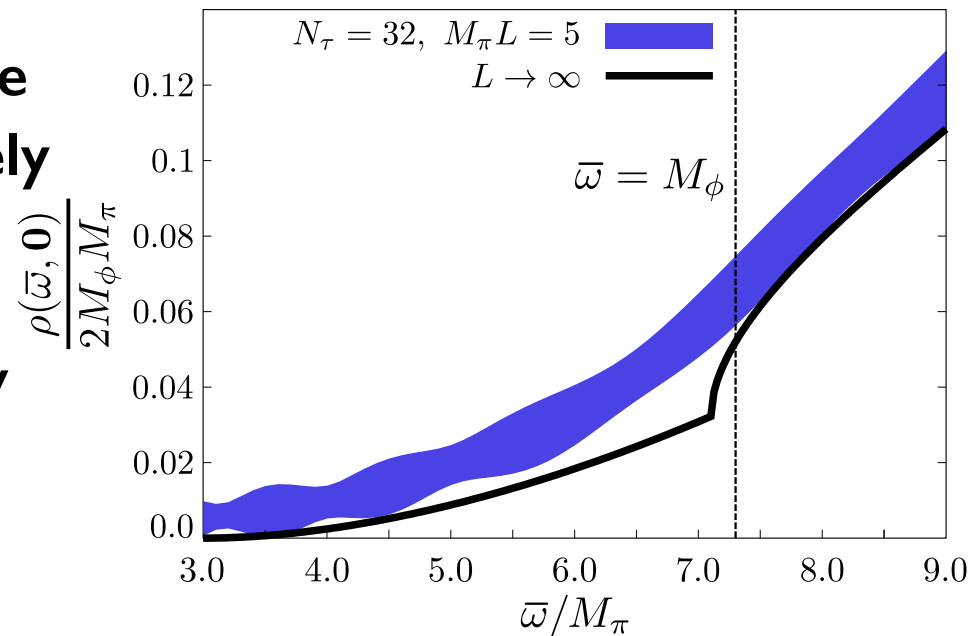
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figures by D. Robaina

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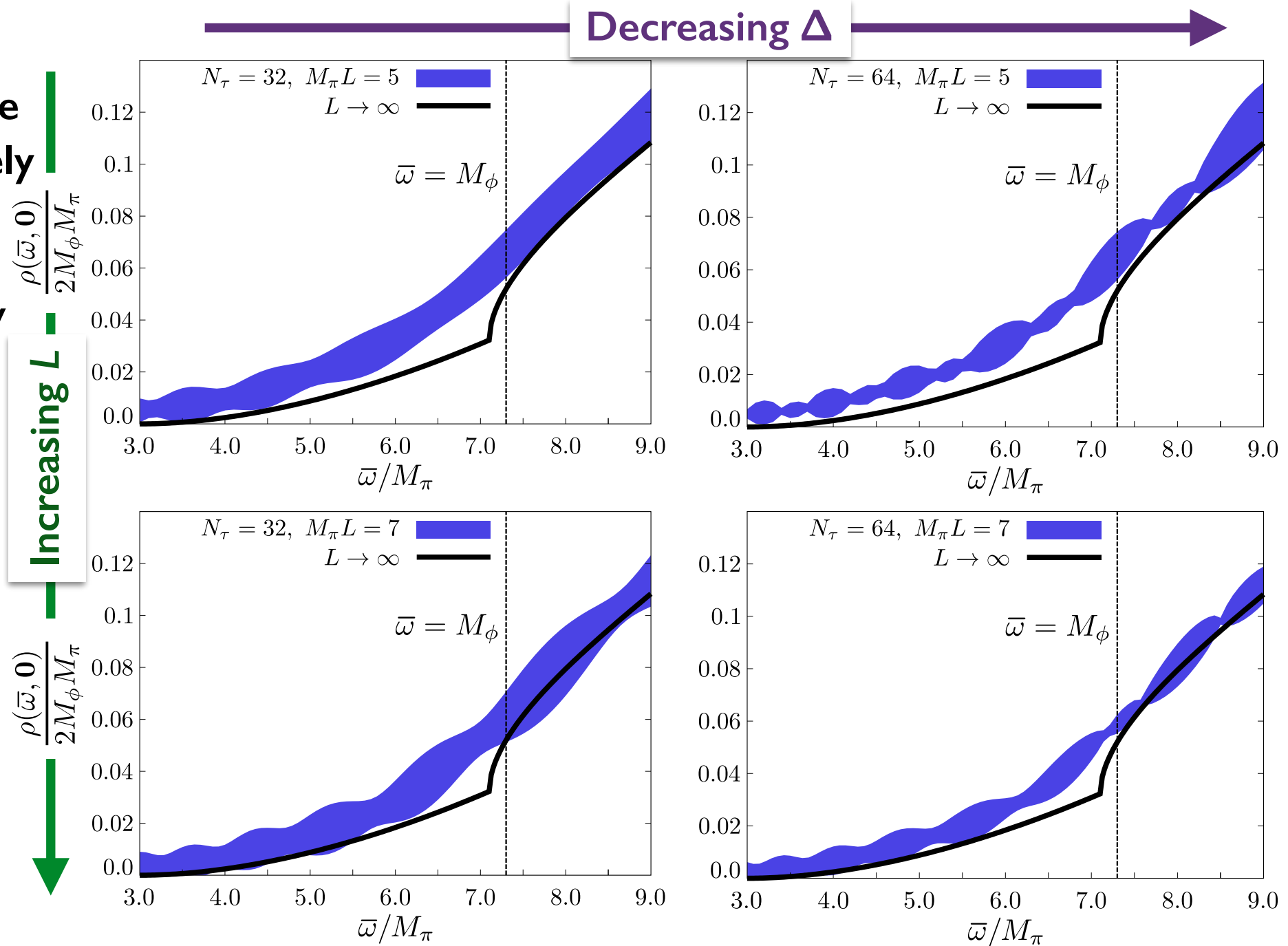
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figures by D. Robaina

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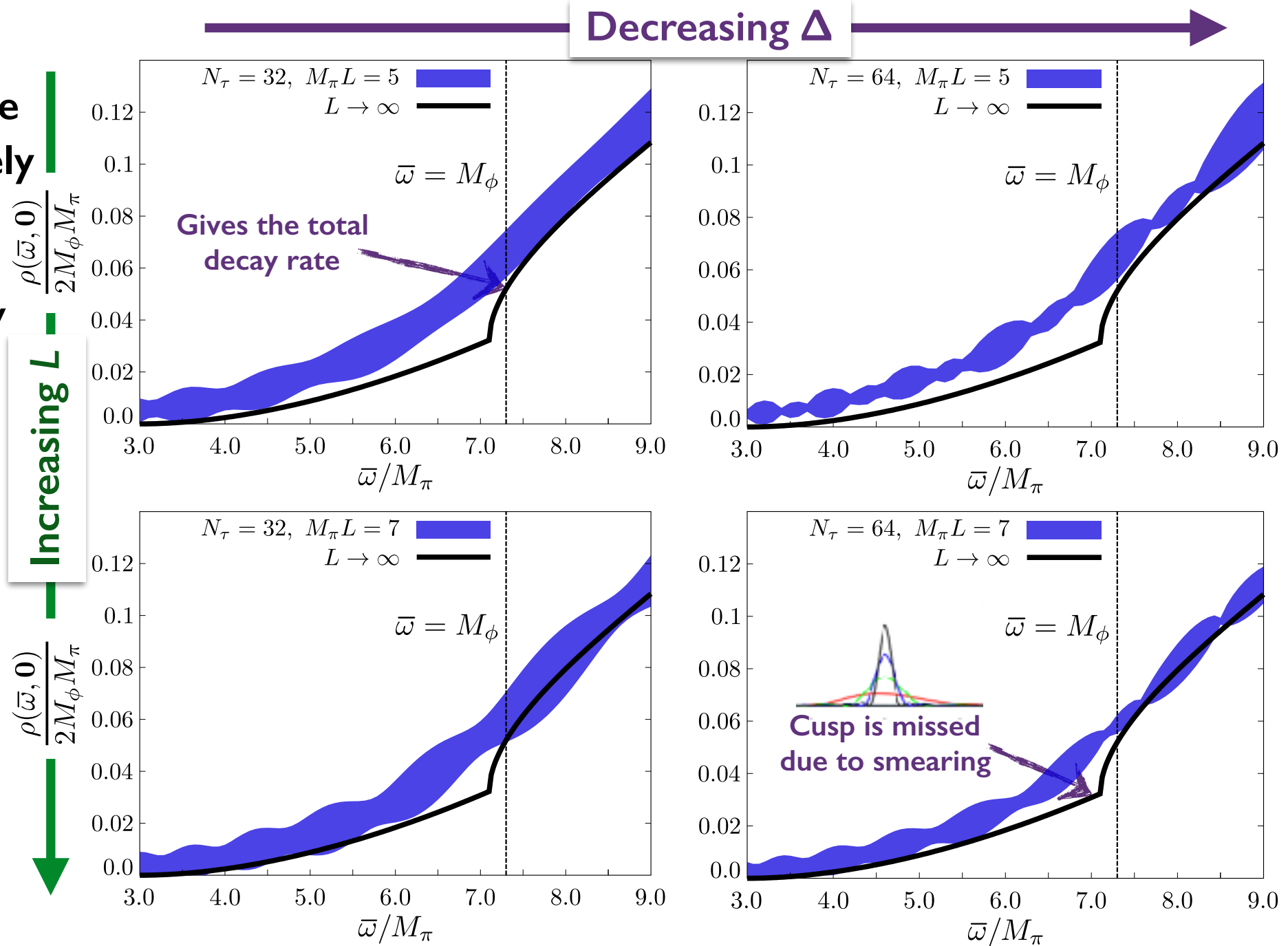
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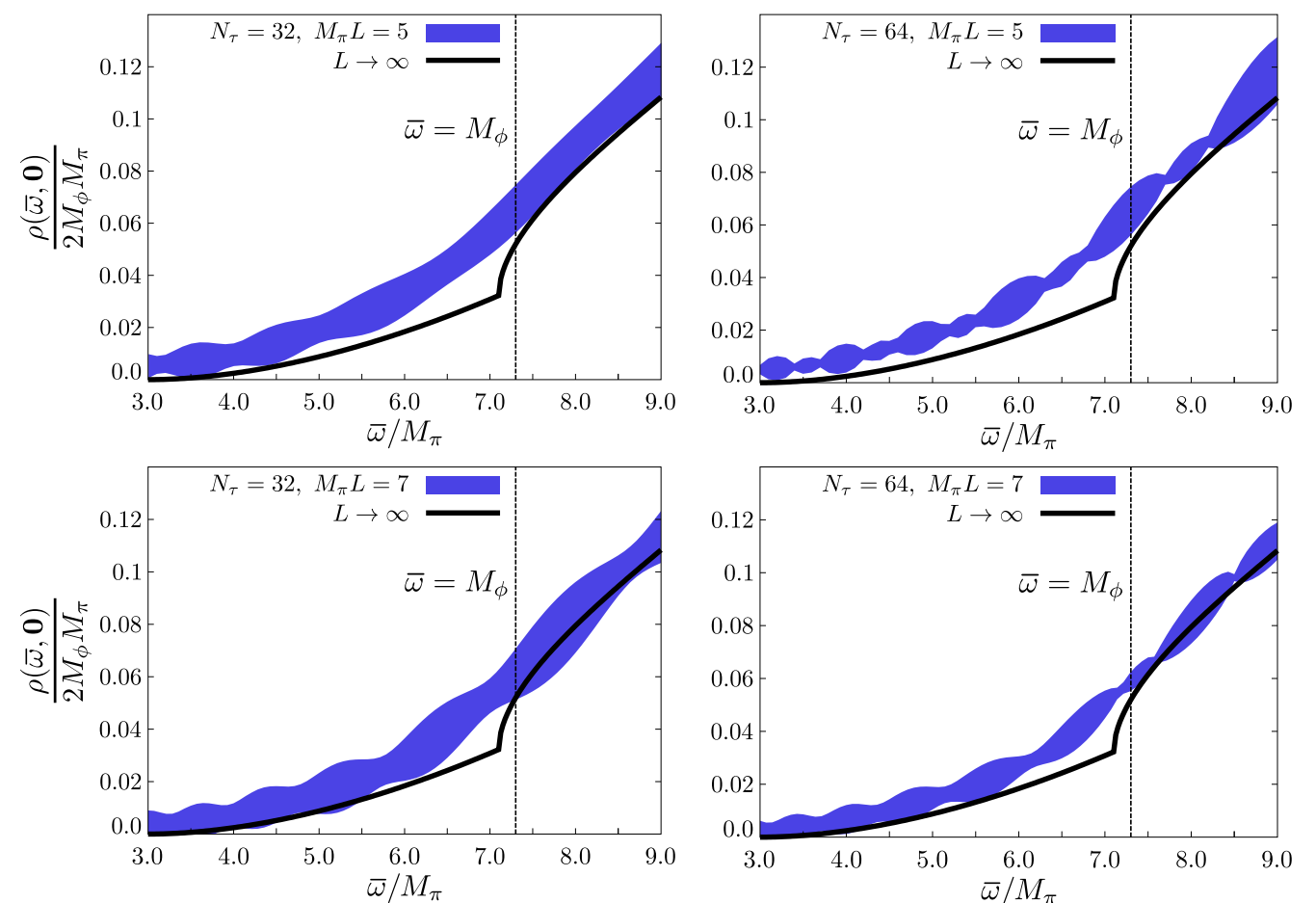
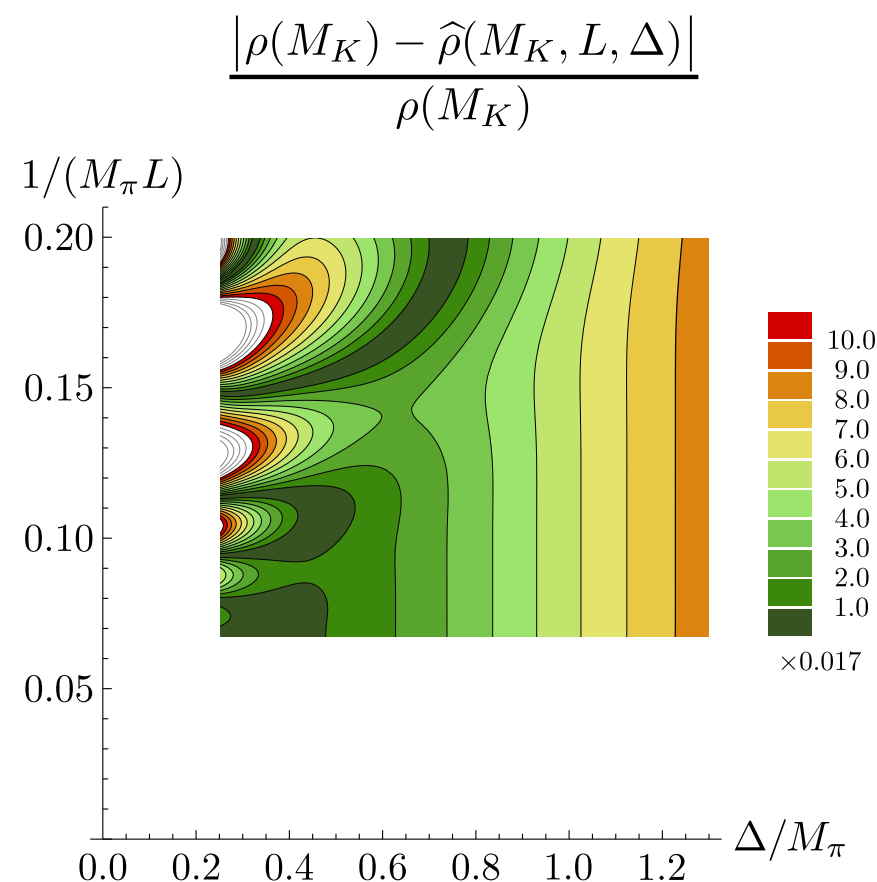


figures by D. Robaina

Still lots to do

- ☐ Develop formal strategies for estimating the ordered double limit
- ☐ Test the approach in a numerical lattice calculation
- ☐ Extend the approach to multiple incoming hadrons, other observables
- ☐ Investigate the role of QED
- ☐ Explore combining this with the exclusive ideas (Lüscher and Lellouch-Lüscher)

Big Picture: New idea that needs to be explored



Thanks for listening!