The hadronic light-by-light scattering contribution to the muon $g-2$ from lattice QCD

Antoine Gérardin

Institut für Kernphysik, Johannes Gutenberg-Universität Mainz

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

PRISMA

Frontiers in Lattice Quantum Field Theory

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$(g-2)_\mu$: current status

Contribution	$a_\mu \times 10^{11}$	
- QED (leptons, $5^{\rm th}$ order)	$116\ 584\ 718.846 \pm 0.037$	[Aoyama et al. '12]
- Electroweak	153.6 ± 1.0	[Gnendiger et al. '13]
- Strong contributions		
HVP (LO)	$6\ 869.9 \pm 42.1$	[Jegerlehner '15]
HVP (NLO)	-98 ± 1	[Hagiwara et al. 11]
HVP (NNLO)	12.4 ± 0.1	[Kurtz et al. '14]
HLbL	102 ± 39	[Jegerlehner '15, Nyffeler '09]
Total (theory)	$116\ 591\ 811 \pm 62$	
Experiment	$116\ 592\ 089 \pm 63$	

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- $\Delta a_\mu = \frac{(g-2)_\mu}{2} = a_\mu^{\text{exp}} - a_\mu^{\text{th}} = 278 \times 10^{11}$

  $\rightarrow$ $\sim 3-4\ \sigma$ discrepancy between experiment and theory
- Future experiments at Fermilab and J-PARC : reduction of the error by a factor of 4
- Theory error is dominated by hadronic contributions
- Motivation for lattice QCD approach : $\rightarrow$ first principle determination

  $\rightarrow$ No reliance on experimental data

  $\rightarrow$ A precision of $\sim 20\ \%$ for HLbL would already be a big step

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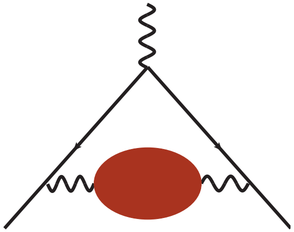
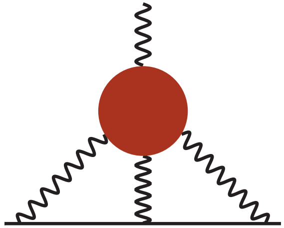
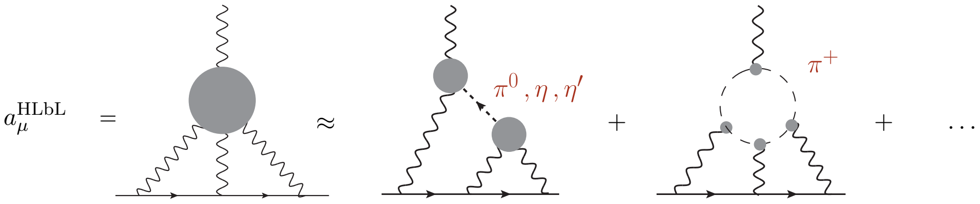
Hadronic Vacuum Polarisation (HVP, α^2)



Hadronic Light-by-Light scattering (HLbL, α^3)



Previous estimates : model calculations



[de Rafael '94]

1) Chiral counting
2) N_c counting

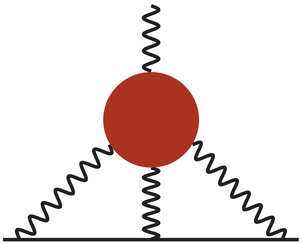
[extracted from A. Nyffeler's slide], units : $a_\mu \times 10^{11}$

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

1) Pseudoscalar contributions dominate numerically : transition form factors
2) Glasgow consensus : $a_\mu^{\rm HLbL} = (105 \pm 26) \times 10^{-11}$
3) Results are in good agreement but errors are difficult to estimate (model calculations)

New strategies : first principle determinations of HLbL



Dispersive approach

[Colangelo et al. '14, '15], [Pauk, Vanderhaeghen '14]

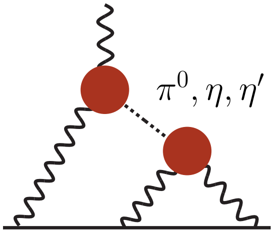
- Similar to the HVP, but more difficult :
  - → Many dispersion relations (19 vs 1 for HVP!)
  - → Experimental data are often missing
- LQCD can provide inputs
  - → pion-pole contribution (dominant), TFFs

Direct lattice calculation

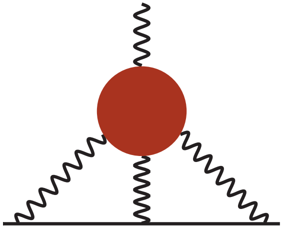
- 4-pt correlation function
  - → HVP : only a 2-pt correlation function
  - → very challenging
  - → but $\mathcal{O}(10\ \%)$ precision needed
- Two groups :
  - [RBC/UKQCD] [Mainz]

Outline

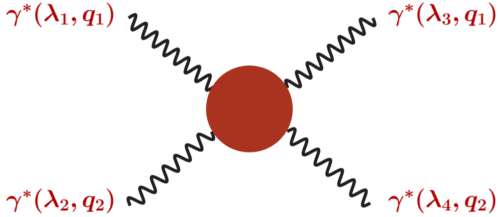
- **Pion-pole contribution on the lattice**
  - ↪ Dominant contribution to the HLbL scattering in $(g-2)_\mu$
  - ↪ Can be used to estimate FSE in the full HLbL calculation
  - ↪ First principle calculation



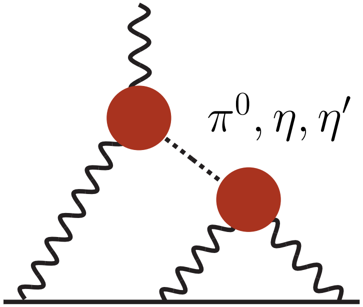
- **Direct lattice QCD calculation**
  - ↪ Only one collaboration has published results so far [Blum et. al 14', 16']
  - ↪ I will present the Mainz strategy



- **HLbL forward scattering amplitudes**
  - ↪ Full HLbL amplitudes contain more info than just a_μ
  - ↪ Extract information about single-meson transition form factor



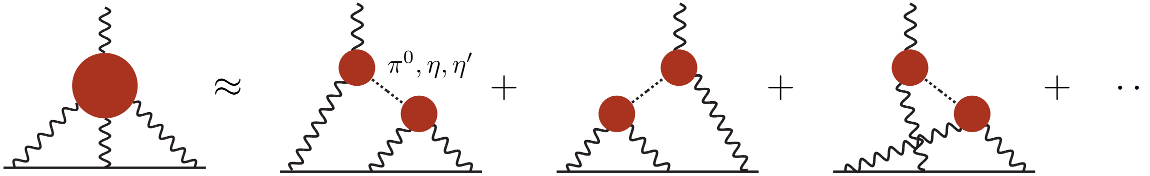
The pion-pole contribution



In collaboration with Harvey Meyer and Andreas Nyffeler

The pion-pole contribution

[Jegerlehner & Nyffeler '09]

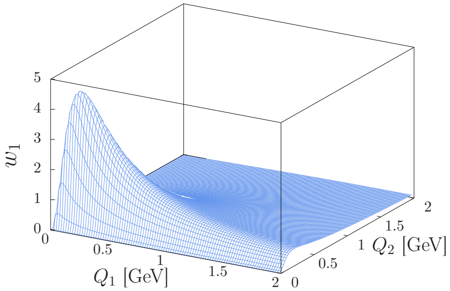


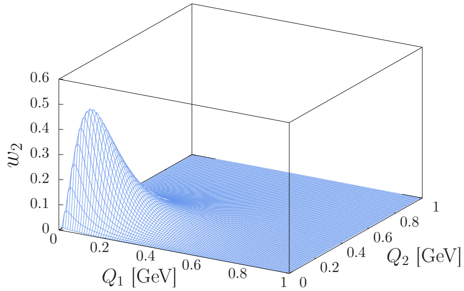
$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) \; + $$
$$w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

→ Product of one single-virtual and one double-virtual transition form factors (spacelike virtualities)

→ $w_{1,2}(Q_1,Q_2,\tau)$ are known model-independent weight functions

→ Weight functions are concentrated at small momenta below 1 GeV (here for $\tau = -0.5$)





The pion-pole contribution

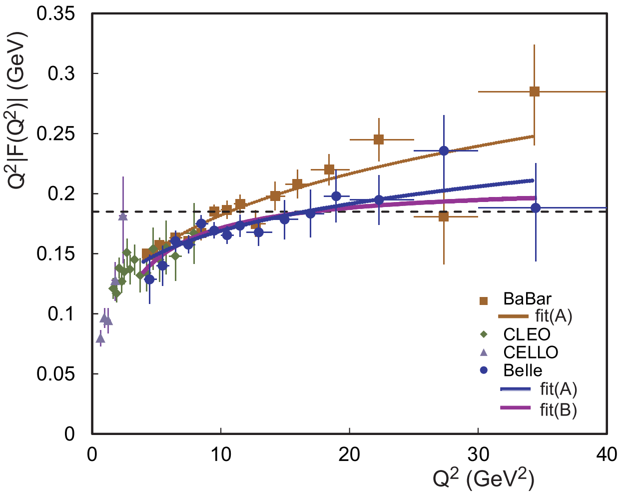
Present status :

- Experimental results available for the single-virtual form factor
- And only for relatively large virtualities $Q^2 > 0.6\ \mathrm{GeV}^2$
- The theory imposes strong constraints for the normalisation and the asymptotic behavior of the TFF

$\hookrightarrow$ Anomaly constraint $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = 1/(4\pi^2 F_\pi)$

$\hookrightarrow$ Brodsky-Lepage, OPE for large virtualities

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2,0) \sim 1/Q^2 \ , \quad \mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2,Q^2) \sim 1/Q^2$$



$\hookrightarrow$ Most evaluations of the pion-pole contribution are therefore based on phenomenological models

$\hookrightarrow$ Systematic errors are difficult to estimate

$\hookrightarrow$ Recent result using dispersion framework [Kubis et al.]

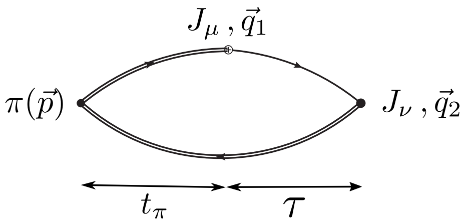
Lattice QCD is particularly well suited to compute the form factor in the energy range relevant to $g-2$!

Lattice calculation of the pion TFF

$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2,q_2^2) = -\int \mathrm{d}\tau\, e^{\omega_1\tau} \int \mathrm{d}^3z\, e^{-i\vec{q}_1\vec{z}}\, \langle 0|T\left\{J_\mu(\vec{z},\tau)J_\nu(\vec{0},0)\right\}|\pi^0(p)\rangle$$

- We consider the following 3-pt correlation function

$$C^{(3)}_{\mu\nu}(\tau,t_\pi;\vec{q}_1) = \sum_{\vec{x},\vec{z}} \langle T\left\{J_\nu(\vec{0},t_f)J_\mu(\vec{z},t_i)P(\vec{x},t_0)\right\}\rangle\, e^{-i\vec{q}_1\vec{z}}$$



$$A_{\mu\nu}(\tau) = \lim_{t_\pi\to\infty} C_{\mu\nu}(\tau,t_\pi)e^{E_\pi t_\pi}$$

$$\mathcal{F}_{\pi^0\gamma\gamma}(q_1^2,q_2^2) \propto \frac{2E_\pi}{Z_\pi}\int_{-\infty}^{\infty} \mathrm{d}\tau\, \widetilde{A}_{\mu\nu}(\tau)\, e^{\omega_1\tau}$$

$$\widetilde{A}_{\mu\nu}(\tau) = \begin{cases} A_{\mu\nu}(\tau) & \tau > 0 \\ A_{\mu\nu}(\tau)\, e^{-E_\pi\tau} & \tau < 0 \end{cases}$$

- Finite time-extent of the lattice :
  - → Fit the 3-pt correlation function at large τ (e.g. assuming a VMD)
  - → Use the model to integrate up to $\tau\to\infty$
- There are also (quark) disconnected contributions
  - → $\mathcal{O}(0.5\ \%)$ on E5 with $m_\pi = 340$ MeV

Lattice setup

- Lattice QCD is not a model : specific regularisation of the theory adapted to numerical simulations
- However there are systematic errors that we need to understand :

1) We used $N_f = 2$ simulations (CLS ensembles)

   → we are currently analysing the $N_f = 2+1$ CLS ensembles with improved statistics

2) Finite lattice spacing : discretisation errors

   → 3 lattice spacings ($a = 0.075, 0.065, 0.048$ fm) : extrapolation to the continuum limit $a = 0$

3) Unphysical quark masses

   → Different simulations with pion mass in the range [190-440] MeV : extrapolation to $m_\pi = m_\pi^{\mathrm{exp}}$

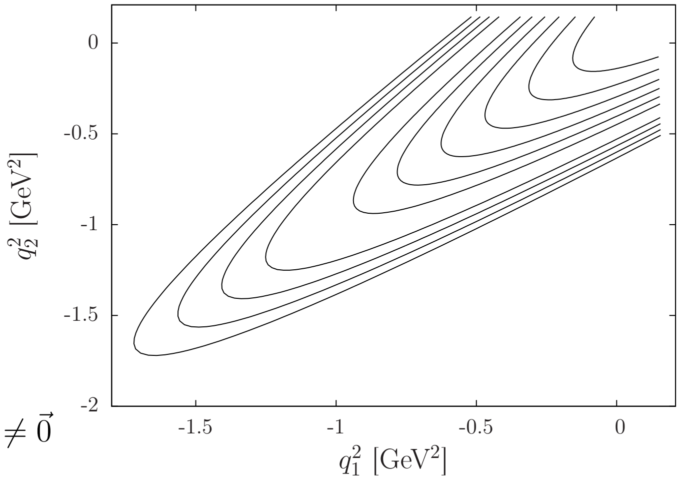
4) Finite volume

   → Discrete spatial momenta $\vec{q} = 2\pi/L\vec{n}$

pion at rest : $q_1 = (\omega_1, \vec{q}_1)$

$$q_2 = (m_\pi - \omega_1, \vec{q}_2)$$

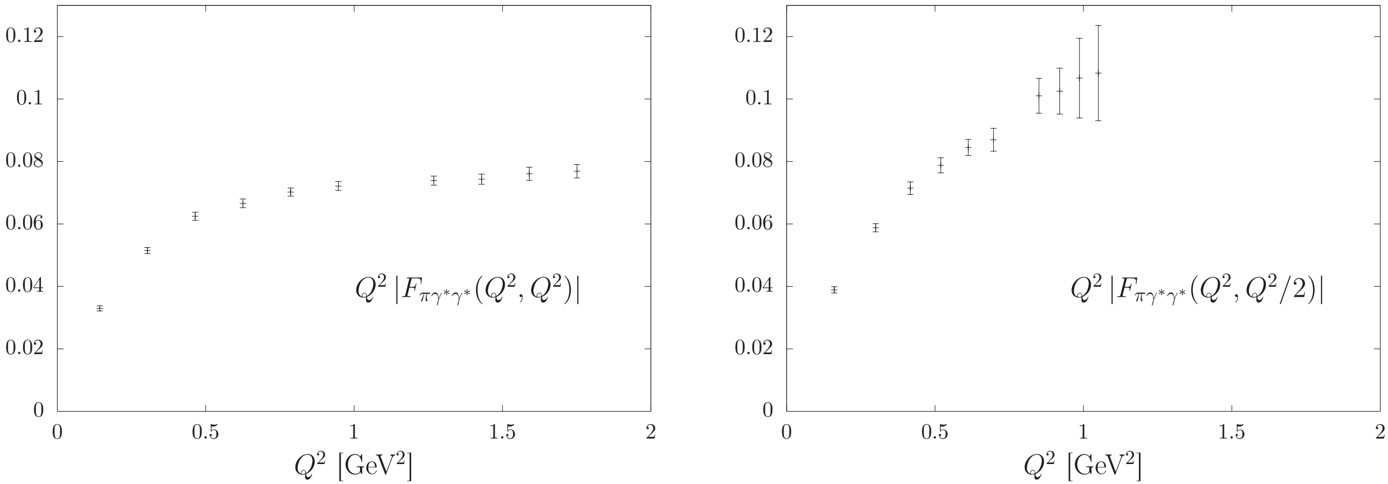
$$q_1^2 = \omega_1^2 - |\vec{q}_1|^2$$
$$q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2$$



→ we are currently adding a new frame with $\vec{p} \neq \vec{0}$

Transition form factor : results

- Results for one of the eight ensembles with $a = 0.048$ fm and $m_\pi = 270$ MeV



- Extrapolation to the physical point
  - Use phenomenological models to describe the lattice data
  - e.g. VMD model , LMD model (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

  - Extrapolate the model parameters to the continuum and chiral limit

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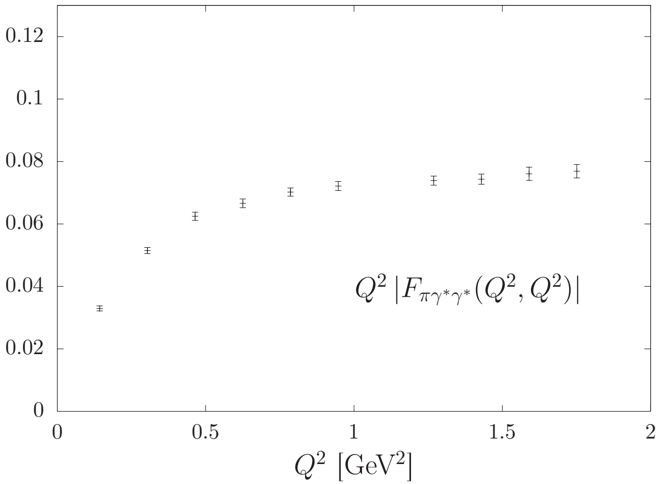
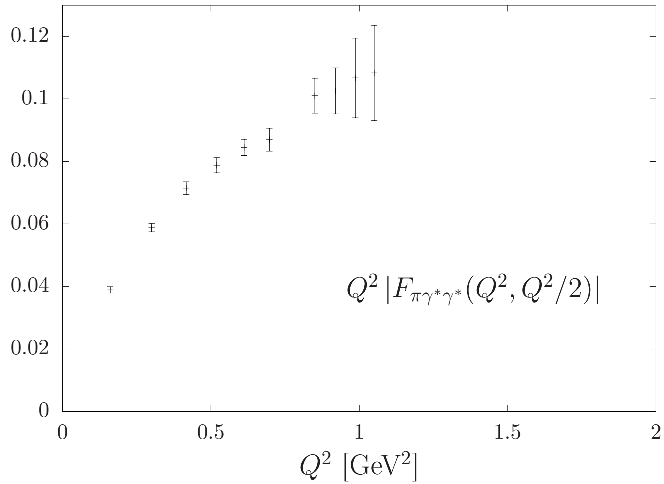


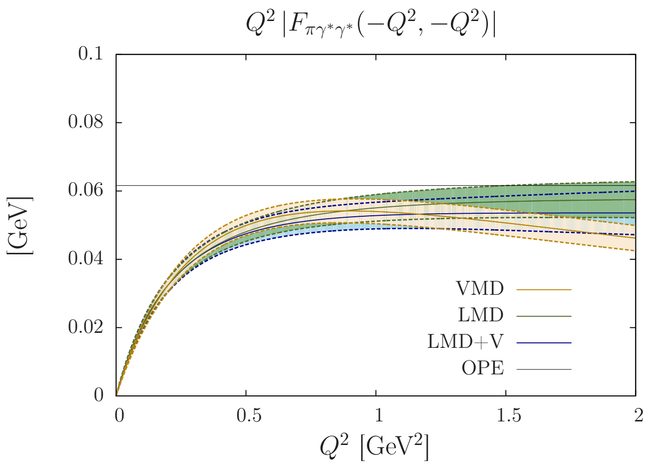


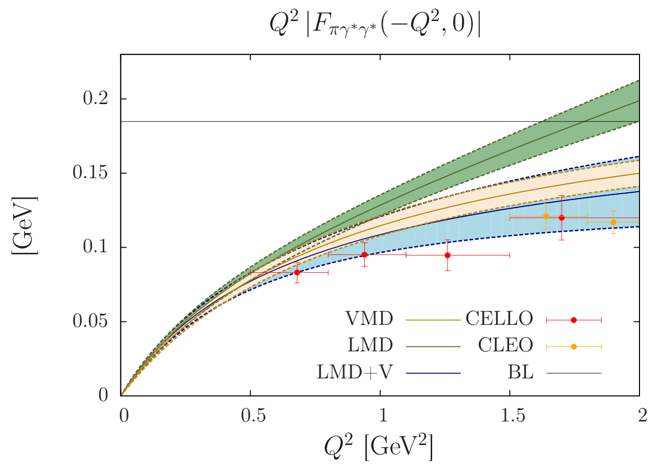
- Extrapolation to the physical point
  - Use phenomenological models to describe the lattice data
  - or the LMD+V model [Knecht, Nyffeler '01]

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2+q_2^2) + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2+q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2-q_1^2)(M_{V_2}^2-q_1^2)(M_{V_1}^2-q_2^2)(M_{V_2}^2-q_2^2)}$$

  - Extrapolate the model parameters to the continuum and chiral limit

Results at the physical point





→ VMD failed to describe our data : bad χ^2

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2,0) \; + $$
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$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$$

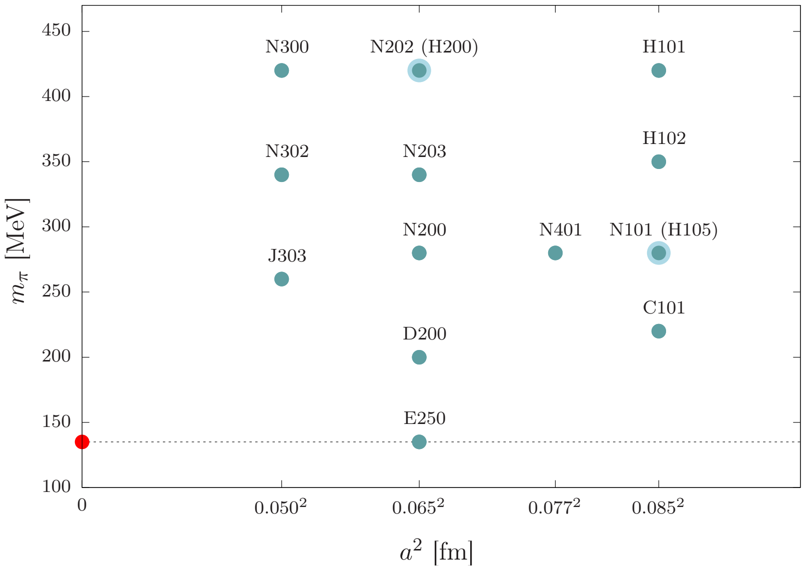
[Gérardin, Nyffeler, Meyer '16]

→ most model calculations yield results in the range : $a_\mu^{\mathrm{HLbL};\pi^0} = (50-80)\times 10^{-11}$

→ In the same ballpark as previous estimate : HLbL is unlikely to explain the 3-4 σ discrepancy

→ Result using the dispersive framework : $a_\mu^{\mathrm{HLbL};\pi^0} = (62.6 \pm 3.0) \times 10^{-11}$ [Hoferichter et al. '18]

$N_f = 2+1$ CLS ensembles



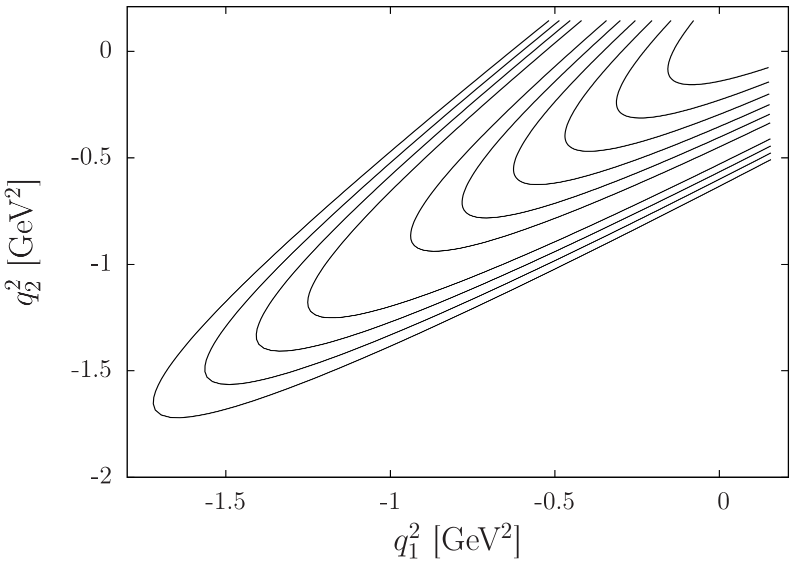
- Four lattice spacings
- One ensemble with physical pion mass (Correlators not yet computed)
- Several volumes

- Full $\mathcal{O}(a)-$improvement of the vector currents :
  - $\rightarrow$ Requires the improvement coefficient c_V (for both the local and the conserved vector currents)
  - $\rightarrow$ Also b_V and $\bar{b}_V$ for the local vector current

$$J_\mu^{R,I}(x) = Z_V \left(1 + 3\,\bar{b}_V\, a\overline{m} + b_V a m_l\right) \left[J_\mu(x) + a c_V\, \partial_\nu T_{\mu\nu}(x)\right]$$

- New frame with $\vec{p} \neq \vec{0}$
  - $\rightarrow$ We can probe much larger virtualities in the single-virtual case

$N_f = 2+1$ CLS ensembles



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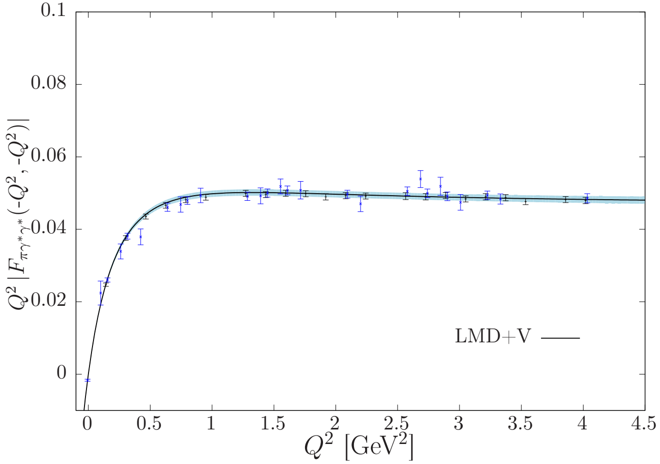
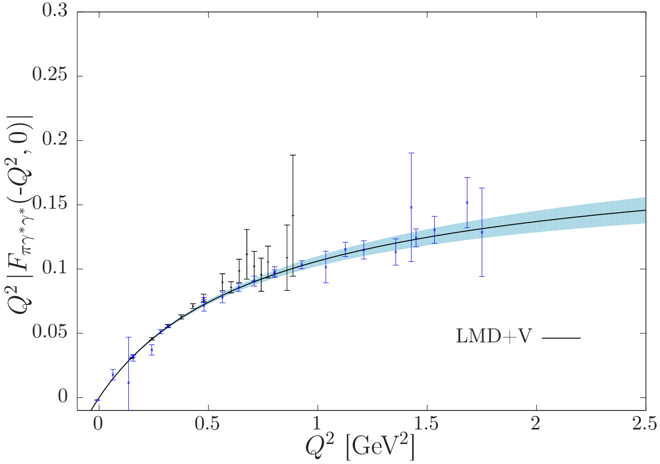
Preliminary results

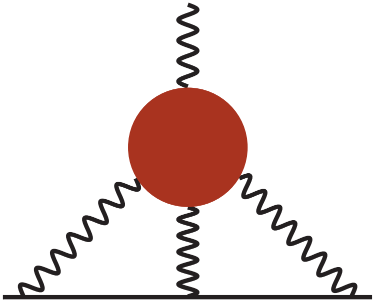
- Preliminary results for one ensemble with $a \approx 0.05$ fm and $m_\pi = 280$ MeV





- Black point : pion rest frame ($\vec{p} = \vec{0}$)
- Blue point : new frame with $\vec{p} = 2\pi/L\vec{z}$

Direct lattice calculation of the hadronic light-by-light scattering contribution



In collaboration with Nils Asmussen, Harvey Meyer and Andreas Nyffeler

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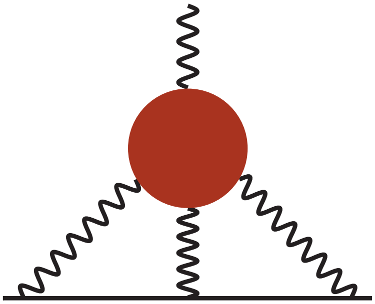
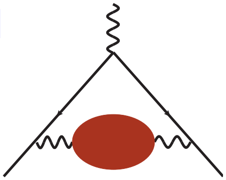


- RBC/UKQCD Collaboration
  - QCD + QED simulations [Hayakawa et al. 2005 ; Blum et al. 2015]
  - QCD + QED kernel estimated stochastically in finite volume [Blum et al. 2016, 2017]
  - Some results on the connected contribution and leading disconnected are already published

- Mainz group
  - Exact QED kernel in position space [Asmussen et al. 2015, 2016, and in prep.]
  - Forward light-by-light scattering amplitudes [A.G et al. 2017]

Exact QED kernel in infinite volume

- For the HVP contribution : time momentum representation (TMR) [Bernecker, Meyer '12]



$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int \mathrm{d}x_0 \; K(x_0) \; G(x_0) \,, \qquad G(x_0) = -\frac{1}{3} \sum_{k=1}^{3} \sum_{\vec{x}} \langle V_k(x) V_k(0) \rangle$$

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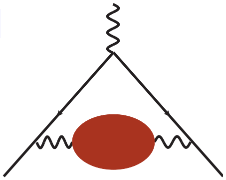

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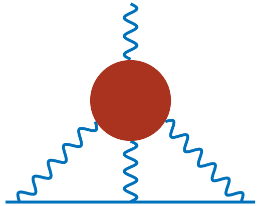
- Compute the QED part perturbatively in the continuum and in infinite volume (position space) [J. Green et al. '16] [N. Asmussen et al. '16 '17]



$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \; \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

$$i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4z \; z_\rho \, \langle J_\mu(x) J_\nu(y) J_\sigma(z) J_\lambda(0) \rangle$$

→ $\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$ is the four-point correlation function computed on the lattice

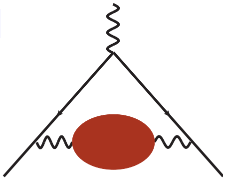
→ $\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is the QED kernel, computed semi-analytically (infra-red finite)

→ Avoid $1/L^2$ finite-volume effects from the massless photons

Exact QED kernel in infinite volume

- For the HVP contribution : time momentum representation (TMR) [Bernecker, Meyer '12]

$$a_\mu^{\rm HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int {\rm d}x_0 \; K(x_0) \; G(x_0) \; , \qquad G(x_0) = -\frac{1}{3}\sum_{k=1}^{3}\sum_{\vec{x}} \langle V_k(x) V_k(0) \rangle$$



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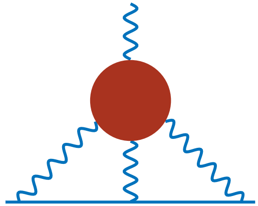

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→ Avoid $1/L^2$ finite-volume effects from the massless photons

- On the lattice :

→ integration over x and z are performed explicitly on the lattice (e.g. using sequential propagators ...)

→ the remaining part depends only on $|y|$

→ one-dimensional integral, can be sampled using different values of $|y|$

→ This is an expensive calculation !

QED Kernel : technical details

The QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ can be decomposed into several tensors

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \sum_{A=\mathrm{I,II,III}} \mathcal{G}^{A}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda}\, T^{(A)}_{\alpha\beta\delta}(x,y)$$

- $\mathcal{G}^{A}_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda}$ = traces of gamma matrices $\rightarrow$ sums of products of Kronecker deltas
- The tensors $T^{(A)}_{\alpha\beta\delta}$ are decomposed into a scalar S, vector V and tensor T part

$$T^{(\mathrm{I})}_{\alpha\beta\delta}(x,y) = \partial^{(x)}_{\alpha}(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta})V_{\delta}(x,y)$$

$$T^{(\mathrm{II})}_{\alpha\beta\delta}(x,y) = m\partial^{(x)}_{\alpha}\left(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta}S(x,y)\right)$$

$$T^{(\mathrm{III})}_{\alpha\beta\delta}(x,y) = m(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta})\left(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta}S(x,y)\right)$$

They are parametrized by six weight functions

$$S(x,y) = 0$$

$$V_{\delta}(x,y) = x_{\delta}\,\bar{\mathfrak{g}}^{(1)} + y_{\delta}\,\bar{\mathfrak{g}}^{(2)}$$

$$T_{\alpha\beta}(x,y) = (x_\alpha x_\beta - \frac{x^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(1)} + (y_\alpha y_\beta - \frac{y^2}{4}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(2)} + (x_\alpha y_\beta + y_\alpha x_\beta - \frac{x\cdot y}{2}\delta_{\alpha\beta})\,\bar{\mathfrak{l}}^{(3)}$$

- the weight functions depend on the three variables x^2, $x\cdot y = |x||y|\cos\beta$ and y^2
- Semi-analytical expressions for the weight functions have been computed to about 5 digits precision

Tests of the QED kernel

$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \, \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y) = \int \mathrm{d}y \, f(y)$$

- Two checks of the QED kernel in the continuum and infinite volume

Tests of the QED kernel

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \, \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \, i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y) = \int \mathrm{d}y \, f(y)$$

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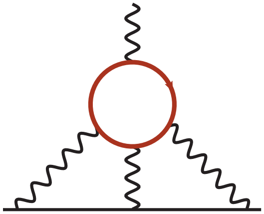

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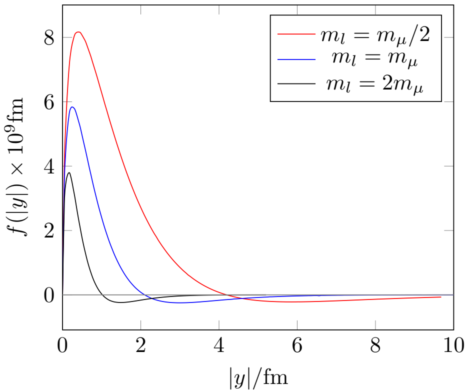

Lepton-loop (analytical result known)

- integrand peaked at small distances
- height of the peak grows when m_{lepton} decreases
- we probe the QED kernel at small distances
- we reproduce the exact result at the percent level

Tests of the QED kernel

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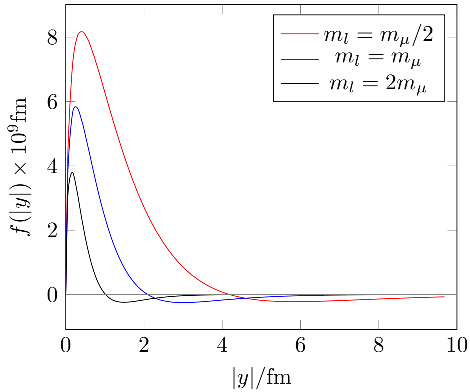

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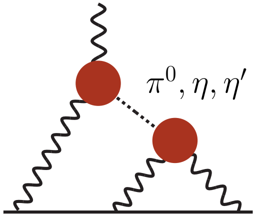

Tests of the QED kernel

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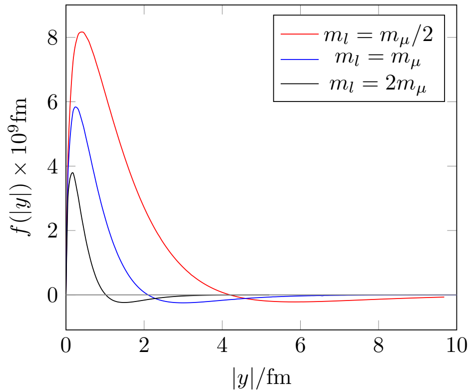
- Two checks of the QED kernel in the continuum and infinite volume



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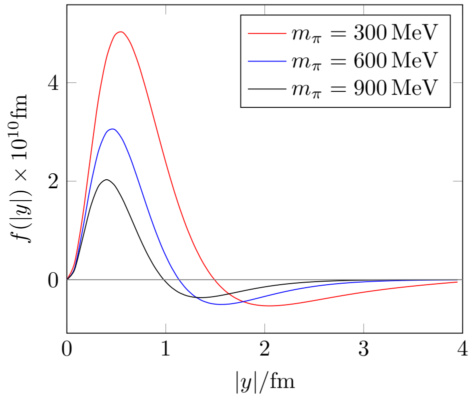

π^0-pole contribution for a specific model

- assume a vector-meson dominance (VMD) model for $\widehat{\Pi}$
  $\rightarrow$ The analytical results is then known
- negative tail at large $|y|$: need large volumes !
- again, we reproduce the exact result at the percent level

[F. Jegerlehner and A. Nyffeler, '09)]

QED kernel : subtractions

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int \mathrm{d}^4 y \int \mathrm{d}^4 x \; \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

- Conservation of the vector current : $\partial_\mu J_\mu(x) = 0 \;\Rightarrow$ The QED kernel is not unique [RBC/UKQCD '17]

$$0 = \sum_x \partial_\mu^{(x)} \left(x_\alpha \widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y) \right) = \sum_x \widehat{\Pi}_{\rho,\alpha\nu\lambda\sigma}(x,y) + \sum_x x_\alpha \, \partial_\mu^{(x)} \widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

$\rightarrow$ we can add any fonction $f(y)$ to the standard QED kernel

$\rightarrow$ differ by volume effects (and discretisation effects for the local vector current)

$\rightarrow$ same argument valid for the other variable x

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- Examples of possible subtractions (idea : subtract very short distance contributions)

$$\mathcal{L}^{(0)}(x,y) = \mathcal{L}(x,y) \qquad \Rightarrow \mathcal{L}^{(1)}(0,0) = 0$$

$$\mathcal{L}^{(1)}(x,y) = \mathcal{L}(x,y) - \frac{1}{2}\mathcal{L}(x,x) - \frac{1}{2}\mathcal{L}(y,y) \qquad \Rightarrow \mathcal{L}^{(1)}(x,x) = 0$$

$$\mathcal{L}^{(2)}(x,y) = \mathcal{L}(x,y) - \mathcal{L}(0,y) - \mathcal{L}(x,0) \qquad \Rightarrow \mathcal{L}^{(2)}(x,0) = \mathcal{L}^{(2)}(0,y) = 0$$

$$\mathcal{L}^{(3)}(x,y) = \mathcal{L}(x,y) - \mathcal{L}(0,y) - \mathcal{L}(x,x) + \mathcal{L}(0,x) \qquad \Rightarrow \mathcal{L}^{(3)}(0,y) = \mathcal{L}^{(3)}(x,x) = 0$$

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- Different definitions may affect :

→ Discretization effects / Finite-size effects / Statistical precision of the estimator

Lepton-loop in the continuum and infinite volume

$$a_\mu^{\rm LbL} = \frac{me^6}{3} \int \, \mathrm{d}^4 y \int \, \mathrm{d}^4 x \; \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

- $\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$ can be computed numerically for the lepton-loop
- The integral reduces to a 3-dimensional integration over the Lorentz invariants x^2, y^2 and $x \cdot y$

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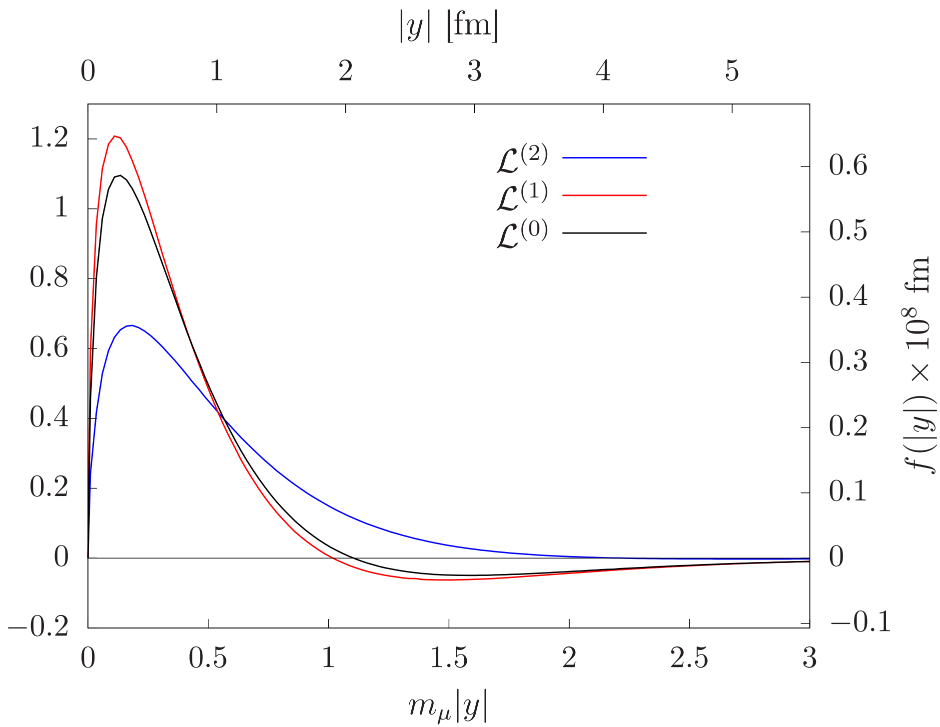

Pion-pole contribution in the continuum and infinite volume

$$a_\mu^{\pi^0-\text{pole}} = \frac{me^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \; \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

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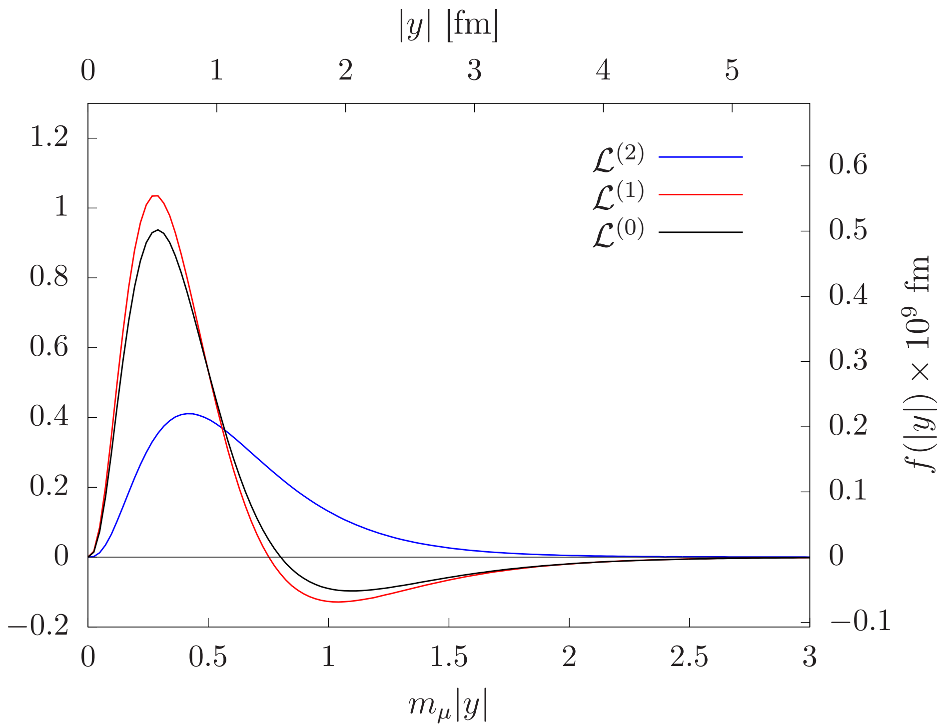

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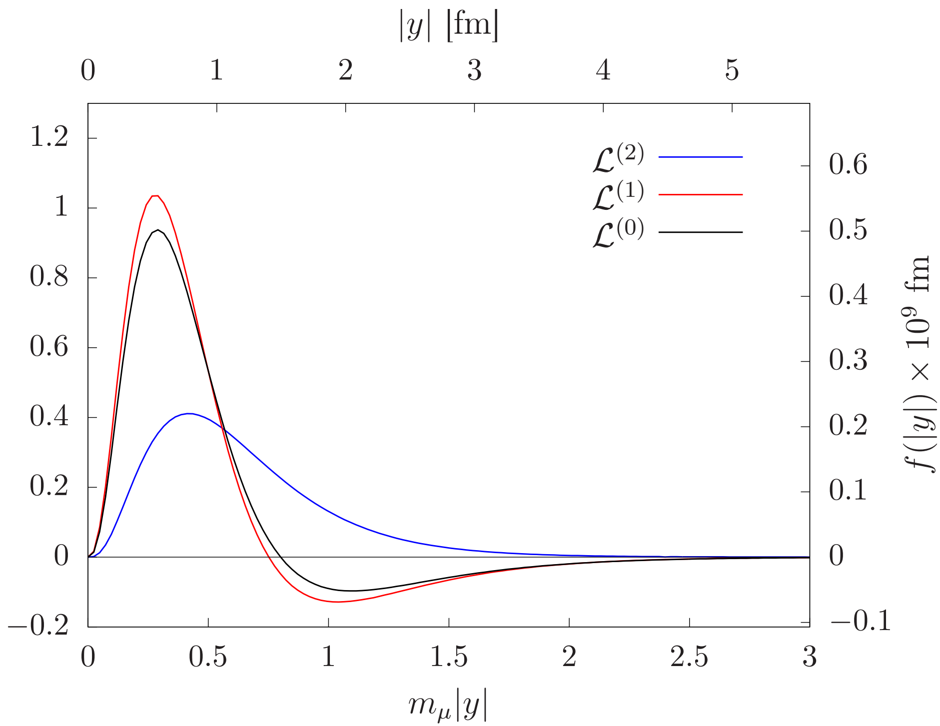
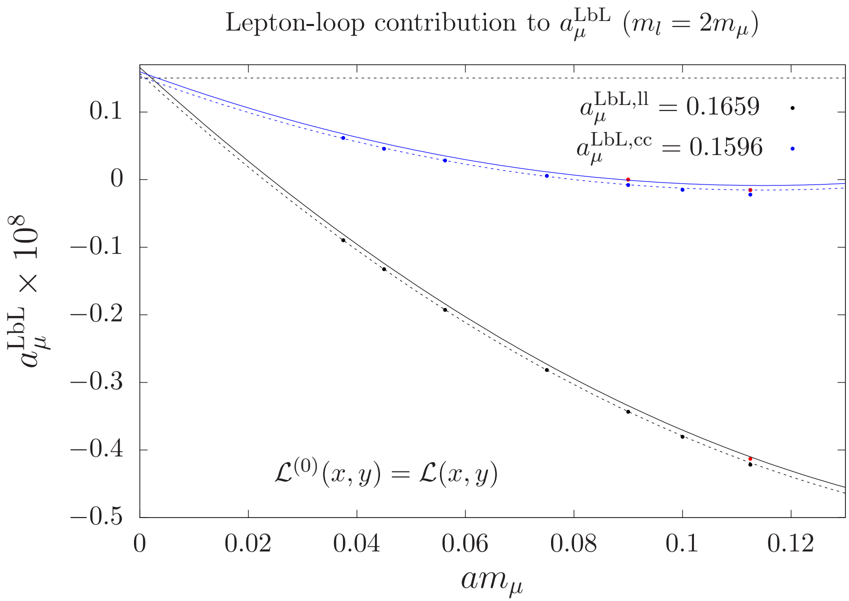


- Until now, everything was done in the continuum and infinite volume (no lattice involved)

Lepton-loop on the lattice

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \; \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

- We now compute $\widehat{\Pi}(x,y)$ on the lattice (unit gauge field, lattice propagators)



→ Use different lattice spacings / volumes

→ blue and black points correspond to two different discretizations of the vector current

→ standard kernel $\mathcal{L}^{(0)}(x,y)$: large discretization effects !

Lepton-loop on the lattice

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \; \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

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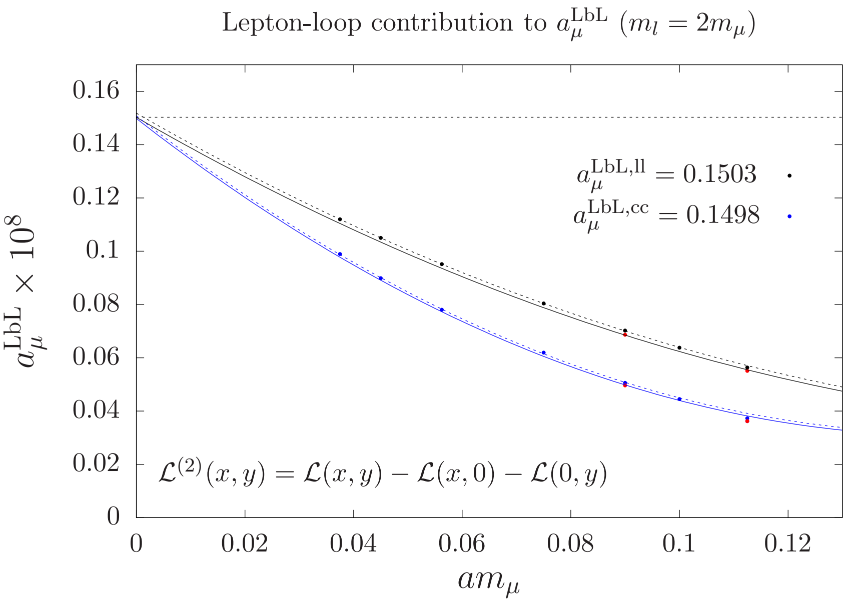


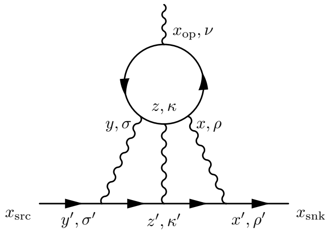
→ $\mathcal{L}^{(2)}(x,y)$ has much smaller discretization effects

→ we can reproduce the known result ($a_\mu^{\mathrm{LbL}} = 0.15031 \times 10^{-8}$) for the lepton-loop with a very good precision

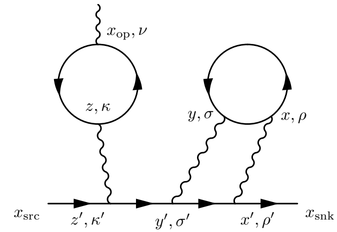
✓ check of the QCD code

The lattice QCD calculation

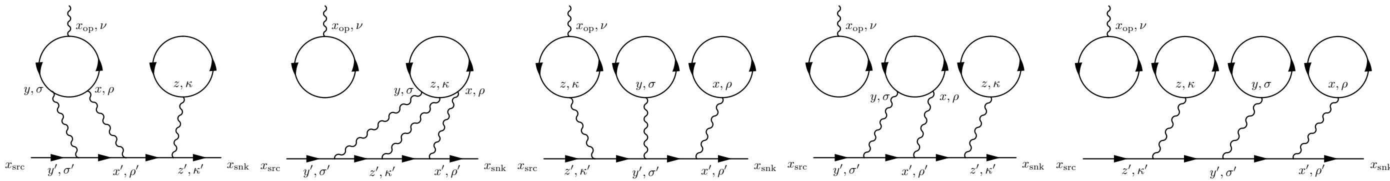
- Fully connected contribution



- Leading 2+2 disconnected contribution



- Sub-dominant disconnected contributions (3+1, 2+1+1, 1+1+1+1)



- 2+2 disconnected diagrams are not negligible !

  → Large-N_c prediction : 2+2 disc ≈ - 50 % × connected [Bijnens '16] , [A. G et al. '17]

  → Disconnected contributions : only $\mathcal{O}(1\ \%)$ for the HVP !

- Other diagrams vanish in the SU(3) limit (at least one quark loop which couple to a single photon)

  → Smaller contributions, but might be relevant for $\mathcal{O}(10\ \%)$ precision

FSE and the pion transition form factor

Pion-pole contribution :

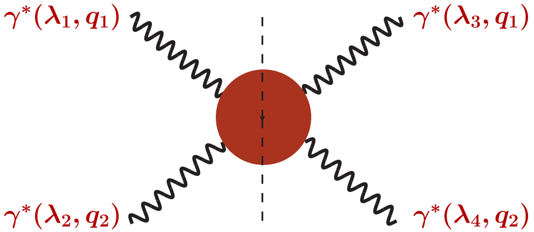
- Dominant contribution (according to model calculation))
- Long-range : source of FSE on the lattice

Idea :

- Use the same set of ensembles as for the pion TFF
  $\rightarrow$ use our result to estimate and correct for the dominant FSE in the lattice calculation
  $\rightarrow$ the pion-pole contributes with a factor $34/9$ in the fully connected piece
  -25/9 in the 2+2 disconnected

1) The QED kernel in infinite volume is now known
2) We have now started the QCD calculation
3) During that time, we studied the forward LbL scattering amplitudes

Forward HLbL scattering amplitudes : axial, scalar and tensor mesons

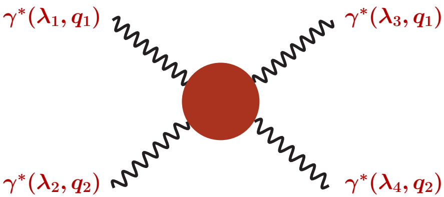


In collaboration with Jeremy Green, Oleksii Gryniuk, Harvey Meyer, Vladimir Pascalutsa and Hartmut Wittig

Light-by-light forward scattering amplitudes

- Pion-pole contribution ✓. Other contributions : more difficult on the lattice (resonances)

- Forward scattering amplitudes $\mathcal{M}_{\lambda_3\lambda_4\lambda_1\lambda_2}$



  - 81 helicity amplitudes ($\lambda_i = 0, \pm 1$)

$$\mathcal{M}_{\lambda_1'\lambda_2'\lambda_1\lambda_2} = \mathcal{M}_{\mu\nu\rho\sigma}\ \epsilon^{*\mu}(\lambda_1')\ \epsilon^{*\nu}(\lambda_2')\ \epsilon^{\rho}(\lambda_1)\ \epsilon^{\sigma}(\lambda_2)$$

  - Photons virtualities : $Q_1^2 = -q_1^2 > 0$ and $Q_2^2 = -q_2^2 > 0$
  - Crossing-symmetric variable : $\nu = q_1 \cdot q_2$

- Using parity and time invariance : only 8 independent amplitudes

$$(\mathcal{M}_{++,++} + \mathcal{M}_{+-,+-}),\ \mathcal{M}_{++,--},\ \mathcal{M}_{00,00},\ \mathcal{M}_{+0,+0},\ \mathcal{M}_{0+,0+},\ (\mathcal{M}_{++,00} + \mathcal{M}_{0+,-0}),$$
$$(\mathcal{M}_{++,++} - \mathcal{M}_{+-,+-}),\ (\mathcal{M}_{++,00} - \mathcal{M}_{0+,-0})$$

↪ Either even or odd with respect to ν

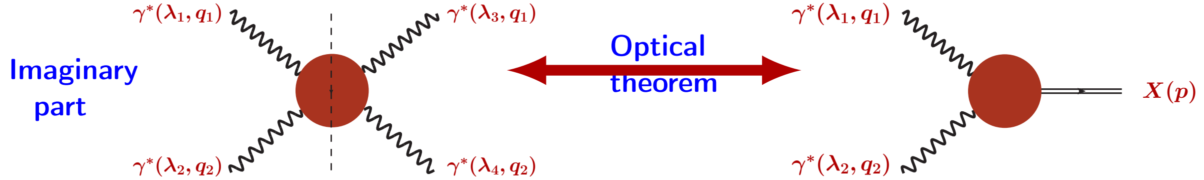
↪ The eight amplitudes have been computed on the lattice for different values of ν, Q_1^2, Q_2^2

Strategy :

1) Compute all the forward LbL scattering amplitudes on the lattice [Green et. al '15]
2) Use a simple model to describe the lattice data (input : TFFs)
3) Extract information about TFFs by fitting the model parameters to lattice data

Dispersion relations

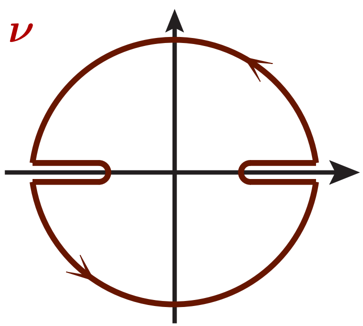
1) Optical theorem



$\hookrightarrow$ X can be any C-even final state

$$W_{\lambda_3\lambda_4,\lambda_1\lambda_2} = \mathrm{Im}\, M_{\lambda_3\lambda_4,\lambda_1\lambda_2} = \frac{1}{2}\int \mathrm{d}\Gamma_X (2\pi)^4 \delta(q_1 + 1_2 - p_X) \mathcal{M}_{\lambda_1\lambda_2}(q_1, q_2, p_X)\, \mathcal{M}^*_{\lambda_3\lambda_4}(q_1, q_2, p_X)$$

2) Dispersion relations [Pascalutsa et. al '12]



Once-subtracted sum rules : crossing-symmetric variable $\nu = q_1 \cdot q_2$

$$\mathcal{M}_{\mathrm{even}}(\nu) = \mathcal{M}_{\mathrm{even}}(0) + \frac{2\nu^2}{\pi}\int_{\nu_0}^{\infty} d\nu' \frac{1}{\nu'(\nu'^2 - \nu^2 - i\epsilon)} W_{\mathrm{even}}(\nu')$$

$$\mathcal{M}_{\mathrm{odd}}(\nu) = \nu\mathcal{M}_{\mathrm{odd}}(\nu) + \frac{2\nu^3}{\pi}\int_{\nu_0}^{\infty} d\nu' \frac{1}{\nu'(\nu'^2 - \nu^2 - i\epsilon)} W_{\mathrm{odd}}(\nu')$$

3) Higher mass singularities are suppressed with ν^2 :

$\hookrightarrow$ Only a few states X are necessary to saturate the sum rules and reproduce the lattice data

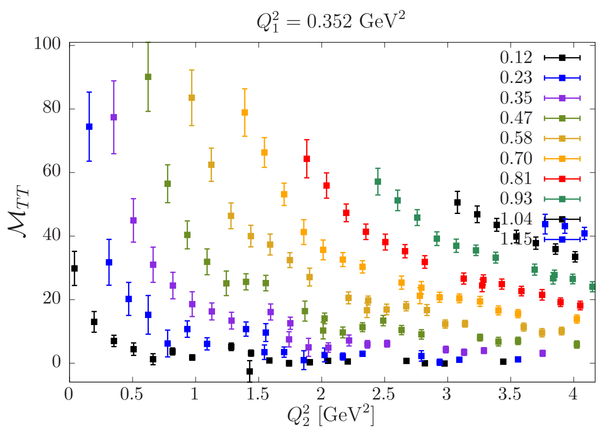
Description of the lattice data using phenomenology

→ For each of the eight amplitudes, we have a dispersion relation :

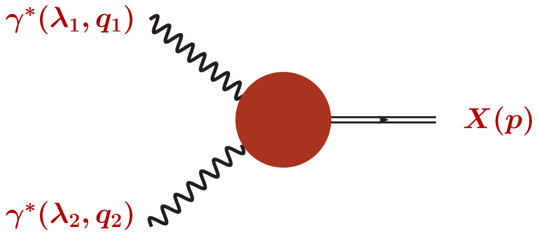
$$\overline{\mathcal{M}}_{\alpha}(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \, \frac{\sqrt{X'} \, \sigma_\alpha / \tau_\alpha(\nu')}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}$$

Lattice calculation

↪ 4-pt correlation function



$\gamma^*(\lambda_1, q_1) + \gamma^*(\lambda_2, q_2) \to \mathrm{X}(p_\mathrm{X})$ fusion cross sections



↪ Main contribution is expected from mesons :

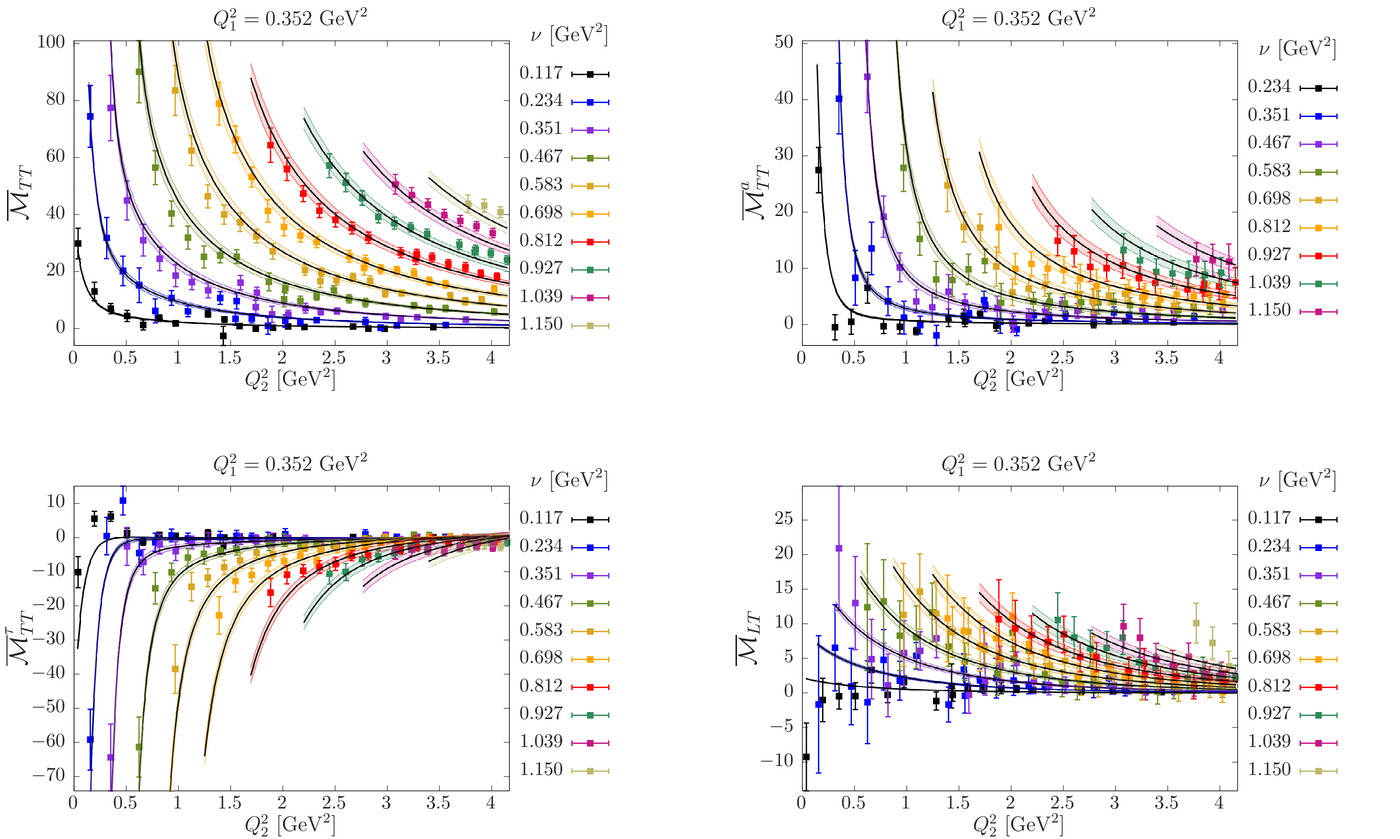
Pseudoscalars (0^{-+}) Axial-vectors (1^{++})
Scalar (0^{++}) Tensors (2^{++})

↪ Input : transition form factors

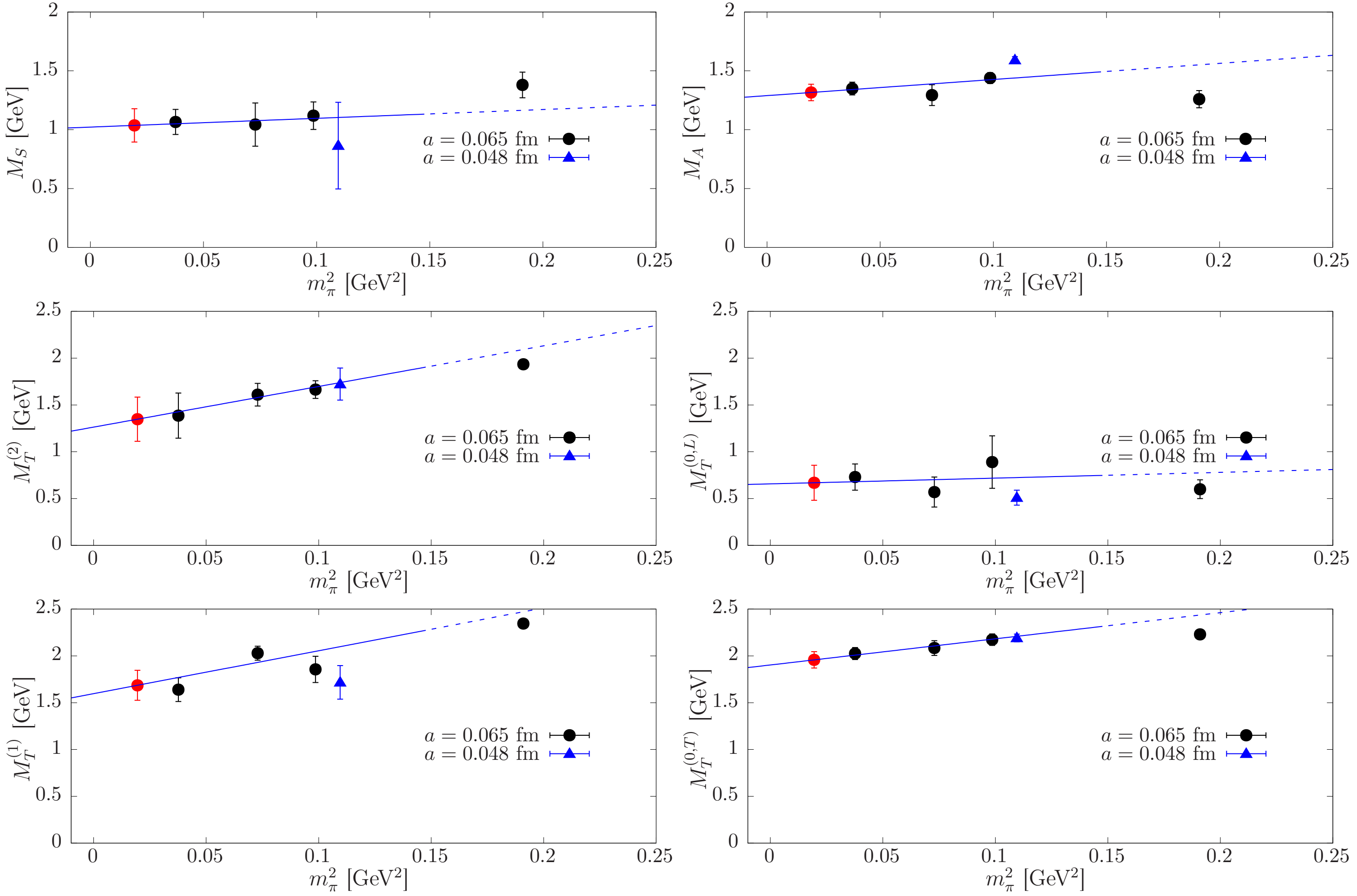
↪ Assume monopole/dipole masses (fit parameters)

Preliminary results : F7 - dependance on ν and Q_2^2

- Each plot correspond to a fixed Q_1^2
- Different colours correspond to different values of $\nu = Q_1^2 \cdot Q_2^2$



Monopole and dipole masses : chiral extrapolations



Conclusion

- There is a persistant $3-4\ \sigma$ discrepancy between theory and experiment for the $(g-2)_\mu$
- Two new experiments (Fermilab and J-PARC) should reduce the experimental error by a factor 4
- The error is dominated by hadronic uncertainties

Pion-pole contribution

→ First lattice QCD determination of $a_\mu^{\mathrm{HLbL};\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$

→ The (dominant) π^0-pole contribution can be computed precisely

→ In progress : new calculation with $N_f = 2+1$, more statistics, full $\mathcal{O}(a)-$improvement ...

Hadronic light-by-light scattering contribution

→ lattice QCD is very promising

→ The QED kernel in infinite volume is now known and checked

→ We are now starting the full QCD calculation. Goal : $20\ \%$ accuracy in the near future

Beyond the pion-pole contribution

→ The forward LbL scattering amplitudes provide more information than a_μ^{HLbL} (single scalar)

→ The lattice data can be described by a simple phenomenological model

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Thank you !