Lattice QCD studies of Muon g-2 and related topics

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Contents

- g-2 HVP since [T. Blum 2003]

  HPQCD
  Riken-BNL-Columbia (RBC) /UKQCD
  Mainz
  ETMC
  BMW
  Regensburg
  HPQCD/FNAL/MILC
  PACS
  ⋮
  ⋮

- adronic Light-by-Light (HLbL) on Lattice since [T. Blum et al 2005]

  RBC/UKQCD
  Mainz

- Inclusive tau decay [if time allowed]
  RBC/UKQCD

Lattice2017

Collaborators / Machines

g-2 DWF
HVP & HLbL

Tom Blum (Connecticut)
Peter Boyle (Edinburgh)
Norman Christ (Columbia)
Vera Guelpers (Southampton)
Masashi Hayakawa (Nagoya)
James Harrison (Southampton)
Taku Izubuchi (BNL/RBRC)
Christoph Lehner (BNL)
Kim Maltman (York)
Chulwoo Jung (BNL)
Andreas Jüttner (Southampton)
Luchang Jin (BNL)
Antonin Portelli (Edinburgh)

HVP Clover on $(8.5\ \mathrm{fm})^3$

Taku Izubuchi (BNL/RBRC)
Christoph Lehner (BNL)
Yoshinobu Kuramashi (Tsukuba/ AICS)
Eigo Shintani (RIKEN AICS)

tau decay

Peter Boyle (Edinburgh)
Taku Izubuchi (BNL/RBRC)
Christoph Lehner (BNL)
Kim Maltman (York)
Antonin Portelli (Edinburgh)
Renwick James Hudspith (York)
Andreas Jüttner(Southampton)
Randy Lewis (Southampton)
Hiroshi Ohki (RBRC/Nara Women)
Matthew Spraggs (Edinburgh)

Part of related calculation are done by resources from
USQCD (DOE), XSEDE, ANL BG/Q Mira (DOE, ALCC), Edinburgh BG/Q,
BNL BG/Q, RIKEN BG/Q and Cluster (RICC, HOKUSAI)

Support from RIKEN, JSPS, US DOE, and BNL

The RBC & UKQCD collaborations

BNL and RBRC

Mattia Bruno
Tomomi Ishikawa
Taku Izubuchi
Luchang Jin
Chulwoo Jung
Christoph Lehner
Meifeng Lin
Hiroshi Ohki
Shigemi Ohta (KEK)
Amarjit Soni
Sergey Syritsyn

Columbia University

Ziyuan Bai
Norman Christ
Duo Guo
Christopher Kelly
Bob Mawhinney
David Murphy
Masaaki Tomii
Jiqun Tu
Bigeng Wang
Tianle Wang

University of Connecticut

Tom Blum
Dan Hoying
Cheng Tu

Edinburgh University

Peter Boyle
Guido Cossu
Luigi Del Debbio
Richard Kenway
Julia Kettle
Ava Khamseh
Brian Pendleton
Antonin Portelli
Tobias Tsang
Oliver Witzel
Azusa Yamaguchi

KEK

Julien Frison

University of Liverpool

Nicolas Garron

Peking University

Xu Feng

University of Southampton

Jonathan Flynn
Vera Guelpers
James Harrison
Andreas Juettner
Andrew Lawson
Edwin Lizarazo
Chris Sachrajda

York University (Toronto)

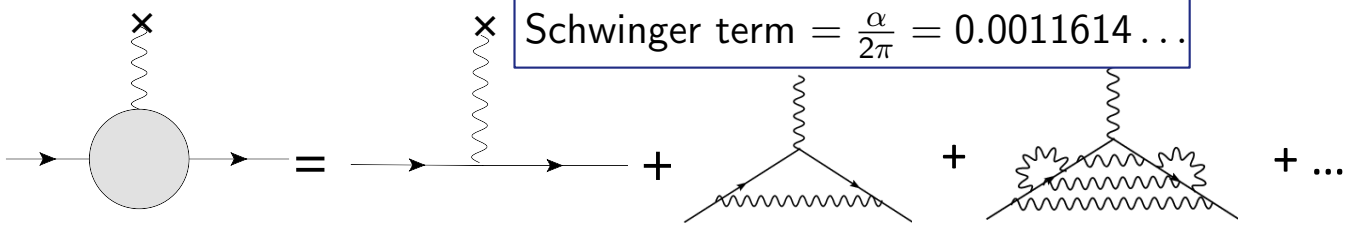
Renwick Hudspith

SM Theory

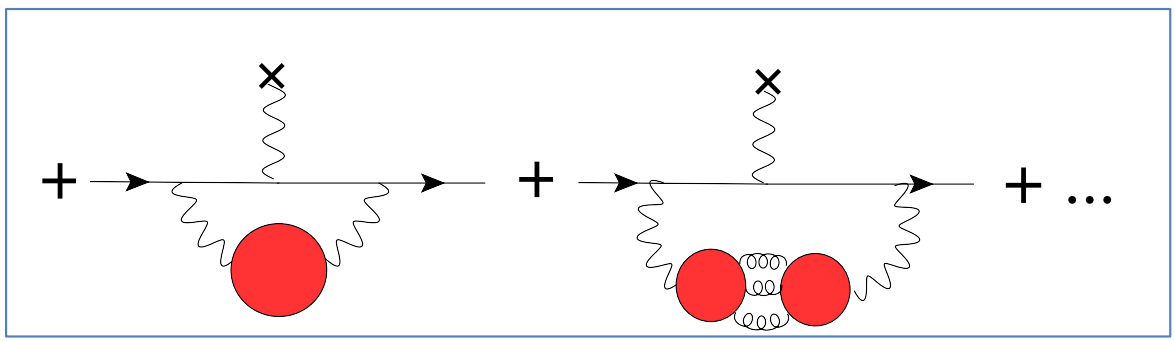
$$\gamma^\mu \;\rightarrow\; \Gamma^\mu(q) = \left(\gamma^\mu F_1(q^2) + \frac{i\,\sigma^{\mu\nu}\,q_\nu}{2m} F_2(q^2)\right)$$

- QED, hadronic, EW contributions

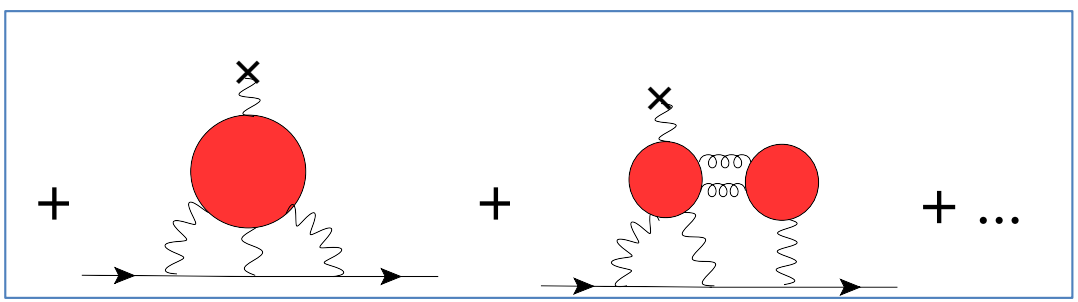
Schwinger term $= \frac{\alpha}{2\pi} = 0.0011614\ldots$



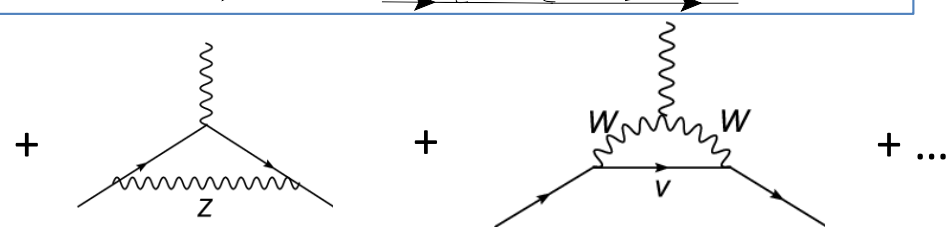
QED (5-loop)
Aoyama et al.
PRL109,111808 (2012)



Hadronic vacuum polarization (HVP)



Hadronic light-by-light (Hlbl)



Electroweak (EW)
Knecht et al 02
Czarnecki et al. 02

$(g\text{-}2)_\mu$ SM Theory vs experiment

- QED, EW, Hadronic contributions

K. Hagiwara et al. , J. Phys. G: Nucl. Part. Phys. 38 (2011) 085003

$$a_\mu^{\rm SM} = (11\ 659\ 182.8\ \pm 4.9\ ) \times 10^{-10}$$

$$a_\mu^{\rm QED} = (11\ 658\ 471.808\ \pm 0.015\ ) \times 10^{-10}$$
$$a_\mu^{\rm EW} = (\ 15.4\ \pm 0.2\ ) \times 10^{-10}$$
$$a_\mu^{\rm had,LOVP} = (\ 694.91\ \pm 4.27\ ) \times 10^{-10}$$
$$a_\mu^{\rm had,HOVP} = (\ -9.84\ \pm 0.07\ ) \times 10^{-10}$$
$$a_\mu^{\rm had,lbl} = (\ 10.5\ \pm 2.6\ ) \times 10^{-10}$$

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 28.8(6.3)_{\rm exp}(4.9)_{\rm SM} \times 10^{-10} \quad [3.6\sigma]$$

- Discrepancy between EXP and SM is larger than EW!
- Currently the dominant uncertainty comes from HVP, followed by HLbL
- x4 or more accurate experiment FNAL , J-PARC
- Our Goal : sub 1% accuracy for HVP, and
  → **10% accuracy for HLbL**

G-2 from BSM sources

- Typical new particle contribute g-2

  $$g\text{-}2 \sim C \ (m_\mu / m_{NP})^2$$

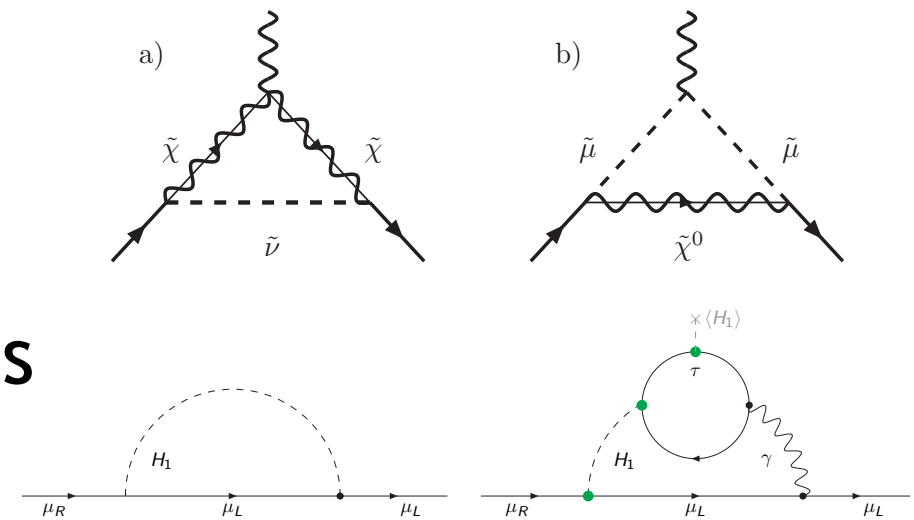
- To explain current discrepancy

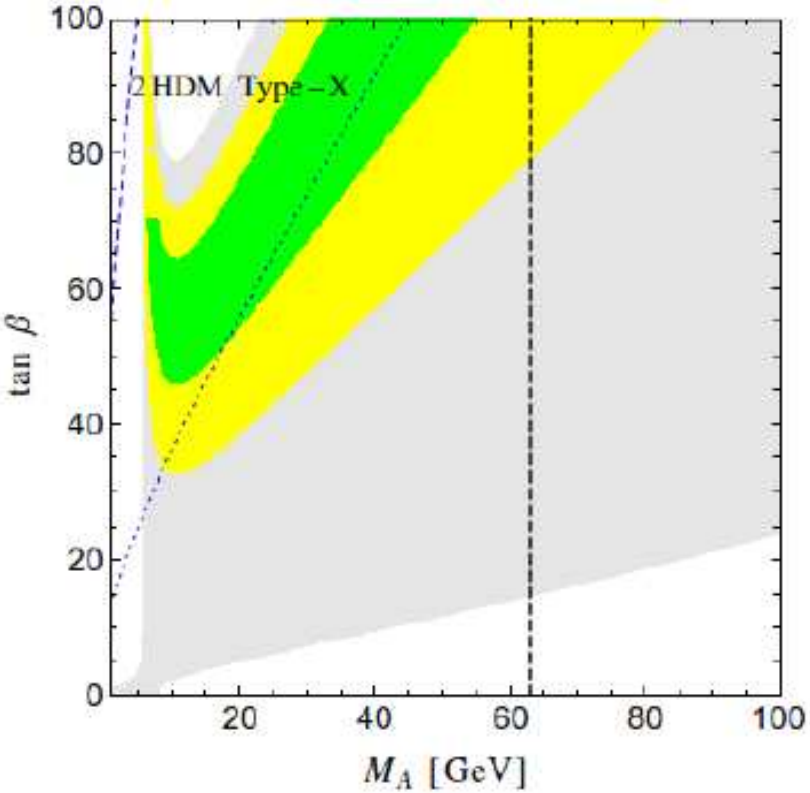
$\mathcal{C}$	1	$\frac{\alpha}{\pi}$	$(\frac{\alpha}{\pi})^2$
$M_{\rm NP}$	$2.0^{+0.4}_{-0.3}$ TeV	100^{+21}_{-13} GeV	5^{+1}_{-1} GeV

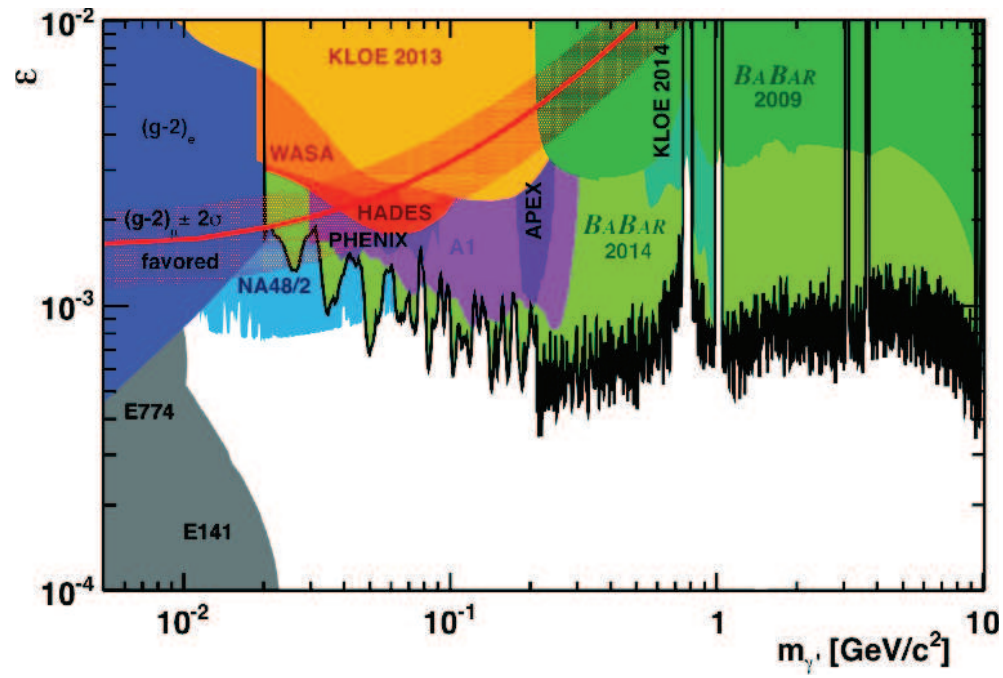
- SUSY (scalar-lepton)
- 2 Higgs doublet models
  Type-X, ….
- Dark photons
  from kinematical mixings

  $$\varepsilon \ F_{\mu\nu} F'_{\mu\nu}$$

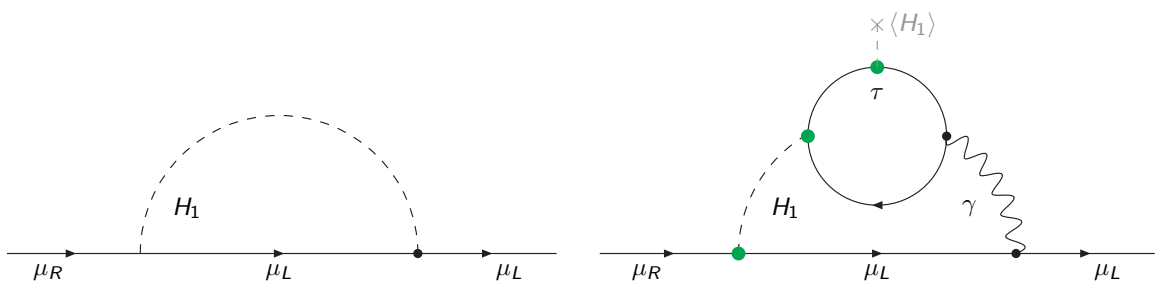
[A. Nyfler]



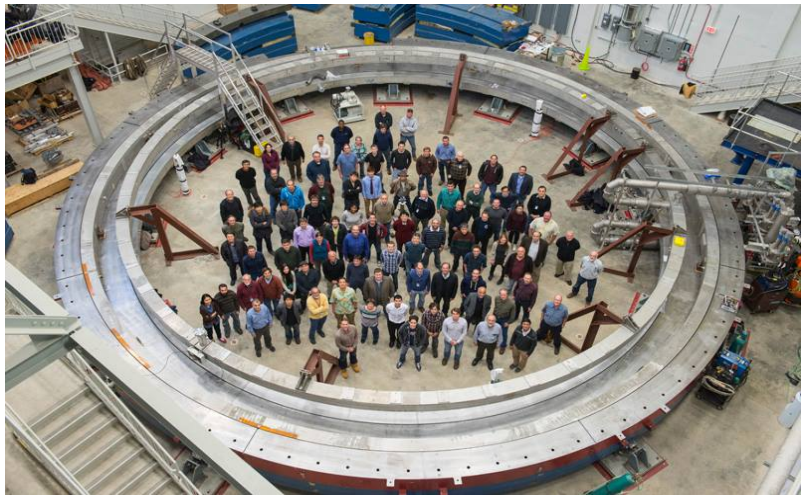




From: F. Curciarello, FCCP15, Capri, September 2015

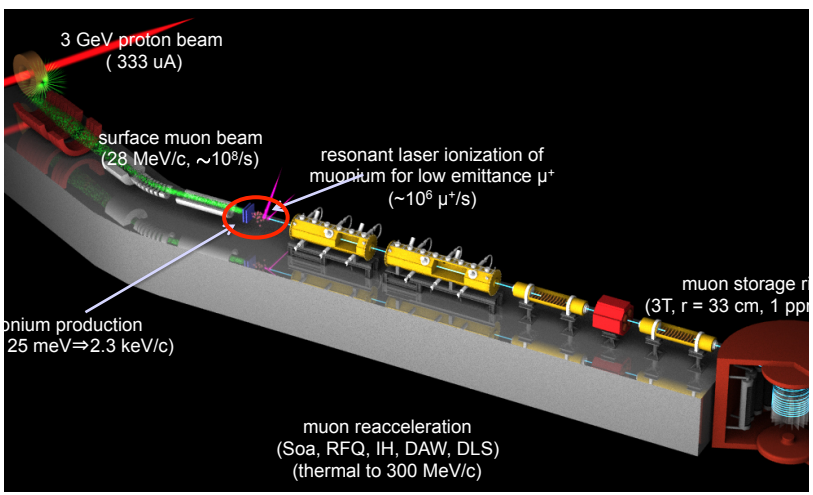


muon anomalous magnetic moment



BNL g-2 till 2004 : ~ 3 σ larger than SM prediction

Contribution	Value $\times 10^{10}$	Uncertainty $\times 10^{10}$
QED (5 loops)	11 658 471.895	0.008
EW	15.4	0.1
HVP LO	692.3	**4.2**
HVP NLO	-9.84	0.06
HVP NNLO	1.24	0.01
Hadronic light-by-light	10.5	**2.6**
Total SM prediction	11 659 181.5	4.9
BNL E821 result	11 659 209.1	6.3
FNAL E989/J-PARC E34 goal		≈ **1.6**



FNAL E989 (began 2017-)
- move storage ring from BNL
- x4 more precise results, 0.14ppm

J-PARC E34
- ultra-cold muon beam
- 0.37 ppm then 0.1 ppm, also EDM

[Luchang Jin's analogy]

Precession of Mercury and GR

discrepancy recognized since 1859

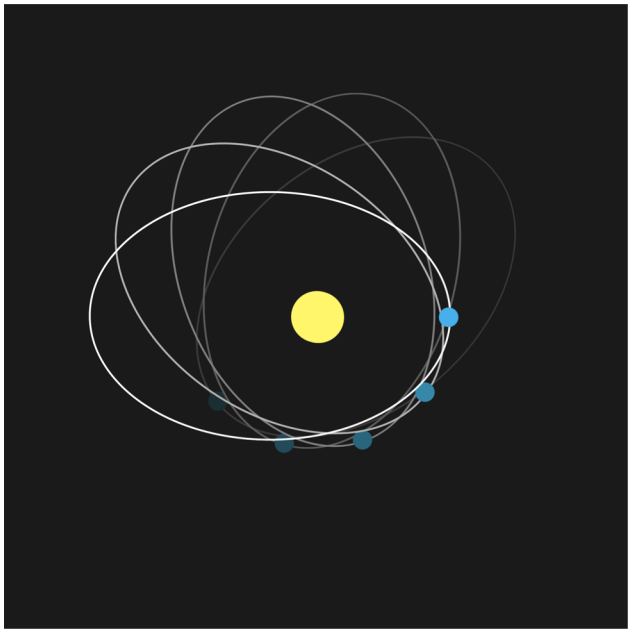
Amount (arc-sec/century)	Cause
5025.6	Coordinate (due to precession of equinoxes)
531.4	Gravitational tugs of the other planets
0.0254	Oblateness of the sun (quadrupole moment)
42.98±0.04	General relativity
5600.0	Total
5599.7	Observed

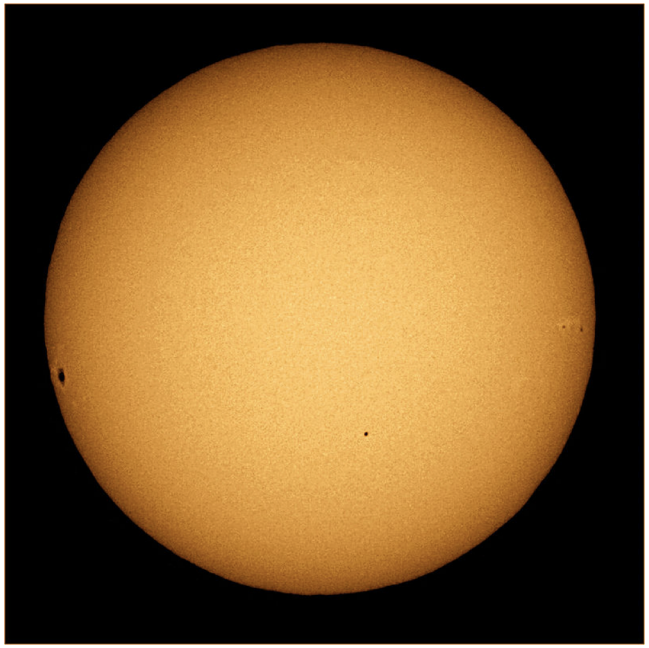
Known physics

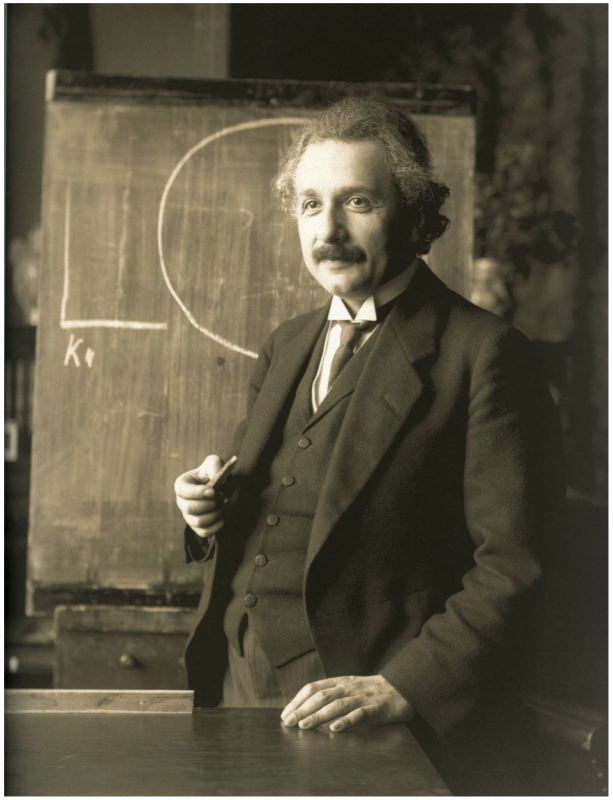
1915 New physics
GR revolution

http://worldnpa.org/abstracts/abstracts_6066.pdf

precession of perihelion

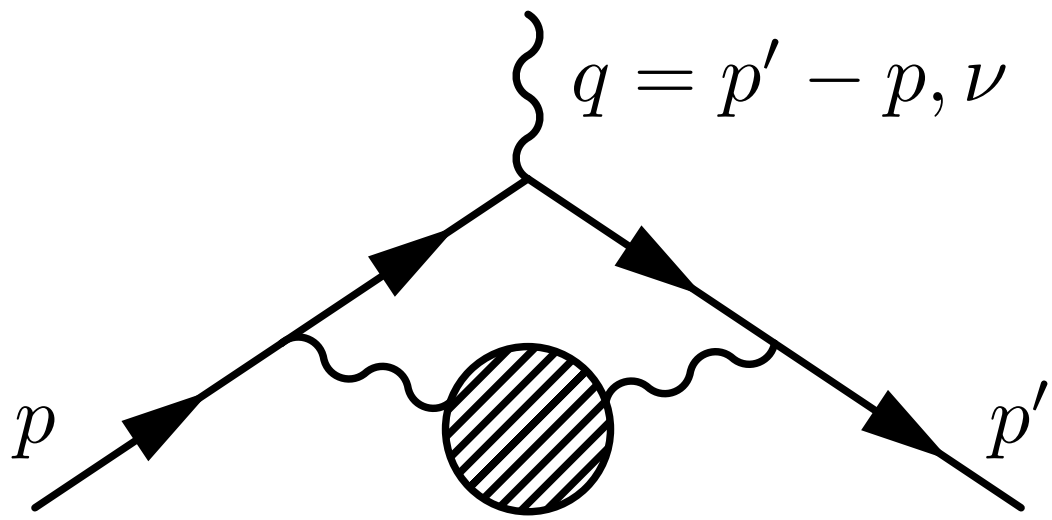




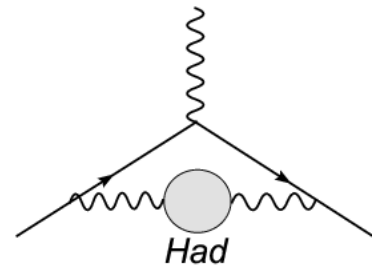


[Christoph Lehner et al. 1801.07224]

Hadronic Vacuum Polarization (HVP) contribution to g-2



Leading order of hadronic contribution (HVP)



- Hadronic vacuum polarization (HVP)

V_μ  V_ν $= (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi_V(q^2)$

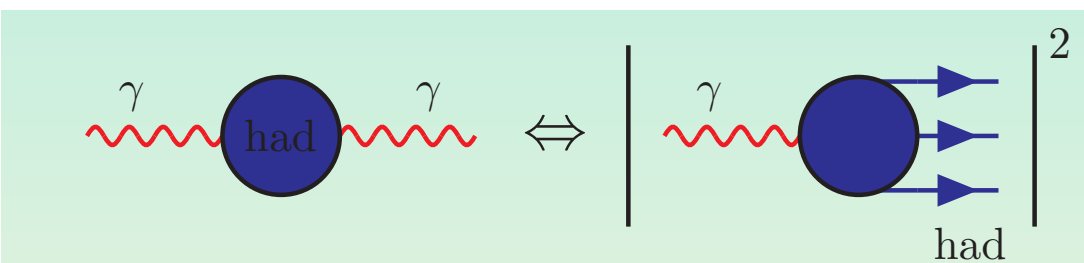
quark's EM current : $V_\mu = \sum_f Q_f \bar{f}\gamma_\mu f$

- Optical Theorem

$$\mathrm{Im}\Pi_V(s) = \frac{s}{4\pi\alpha}\sigma_{\mathrm{tot}}(e^+e^- \to X)$$

- Analycity

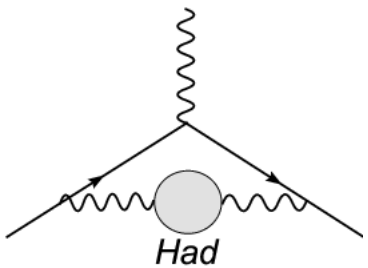
$$\Pi_V(s) - \Pi_V(0) = \frac{k^2}{\pi}\int_{4m_\pi^2}^{\infty} ds \frac{\mathrm{Im}\Pi_V(s)}{s(s-k^2-i\epsilon)}$$



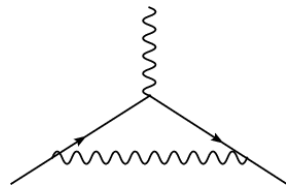
[F. Jegerlehner's lecture]

Leading order of hadronic contribution (HVP)

- Hadronic vacuum polarization (HVP)



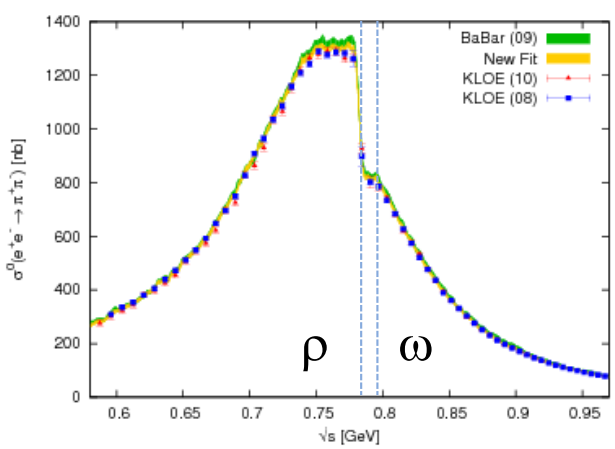
$$a_\mu^{\rm had} = \int ds$$

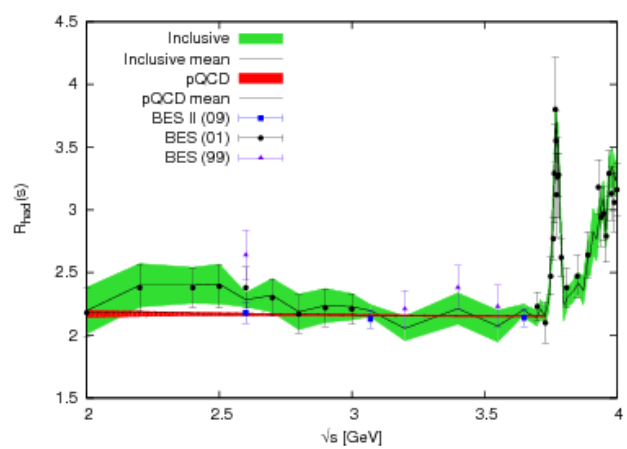


$\times$

$$= \frac{\alpha}{\pi^2}\int_{m_\pi^2}^{\infty}\frac{ds}{s}{\rm Im}\Pi(s)K(s) \qquad K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2+(s/m_\mu^2)(1-x)}$$

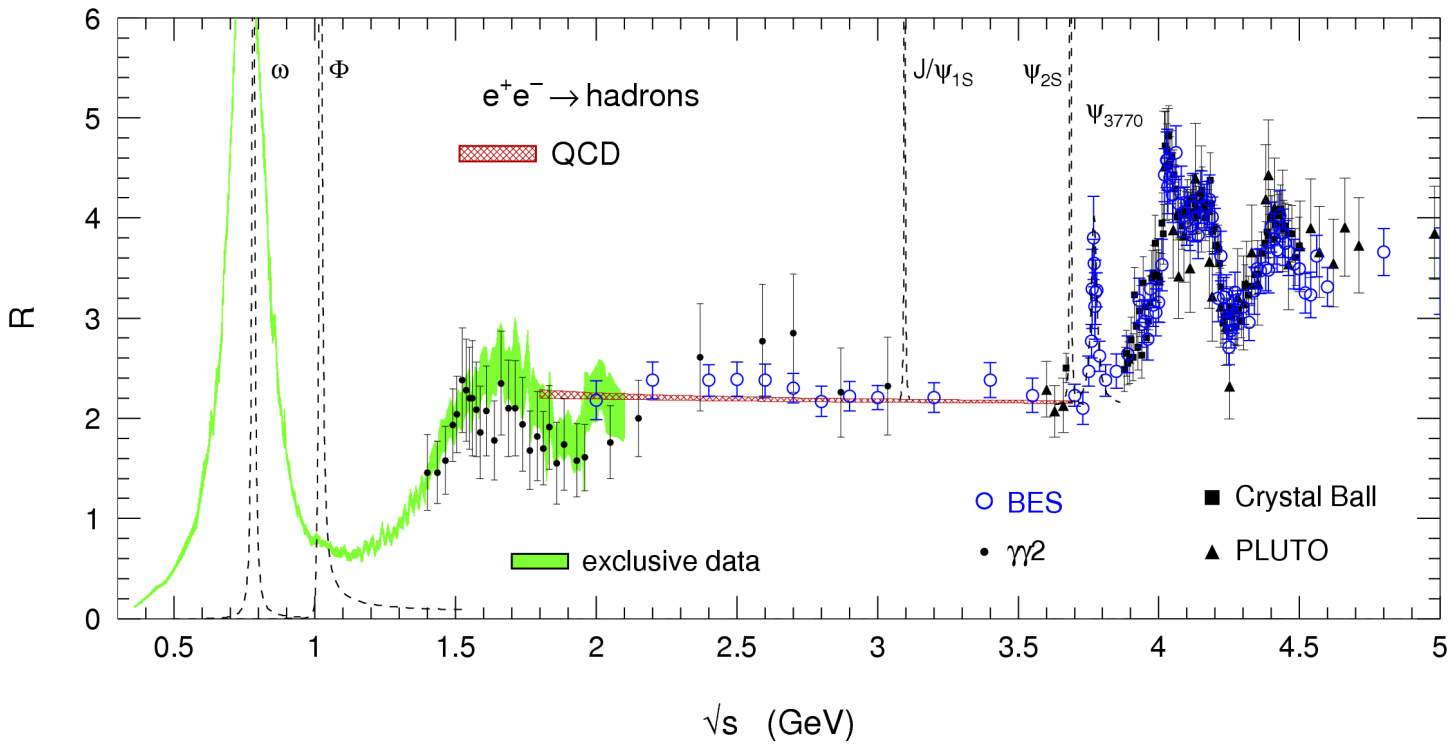
$$= \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2\left[\int_{m_\pi^2}^{s_{\rm cut}} ds\frac{K(s)}{s}R_{\rm had}^{\rm data}(s) + \int_{s_{\rm cut}}^{\infty} ds\frac{K(s)}{s}R_{\rm had}^{\rm pQCD}(s)\right]$$

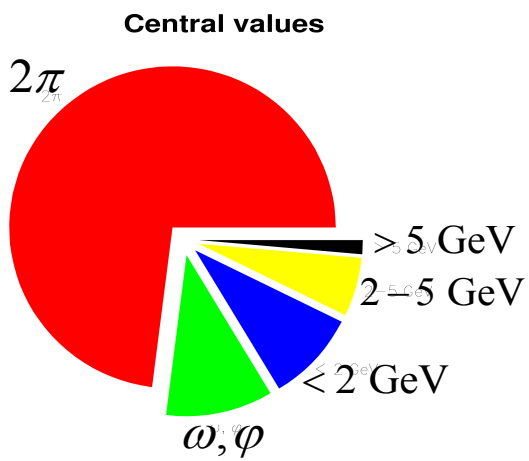


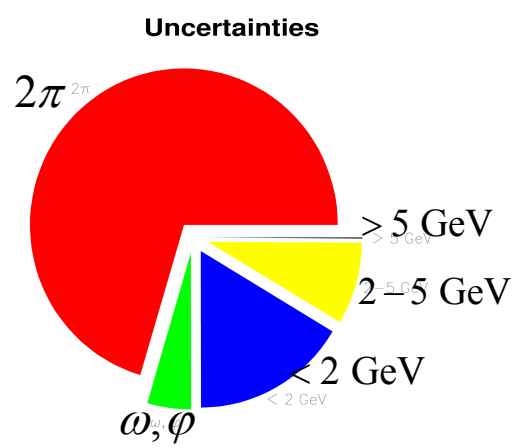


Hagiwara, et al. J.Phys. G38,085003 (2011)

g-2 from R-ratio



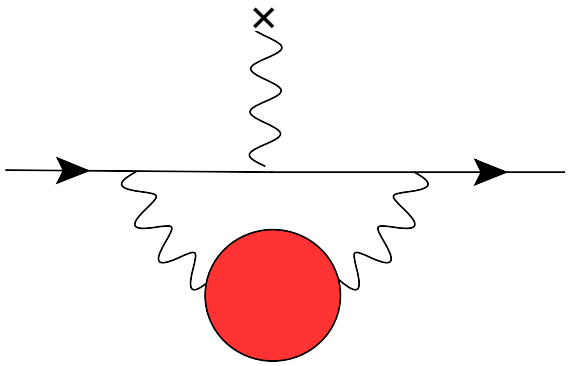




HVP from experimental data

- From experimental e+ e- total cross section ¤$_{total}$(e+e-) and dispersion relation

$$a_\mu^{\mathrm{HVP}} = \frac{1}{4\pi^2}\int_{4m_\pi^2}^{\infty} ds K(s)\sigma_{\mathrm{total}}(s)$$

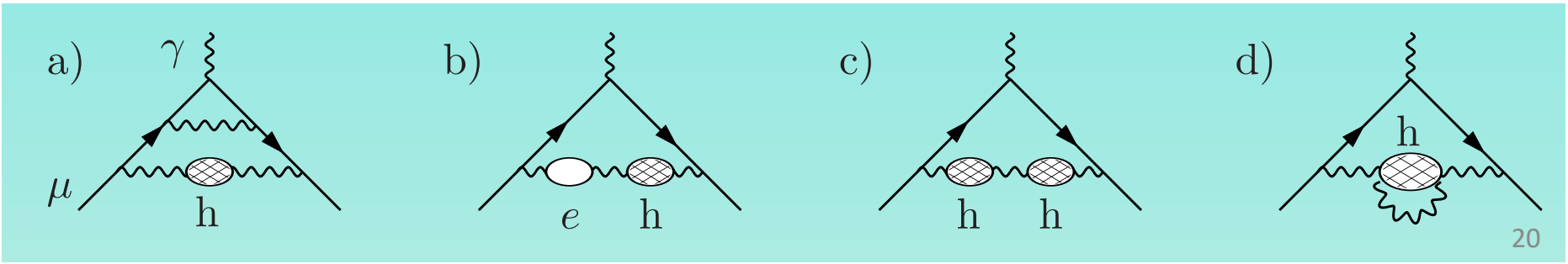


time like $q^2 = s >= 4\, m_\pi^2$

$$a_\mu^{\mathrm{HVP,LO}} = (694.91 \pm 4.27) \times 10^{-10}$$

$$a_\mu^{\mathrm{HVP,HO}} = (-9.84 \pm 0.07) \times 10^{-10}$$

[~ 0.6 % err]



KNT18 a_μ^{SM} update

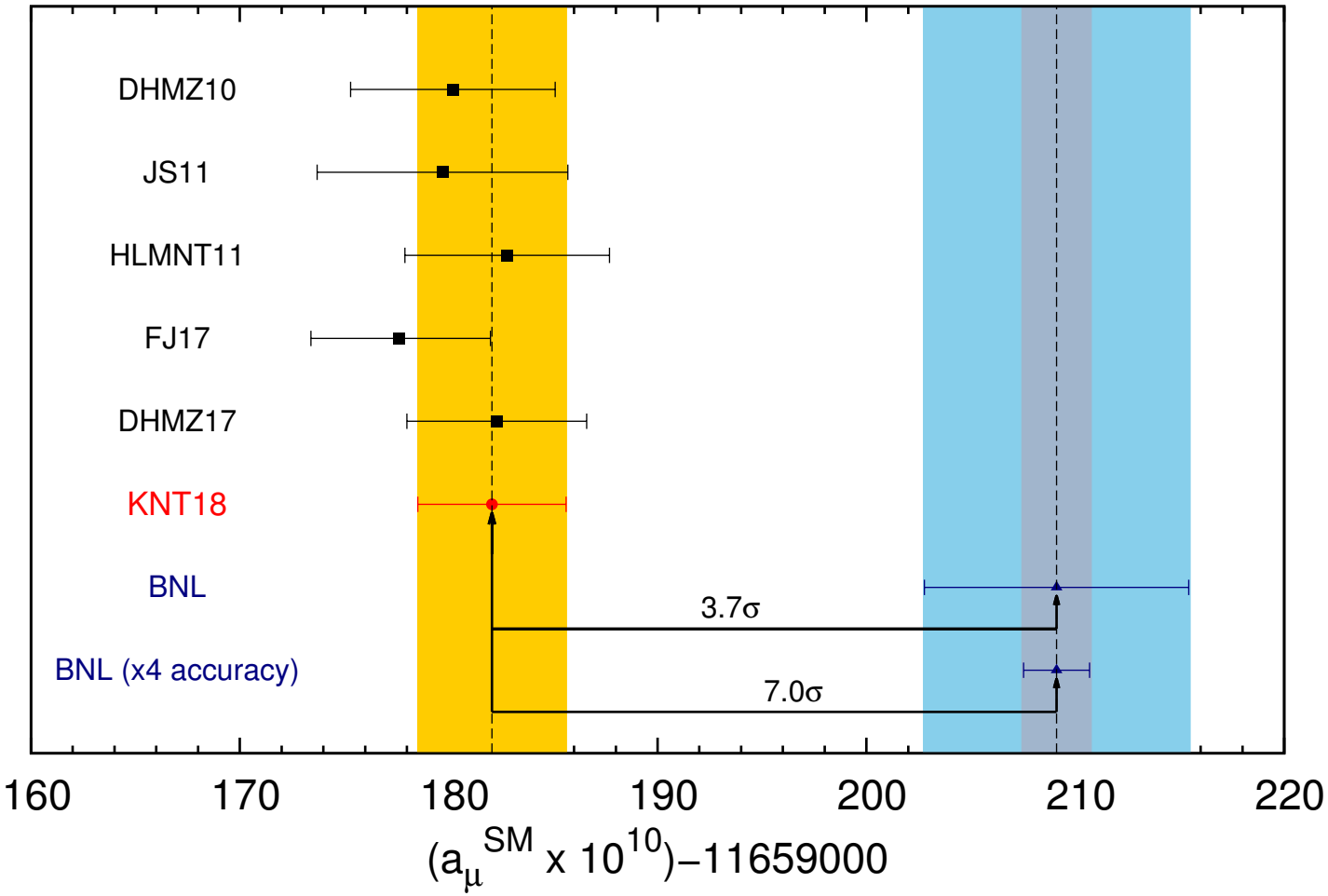
	2011		2017	
QED	11658471.81 (0.02)	$\longrightarrow$	11658471.90 (0.01)	[arXiv:1712.06060]
EW	15.40 (0.20)	$\longrightarrow$	15.36 (0.10)	[Phys. Rev. D 88 (2013) 053005]
LO HLbL	10.50 (2.60)	$\longrightarrow$	9.80 **(2.60)**	[EPJ Web Conf. 118 (2016) 01016]
NLO HLbL			0.30 (0.20)	[Phys. Lett. B 735 (2014) 90]
	HLMNT11		**KNT18**	
LO HVP	694.91 (4.27)	$\longrightarrow$	693.27 **(2.46)**	this work
NLO HVP	-9.84 (0.07)	$\longrightarrow$	-9.82 (0.04)	this work
NNLO HVP			1.24 (0.01)	[Phys. Lett. B 734 (2014) 144]
Theory total	11659182.80 (4.94)	$\longrightarrow$	11659182.05 (3.56)	this work
Experiment			11659209.10 (6.33)	world avg
Exp - Theory	26.1 (8.0)	$\longrightarrow$	27.1 (7.3)	this work
Δa_μ	3.3σ	$\longrightarrow$	3.7σ	this work

Alex Keshavarzi's talk at "HVP working group Muon g-2 Theory Initiative" @ KEK

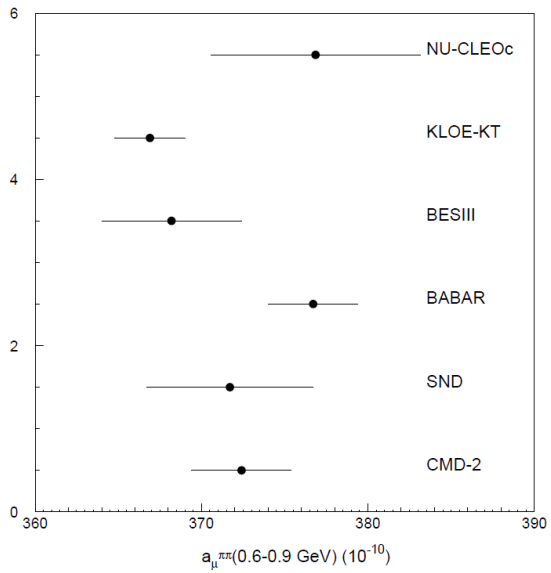
LO HVP : error 2.54 x10^{-10} [0.37%]

full covariance matrix will be public soon

KNT18 a_μ^{SM} update



The BABAR/KLOE discrepancy for $\pi\pi\gamma(\gamma)$



- BABAR and KLOE measurements most precise to date, but in poor agreement
- Others are in between, but not precise enough to decide
- No progress achieved in understanding the reason(s) of the discrepancy
- consequence: accuracy of combined results degraded
- imperative to improve accuracy of prediction (forthcoming g-2 results at FNAL, J-PARC)
- Other efforts at VEPP-2000 underway
- Design a new independent BABAR analysis

Idea : Cross check, combine, and improve by LQCD data

[T. Blum PRL91 (2003) 052001]

HVP from Lattice

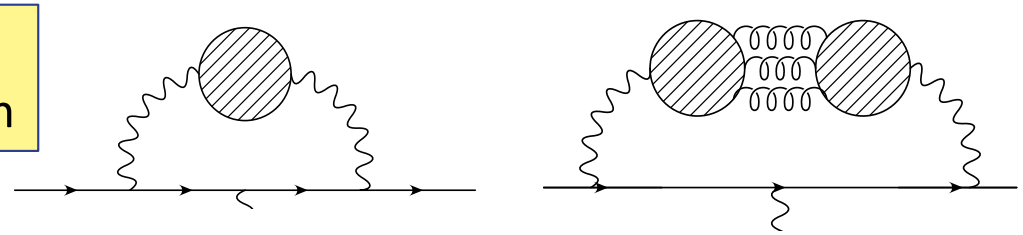
- Analytically continue to Euclidean/space-like momentum $K^2 = -q^2 > 0$
- Vector current 2pt function

$$a_\mu = \frac{g-2}{2} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2 f(K^2)\hat{\Pi}(K^2)$$

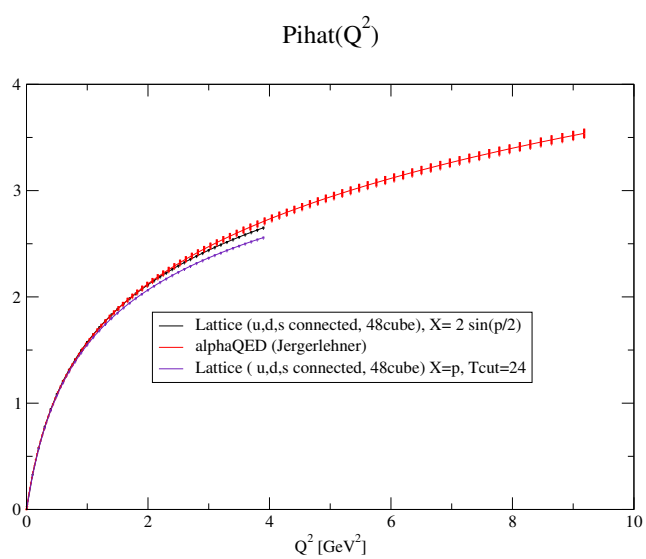
$$\Pi^{\mu\nu}(q) = \int d^4x e^{iqx} \langle J^\mu(x) J^\nu(0)\rangle$$

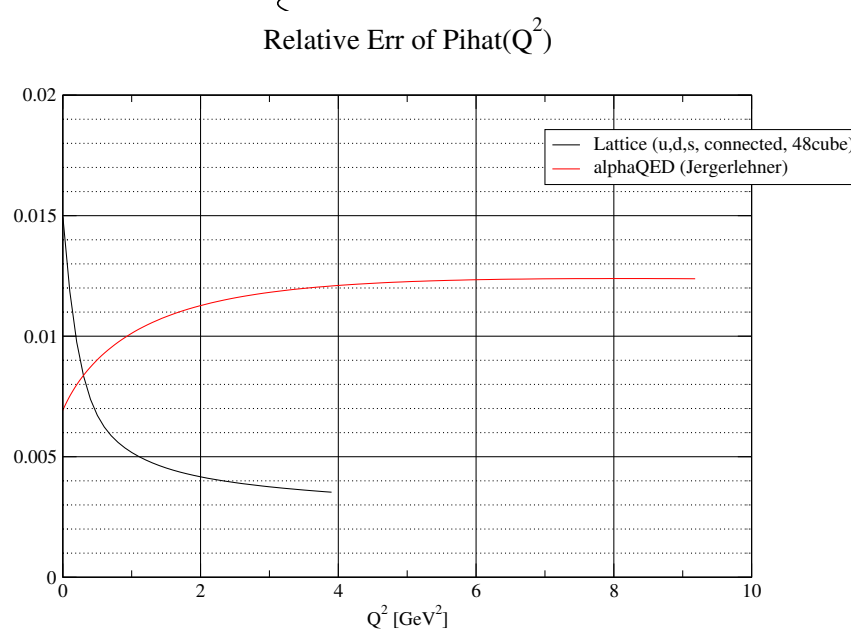
- Low Q2, or long distance, part of Π (Q2) is relevant for g-2

connected contribution



disconnected contribution





Simulation details [RBC/UKQCD 2015]

two gauge field ensembles generated by RBC/UKQCD collaborations

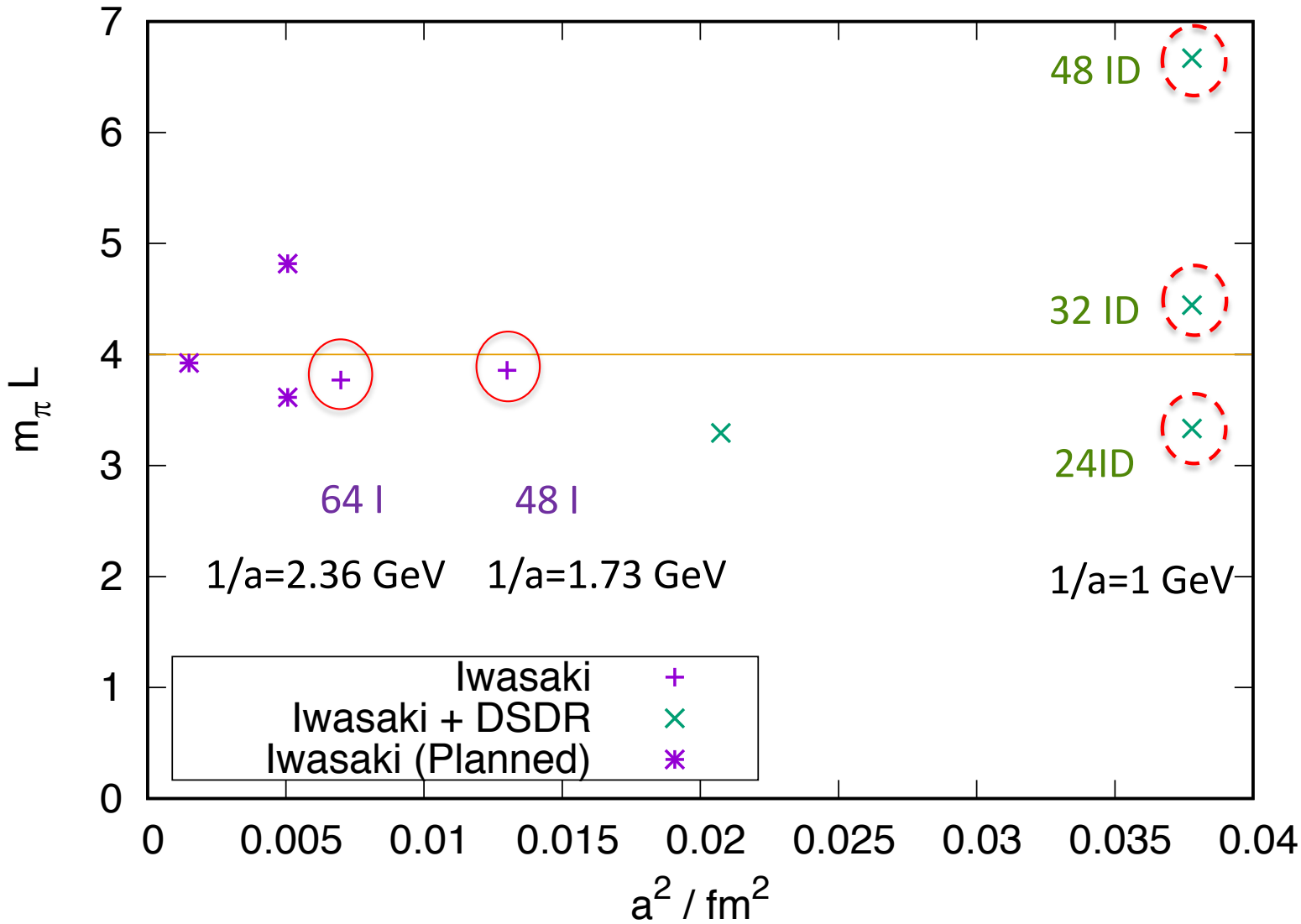
Domain wall fermions: chiral symmetry at finite a

Iwasaki Gauge action (gluons)

- pion mass $m_\pi = 139.2(2)$ and 139.3(3) MeV ($m_\pi L \lesssim 4$)
- lattice spacings $a = 0.114$ and 0.086 fm
- lattice scale $a^{-1} = 1.730$ and 2.359 GeV
- lattice size $L/a = 48$ and 64
- lattice volume $(5.476)^3$ and $(5.354)^3$ fm^3

Use all-mode-average (AMA) [Blum et al 2012] and low-mode- averaging (LMA) [Giusti et al, 2004, Degrand et al 2005, Lehner 2016 for HVP] techniques for improved statistics by more than three orders of magnitudes compared to basic CG, and $\times 10$ smaller memory via multigrid-Lanczos [Lehner 2017] .

Nf=2+1 DWF QCD ensemble at physical quark mass



Euclidean Time Momentum Representation

[Bernecker Meyer 2011, Feng et al. 2013]

In Euclidean space-time, project verctor 2 pt to zero spacial momentum, $\vec{p} = 0$:

$$C(t) = \frac{1}{3}\sum_{x,i}\langle j_i(x) j_i(0)\rangle$$

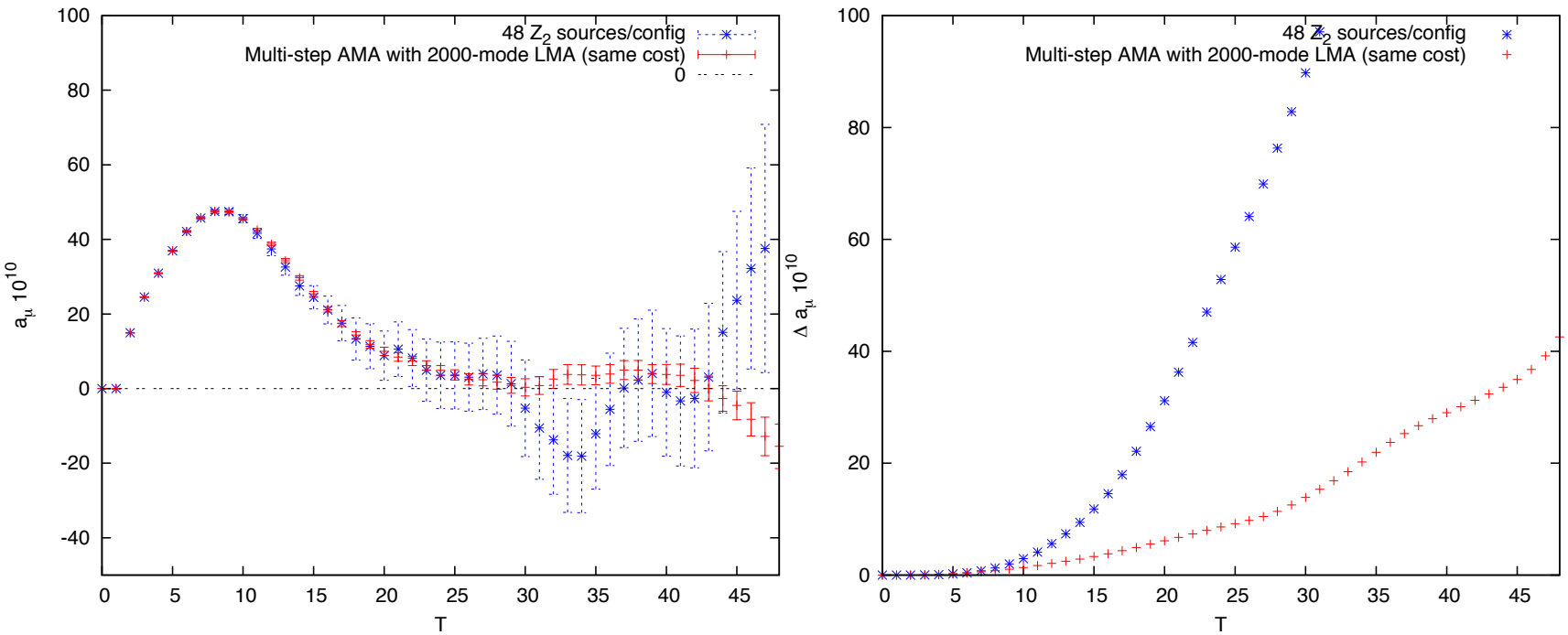
g-2 HVP contribution is

$$a_\mu^{HVP} = \sum_t w(t) C(t) \qquad \text{w(t) ~ t}^4$$

$$w(t) = 2\int_0^\infty \frac{d\omega}{\omega} f_{\text{QED}}(\omega^2)\left[\frac{\cos\omega t - 1}{\omega^2} + \frac{t^2}{2}\right]$$

- Subtraction $\Pi(0)$ is performed.
  Noise/Signal $\sim e^{(E_{\pi\pi} - m_\pi)t}$, is improved [Lehner et al. 2015] .
- Corresponding $\hat{\Pi}(Q^2)$ has exponentially small volume error [Portelli et al. 2016] . $w(t)$ includes the continuum QED part of the diagram

DWF light HVP [2016 Christoph Lehner]



120 conf (a=0.11fm), 80 conf (a=0.086fm) physical point Nf=2+1 Mobius DWF
4D full volume LMA with 2,000 eigen vector (of e/o preconditioned zMobius D^+D)
EV compression (1/10 memory) using local coherence [C. Lehner Lat2017 Poster]
In addition, 50 sloppy / conf via multi-level AMA
more than x 1,000 speed up compared to simple CG

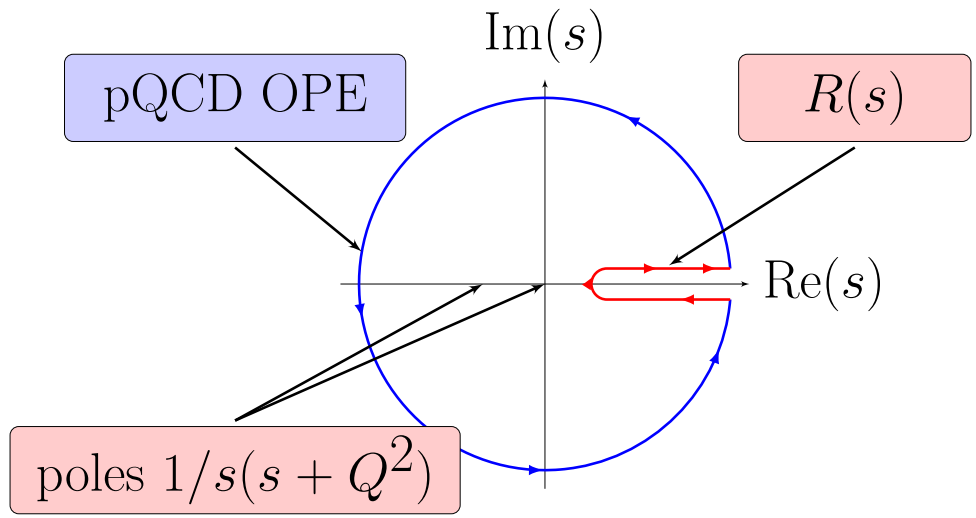
Euclidean time correlation from e^+e^- $R(s)$ data

From e^+e^- $R(s)$ ratio, using disparsive relation, zero-spacial momentum projected Euclidean correlation function $C(t)$ is obtained

$$\hat{\Pi}(Q^2) = Q^2 \int_0^\infty ds \frac{R(s)}{s(s+Q^2)}$$

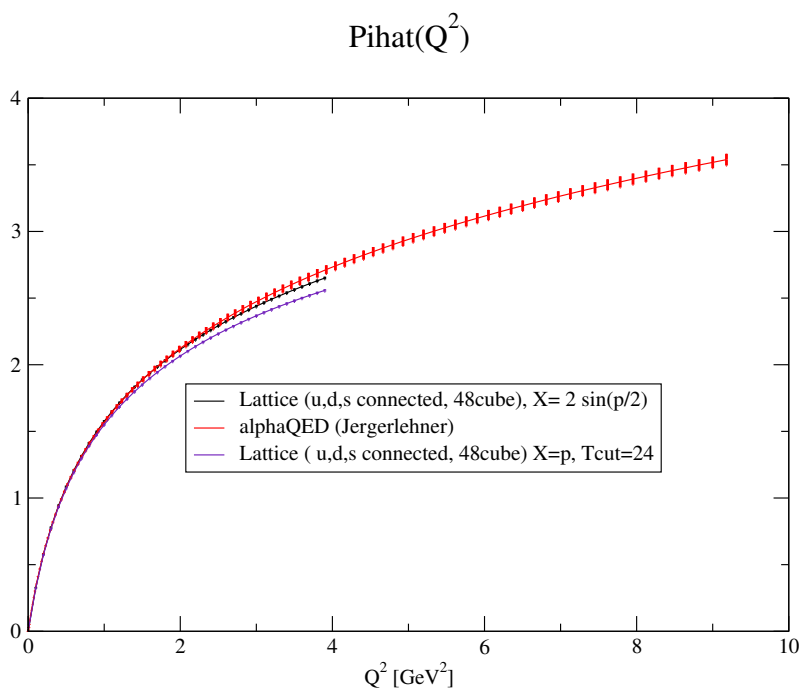
$$C^{\text{R-ratio}}(t) = \frac{1}{12\pi^2} \int_0^\infty \frac{d\omega}{2\pi} \hat{\Pi}(\omega^2) = \frac{1}{12\pi^2} \int_0^\infty ds \sqrt{s} R(s) e^{-\sqrt{s}t}$$

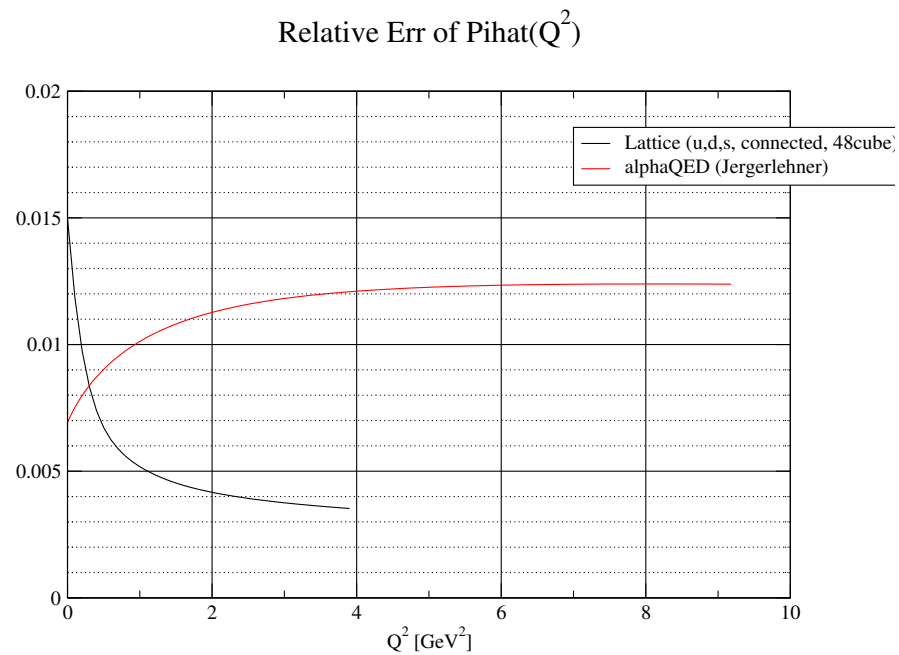
- $C(t)$ or $w(t)C(t)$ are directly comparable to Lattice results with the proper limits ($m_q \to m_q^{\text{phys}}, a \to 0, V \to \infty$, QED ...)
- Lattice: long distance has large statistical noise, (short distance: discretization error, removed by $a \to 0$ and/or pQCD)
- R-ratio : short distance has large error



$$\hat{\Pi}(Q^2) = Q^2 \int_0^\infty ds \frac{R(s)}{s(s+Q^2)}$$

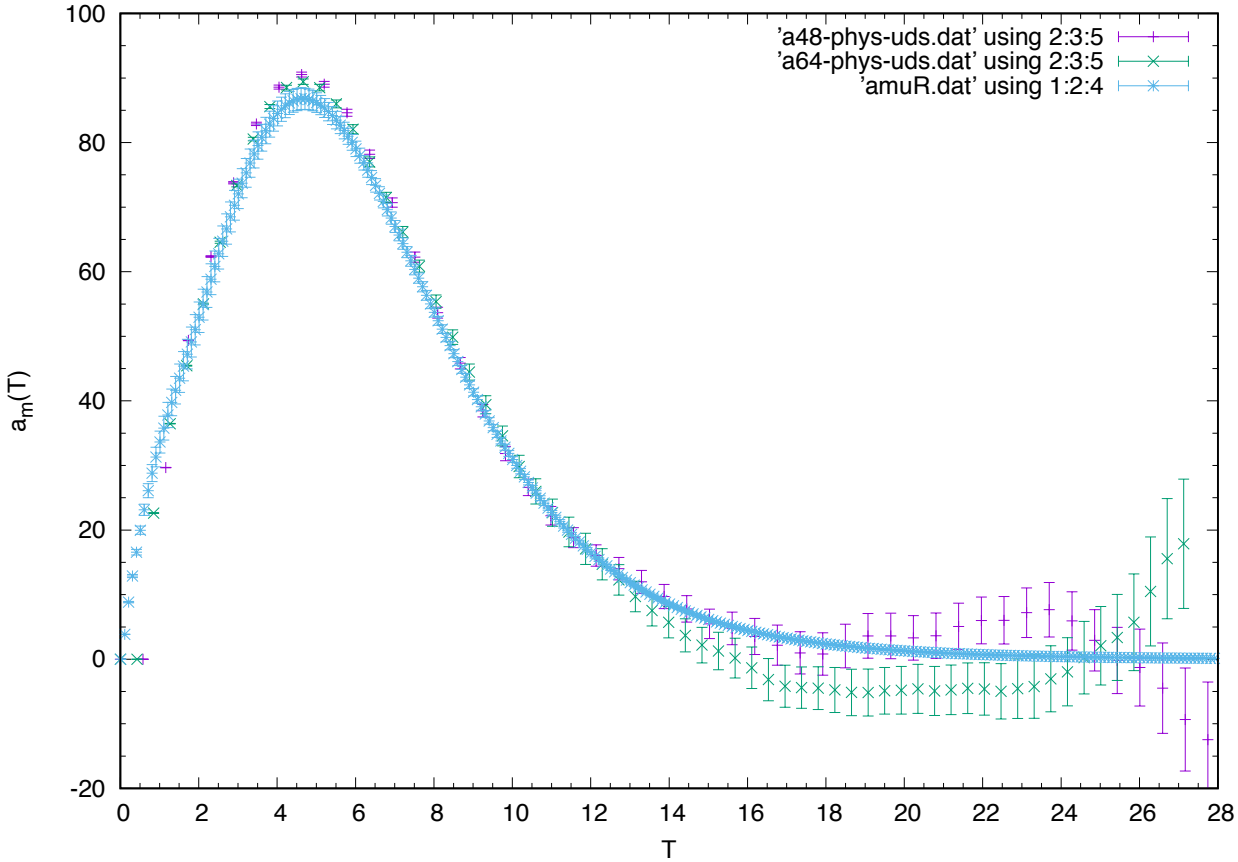
($1/a = 1.78$ GeV, Relative statistical error)



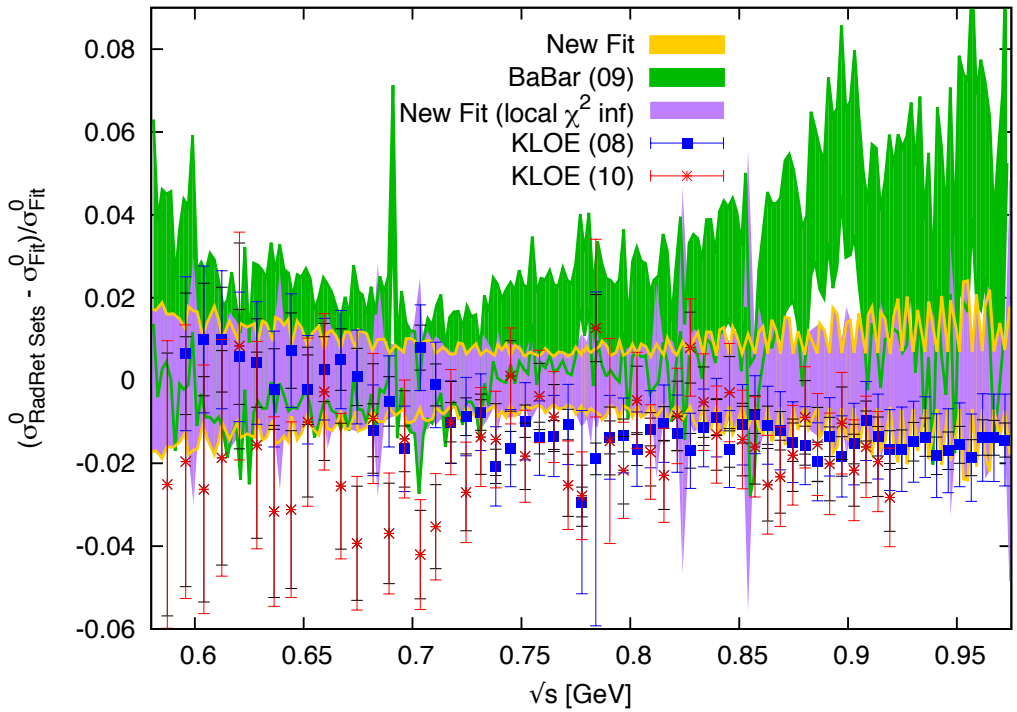


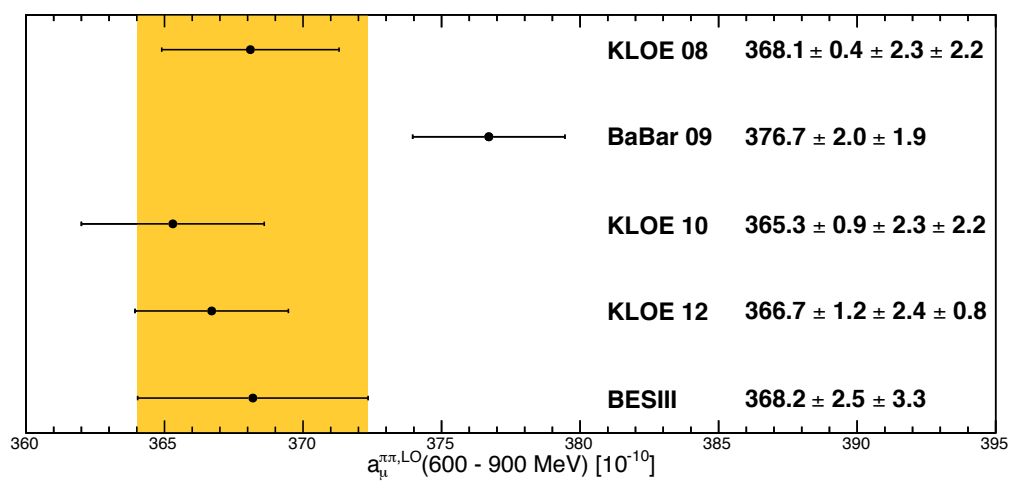
Comparison of R-ratio and Lattice
[F. Jegerlehner alphaQED 2016]

- Covariance matrix among energy bin in R-ratio is not available, assumes 100% correlated

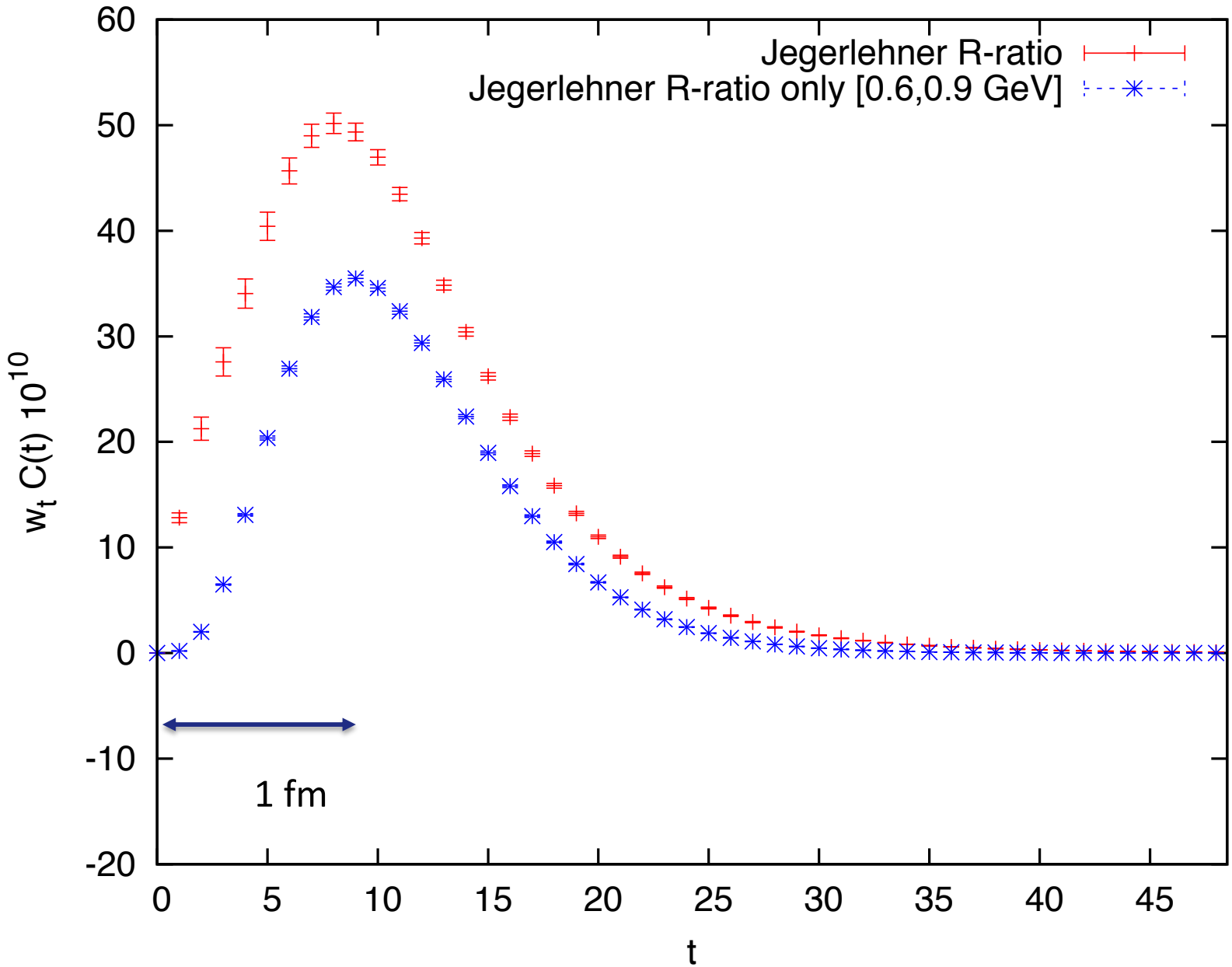


Near ρ peak, KLOE and Babar disagree





Careful comparison of R-ratio with lattice results may help



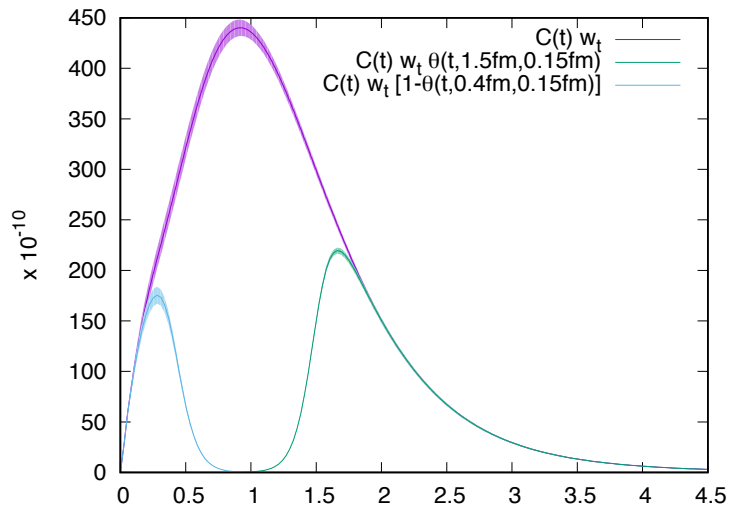
Combine R-ratio and Lattice

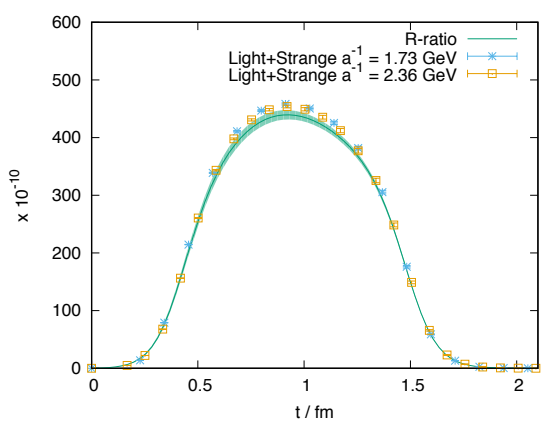
- Use short and long distance from R-ratio using smearing function, and mid-distance from lattice

$$\Theta(t,\mu,\sigma) \equiv [1+\tanh[(t-\mu)/\sigma]]/2$$

$$a_\mu = \sum_t w_t C(t) \equiv a_\mu^{\rm SD} + a_\mu^{\rm W} + a_\mu^{\rm LD}$$

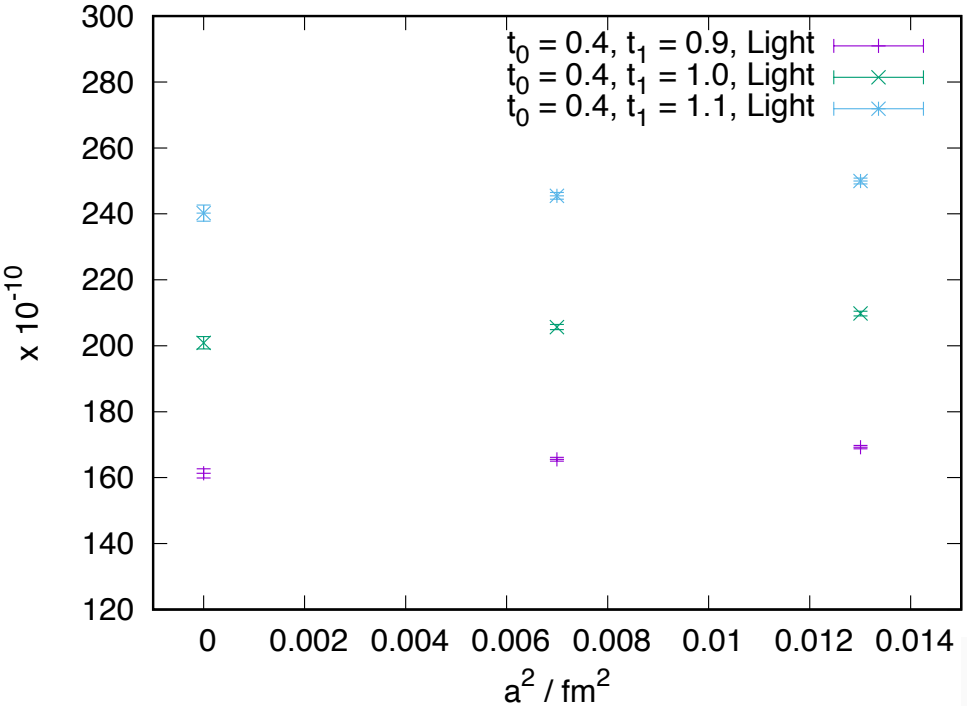
$$a_\mu^{\rm SD} = \sum_t C(t) w_t [1-\Theta(t,t_0,\Delta)]\,,$$
$$a_\mu^{\rm W} = \sum_t C(t) w_t [\Theta(t,t_0,\Delta)-\Theta(t,t_1,\Delta)]\,,$$
$$a_\mu^{\rm LD} = \sum_t C(t) w_t \Theta(t,t_1,\Delta)$$





Continuum limit of a^W

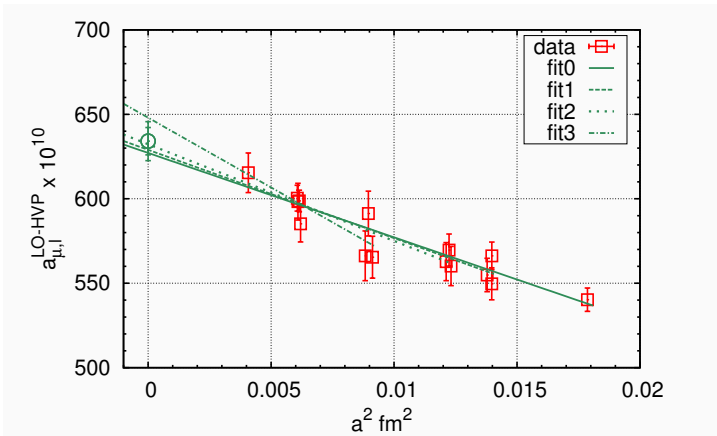
Continuum limit of a_μ^W from our lattice data; below $t_0 = 0.4$ fm and $\Delta = 0.15$ fm



RBC/UKQCD [C. Lehner Lat17]

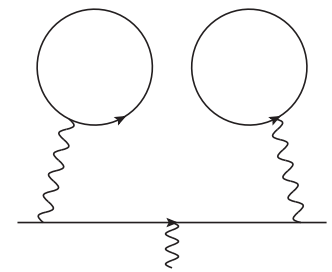
Continuum extrapolation is mild

c.f BMWc [K. Miura Lat17]



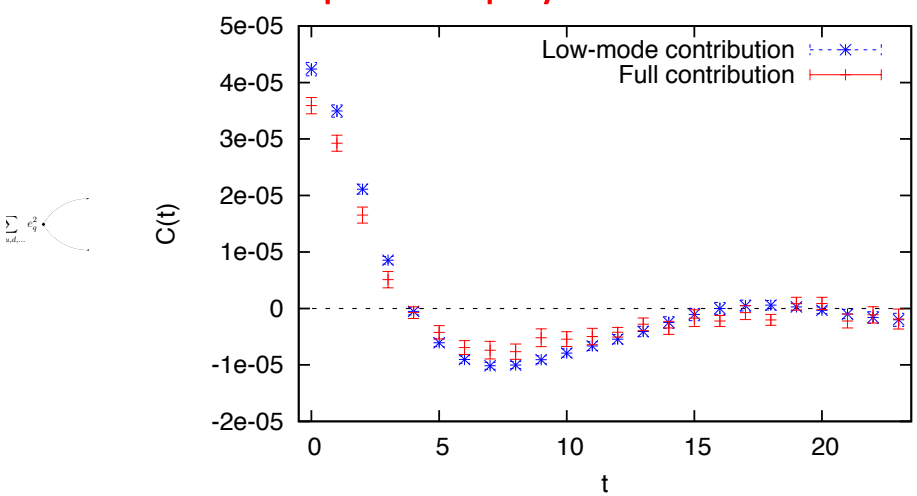
disconnected quark loop contribution

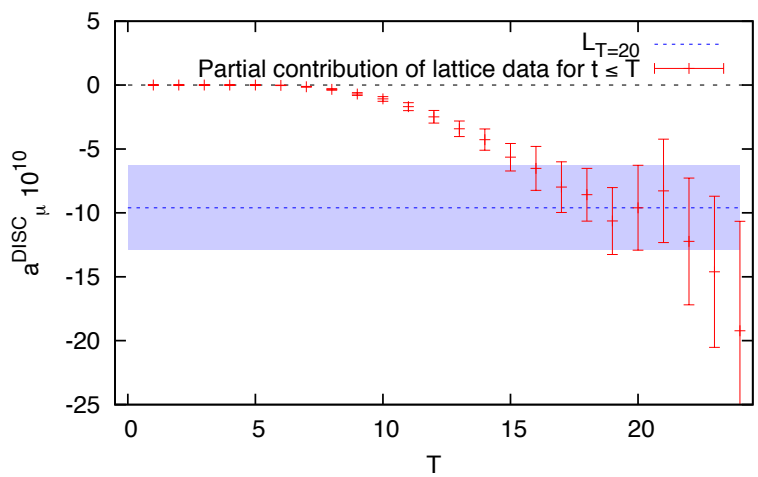
- [C. Lehner et al. (RBC/UKQCD 2015, arXiv:1512.09054, PRL)]
- Very challenging calculation due to statistical noise
- Small contribution, vanishes in SU(3) limit,
  Qu+Qd+Qs = 0
- Use low mode of quark propagator, treat it exactly
  (all-to-all propagator with sparse random source)
- First non-zero signal



$$a_\mu^{\mathrm{HVP\ (LO)\ DISC}} = -9.6(3.3)_{\mathrm{stat}}(2.3)_{\mathrm{sys}} \times 10^{-10}$$

Sensitive to m_π
crucial to compute at physical mass

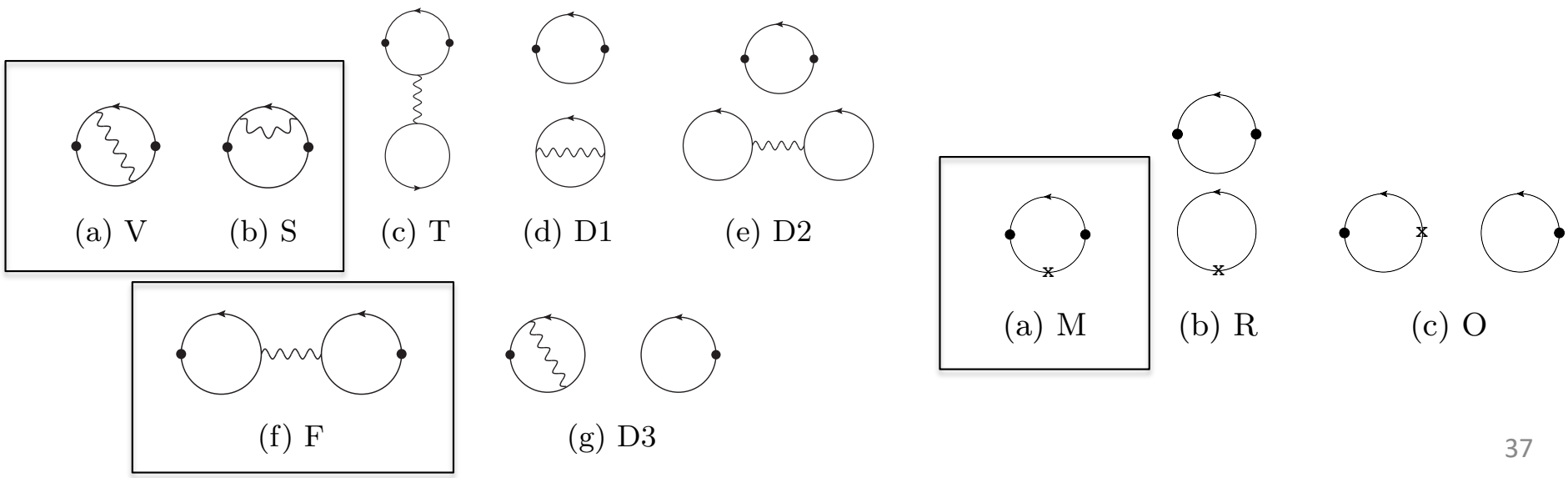




HVP QED+ strong IB corrections

[V. Gulpers's talk]

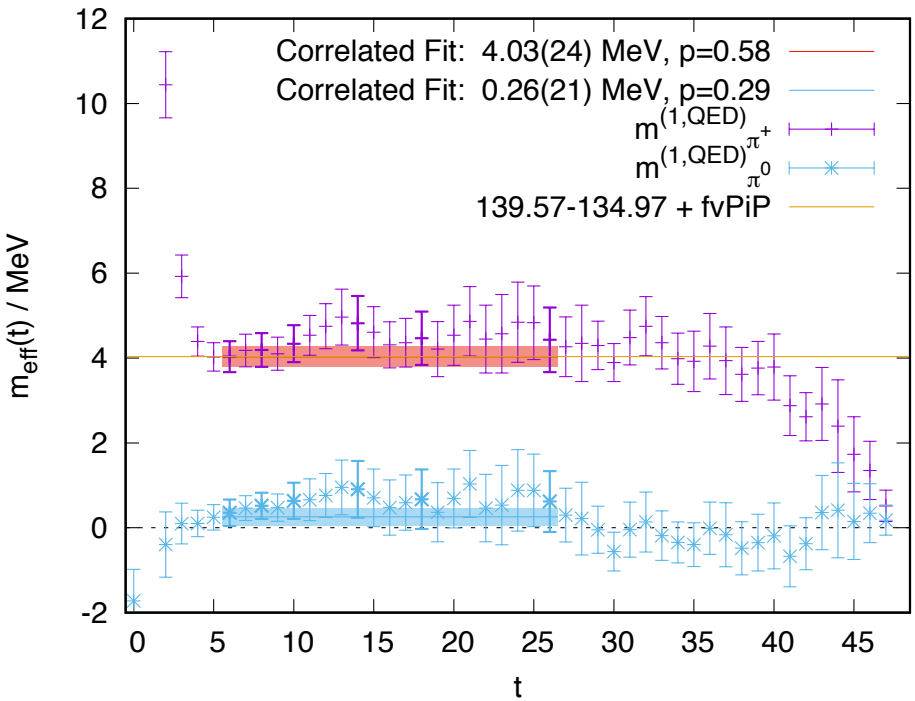
- HVP is computed so far at Iso-symmetric quark mass, needs to compute isospin breaking corrections : Qu, Qd, mu-md ≠0
- u,d,s quark mass and lattice spacing are re-tuned using {charge,neutral} x{pion,kaon} and (Omega baryon masses)
- For now, V, S, F, M are computed : assumes EM and IB of sea quark and also shift to lattice spacing is small (correction to disconnected diagram)
- Point-source method : stochastically sample pair of 2 EM vertices a la important sampling with exact photon



QED+IB retuning [2017 C. Lehner]

- Use QED_L for photon propagator, universal finite volume correction,
  => - 0.57 MeV shift
- 30 conf, a=0.11 fm, AMA per conf : 50x50 sloppy measurements for long distance, 25x25 for short distance.

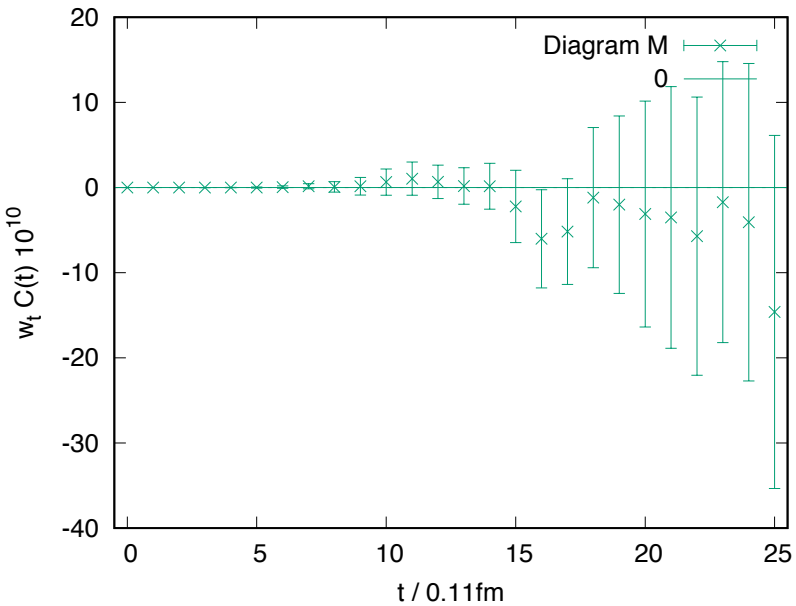
$$\Delta m^{FV} = -m_\pi \alpha_{\rm QED} \left(\frac{\kappa}{2 m_\pi L} \left(1 + \frac{2}{m_\pi L} \right) \right)$$

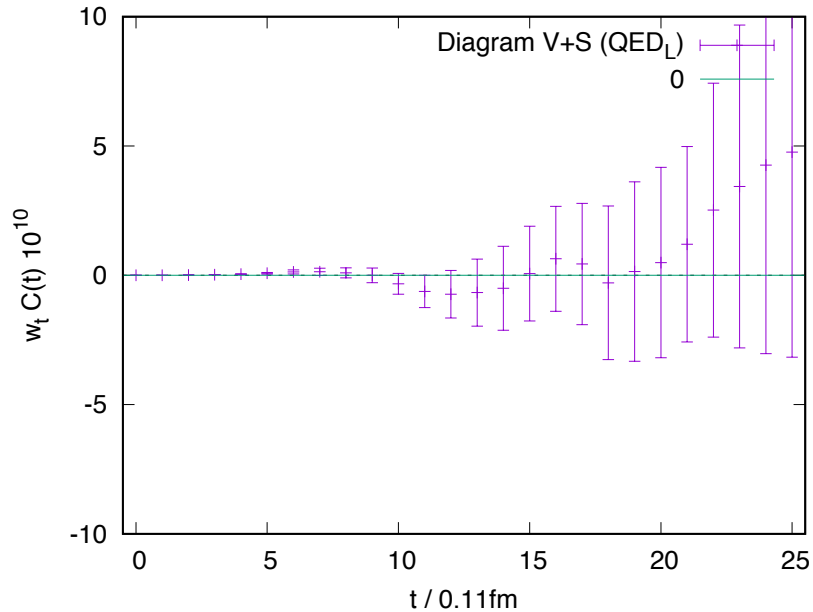


$$\Delta m_u = -0.000678(83)\,,$$
$$\Delta m_d = 0.000519(83)\,,$$
$$\Delta m_s = -0.000431(32)\,,$$
$$\frac{m_{res} + m_l + \Delta m_u}{m_{res} + m_l + \Delta m_d} = 0.373(59)\,,$$
$$\frac{m_u^{\rm PDG}}{m_d^{\rm PDG}} = 0.48(11)\,.$$

HVP IB+QED corrections

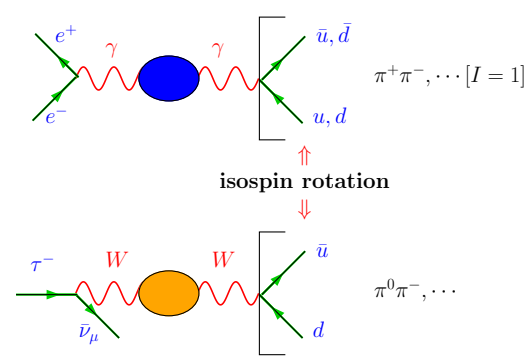
- Strong IB effect (left), EM effect (right)



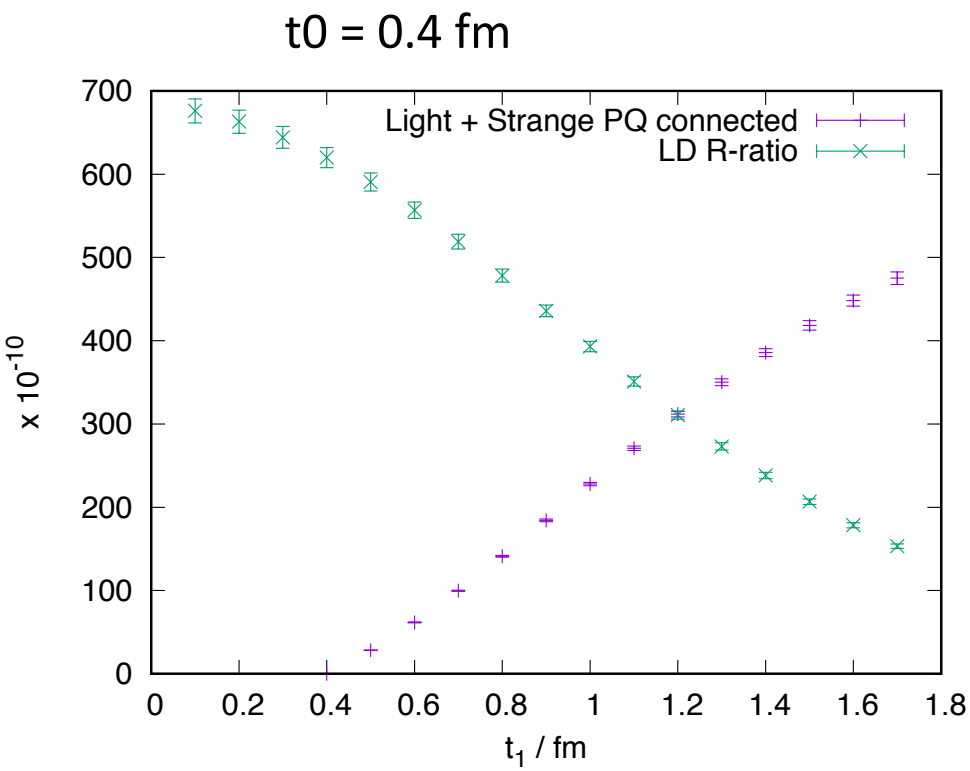


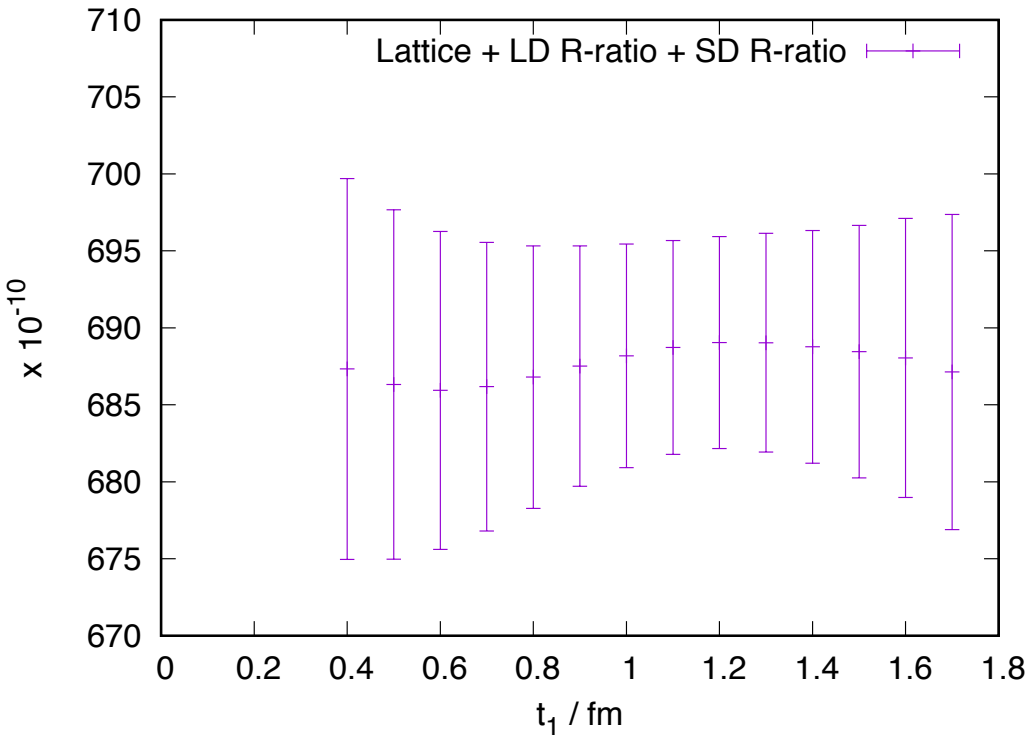
- Could also compute the difference IB correction of
  $a_\mu(e+e-) - a_\mu(\tau) \sim O(10) \times 10^{-10}$

[M. Bruno's talk]



R-ratio + Lattice
[Christoph Lehner Lat17]



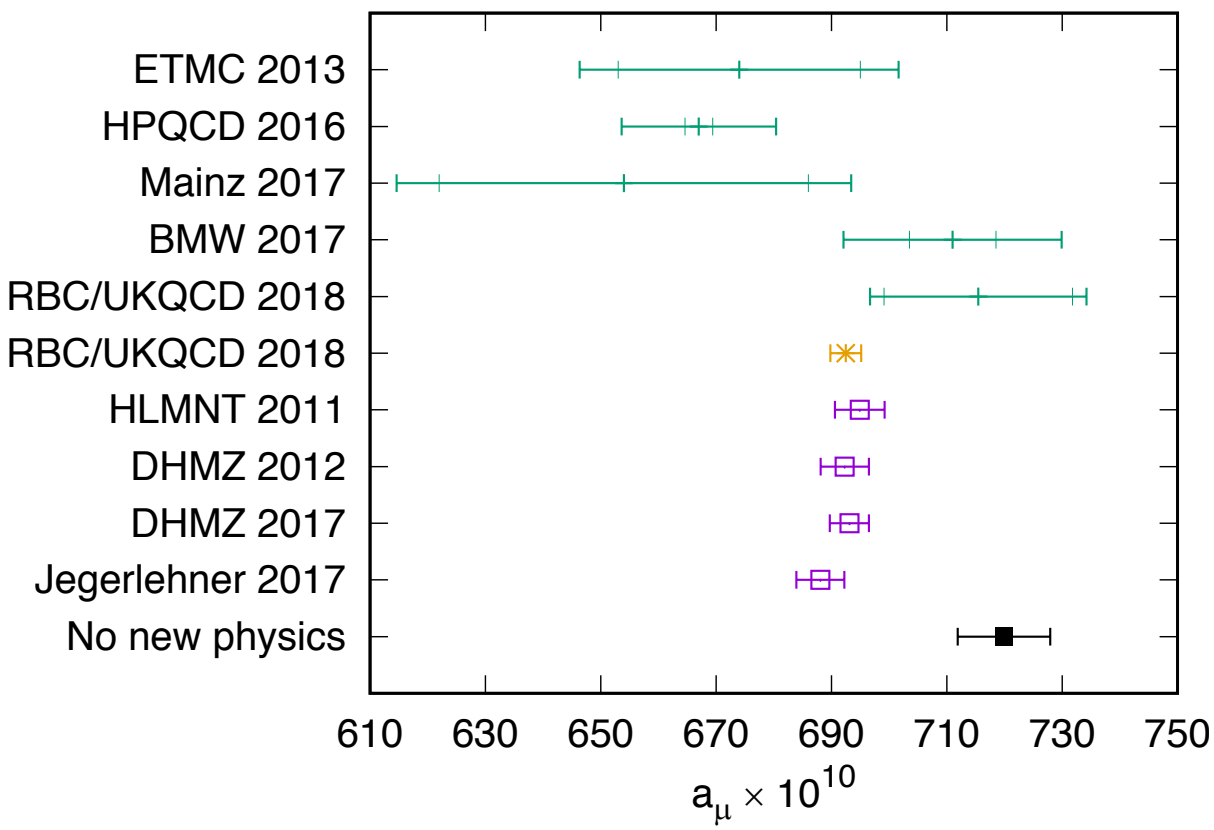


t1 dependence is flat => a consistency between R-ratio and Lattice

t1 = 1.2 fm, R-ratio : Lattice = 50:50

t1=1.2 fm current error (note 100% correlation in R-ratio) is minimum

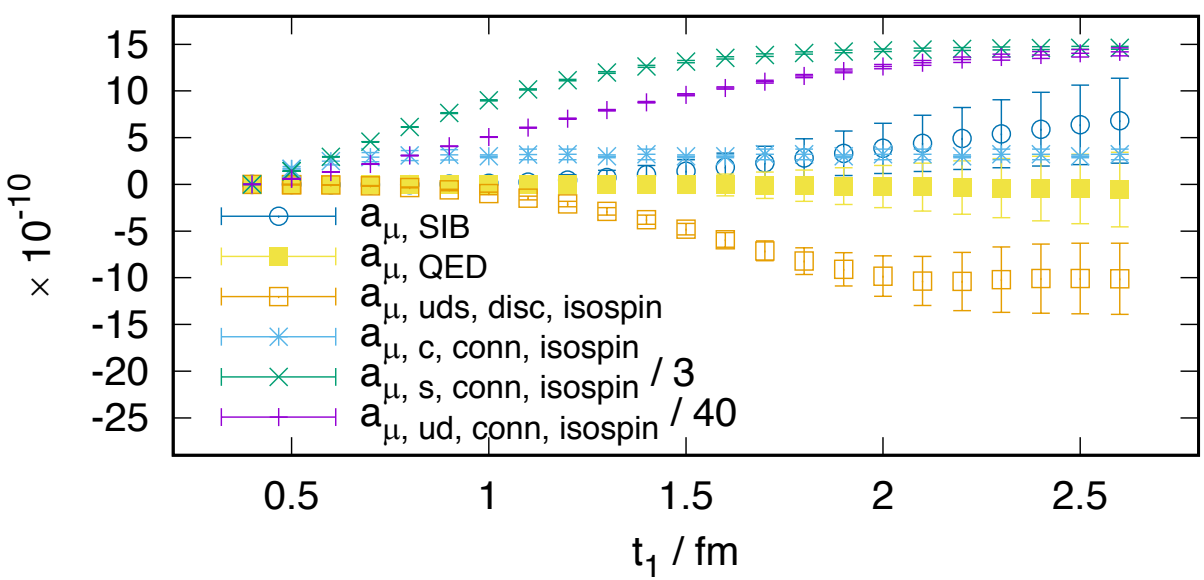
HVP Preliminary results
[Christoph Lehner et al. 1801.07224]

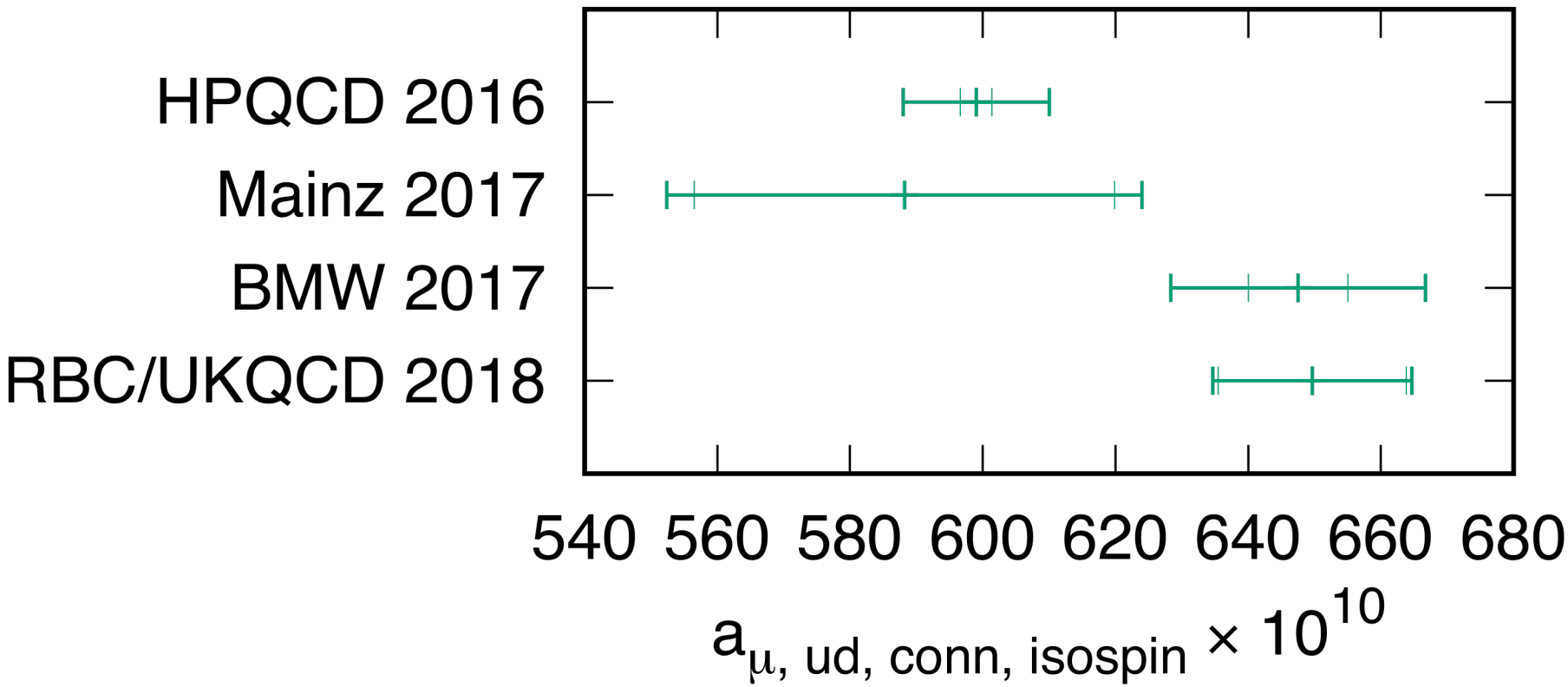


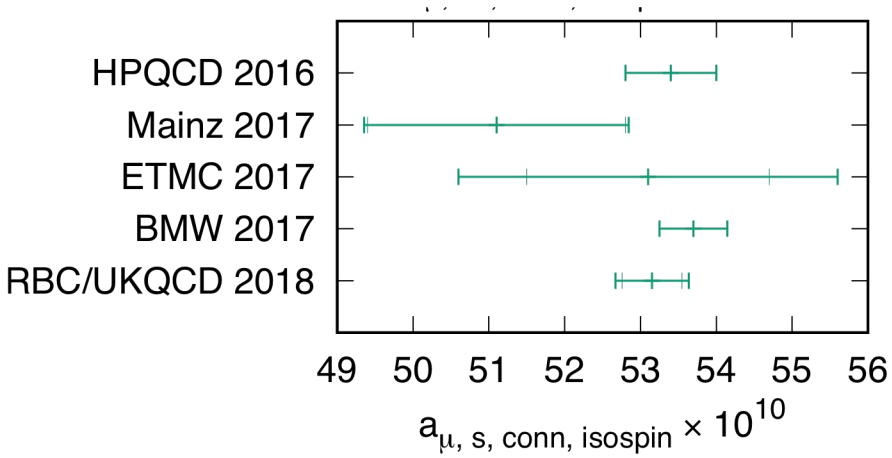
- Combined R-ratio +Lattice in window [0.4 fm, 1.2 fm] =>error 6.8e-10 [1 %]
- central value contributions R-ratio:Lattice = 2:1
- Finite Volume correction 3(3) e-10
- scale error : 0.2% => ~ 3e-10
- valence quark mass, a^4 error ~ negligible

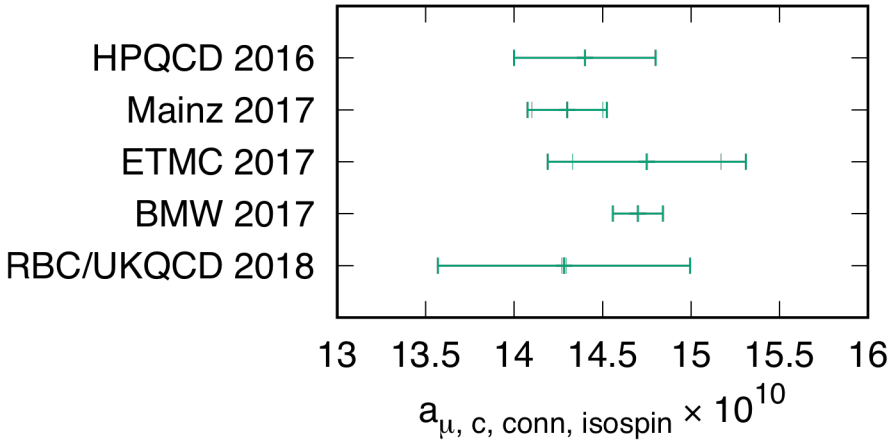
$a_\mu^{\text{ud, conn, isospin}}$	$202.9(1.4)_{\rm S}(0.2)_{\rm C}(0.1)_{\rm V}(0.2)_{\rm A}(0.2)_{\rm Z}$	$649.7(14.2)_{\rm S}(2.8)_{\rm C}(3.7)_{\rm V}(1.5)_{\rm A}(0.4)_{\rm Z}(0.1)_{\rm E48}(0.1)_{\rm E64}$
$a_\mu^{\text{s, conn, isospin}}$	$27.0(0.2)_{\rm S}(0.0)_{\rm C}(0.1)_{\rm A}(0.0)_{\rm Z}$	$53.2(0.4)_{\rm S}(0.0)_{\rm C}(0.3)_{\rm A}(0.0)_{\rm Z}$
$a_\mu^{\text{c, conn, isospin}}$	$3.0(0.0)_{\rm S}(0.1)_{\rm C}(0.0)_{\rm Z}(0.0)_{\rm M}$	$14.3(0.0)_{\rm S}(0.7)_{\rm C}(0.1)_{\rm Z}(0.0)_{\rm M}$
$a_\mu^{\text{uds, disc, isospin}}$	$-1.0(0.1)_{\rm S}(0.0)_{\rm C}(0.0)_{\rm V}(0.0)_{\rm A}(0.0)_{\rm Z}$	$-11.2(3.3)_{\rm S}(0.4)_{\rm V}(2.3)_{\rm L}$
$a_\mu^{\text{QED, conn}}$	$0.2(0.2)_{\rm S}(0.0)_{\rm C}(0.0)_{\rm V}(0.0)_{\rm A}(0.0)_{\rm Z}(0.0)_{\rm E}$	$5.9(5.7)_{\rm S}(0.3)_{\rm C}(1.2)_{\rm V}(0.0)_{\rm A}(0.0)_{\rm Z}(1.1)_{\rm E}$
$a_\mu^{\text{QED, disc}}$	$-0.2(0.1)_{\rm S}(0.0)_{\rm C}(0.0)_{\rm V}(0.0)_{\rm A}(0.0)_{\rm Z}(0.0)_{\rm E}$	$-6.9(2.1)_{\rm S}(0.4)_{\rm C}(1.4)_{\rm V}(0.0)_{\rm A}(0.0)_{\rm Z}(1.3)_{\rm E}$
a_μ^{SIB}	$0.1(0.2)_{\rm S}(0.0)_{\rm C}(0.2)_{\rm V}(0.0)_{\rm A}(0.0)_{\rm Z}(0.0)_{\rm E48}$	$10.6(4.3)_{\rm S}(0.6)_{\rm C}(6.6)_{\rm V}(0.1)_{\rm A}(0.0)_{\rm Z}(1.3)_{\rm E48}$
$a_\mu^{\text{udsc, isospin}}$	$231.9(1.4)_{\rm S}(0.2)_{\rm C}(0.1)_{\rm V}(0.3)_{\rm A}(0.2)_{\rm Z}(0.0)_{\rm M}$	$705.9(14.6)_{\rm S}(2.9)_{\rm C}(3.7)_{\rm V}(1.8)_{\rm A}(0.4)_{\rm Z}(2.3)_{\rm L}(0.1)_{\rm E48}$ $(0.1)_{\rm E64}(0.0)_{\rm M}$
$a_\mu^{\text{QED, SIB}}$	$0.1(0.3)_{\rm S}(0.0)_{\rm C}(0.2)_{\rm V}(0.0)_{\rm A}(0.0)_{\rm Z}(0.0)_{\rm E}(0.0)_{\rm E48}$	$9.5(7.4)_{\rm S}(0.7)_{\rm C}(6.9)_{\rm V}(0.1)_{\rm A}(0.0)_{\rm Z}(1.7)_{\rm E}(1.3)_{\rm E48}$
$a_\mu^{\text{R-ratio}}$	$460.4(0.7)_{\rm RST}(2.1)_{\rm RSY}$	
a_μ	$692.5(1.4)_{\rm S}(0.2)_{\rm C}(0.2)_{\rm V}(0.3)_{\rm A}(0.2)_{\rm Z}(0.0)_{\rm E}(0.0)_{\rm E48}$ $(0.0)_{\rm b}(0.1)_{\rm c}(0.0)_{\overline{\rm S}}(0.0)_{\overline{\rm Q}}(0.0)_{\rm M}(0.7)_{\rm RST}(2.1)_{\rm RSY}$	$715.4(16.3)_{\rm S}(3.0)_{\rm C}(7.8)_{\rm V}(1.9)_{\rm A}(0.4)_{\rm Z}(1.7)_{\rm E}(2.3)_{\rm L}$ $(1.5)_{\rm E48}(0.1)_{\rm E64}(0.3)_{\rm b}(0.2)_{\rm c}(1.1)_{\overline{\rm S}}(0.3)_{\overline{\rm Q}}(0.0)_{\rm M}$

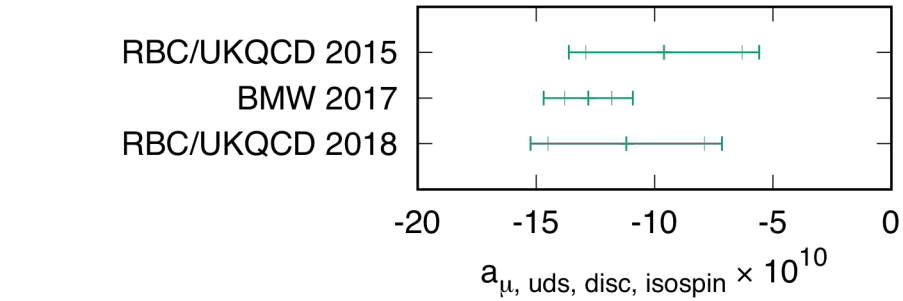
TABLE I. Individual and summed contributions to a_μ multiplied by 10^{10}. The left column lists results for the window method with $t_0 = 0.4$ fm and $t_1 = 1$ fm. The right column shows results for the pure first-principles lattice calculation. The respective uncertainties are defined in the main text.

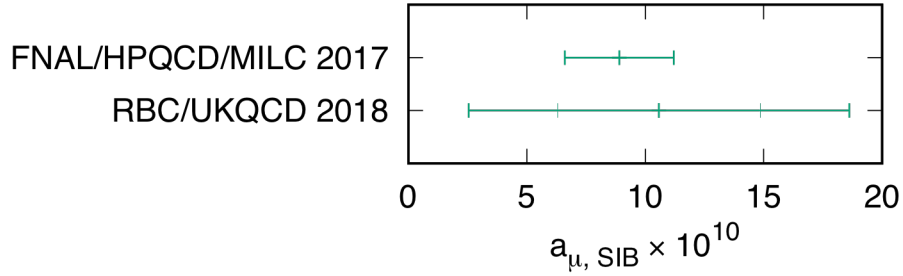


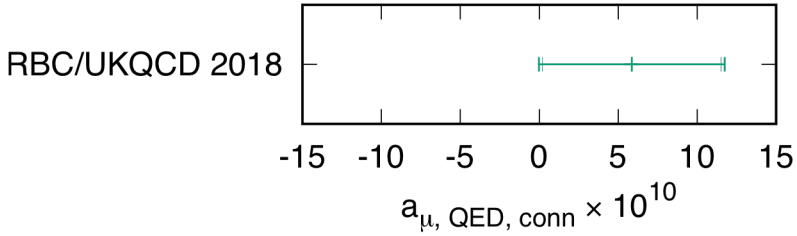

HPQCD 2016
Mainz 2017
BMW 2017
RBC/UKQCD 2018
540 560 580 600 620 640 660 680
$a_{\mu,\ ud,\ conn,\ isospin} \times 10^{10}$

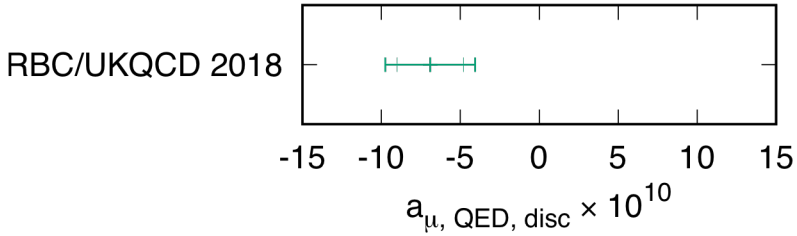

HPQCD 2016
Mainz 2017
ETMC 2017
BMW 2017
RBC/UKQCD 2018
49 50 51 52 53 54 55 56
$a_{\mu,\ s,\ conn,\ isospin} \times 10^{10}$

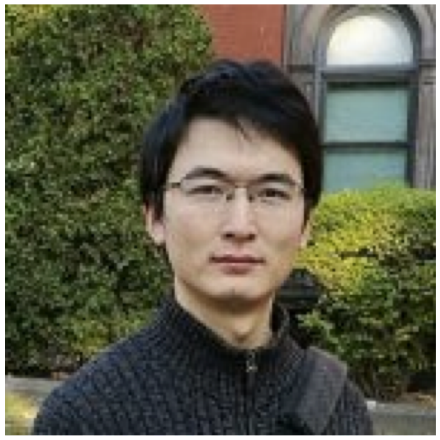

HPQCD 2016
Mainz 2017
ETMC 2017
BMW 2017
RBC/UKQCD 2018
13 13.5 14 14.5 15 15.5 16
$a_{\mu,\ c,\ conn,\ isospin} \times 10^{10}$


RBC/UKQCD 2015
BMW 2017
RBC/UKQCD 2018
-20 -15 -10 -5 0
$a_{\mu,\ uds,\ disc,\ isospin} \times 10^{10}$


FNAL/HPQCD/MILC 2017
RBC/UKQCD 2018
0 5 10 15 20
$a_{\mu,\ SIB} \times 10^{10}$


RBC/UKQCD 2018
-15 -10 -5 0 5 10 15
$a_{\mu,\ QED,\ conn} \times 10^{10}$

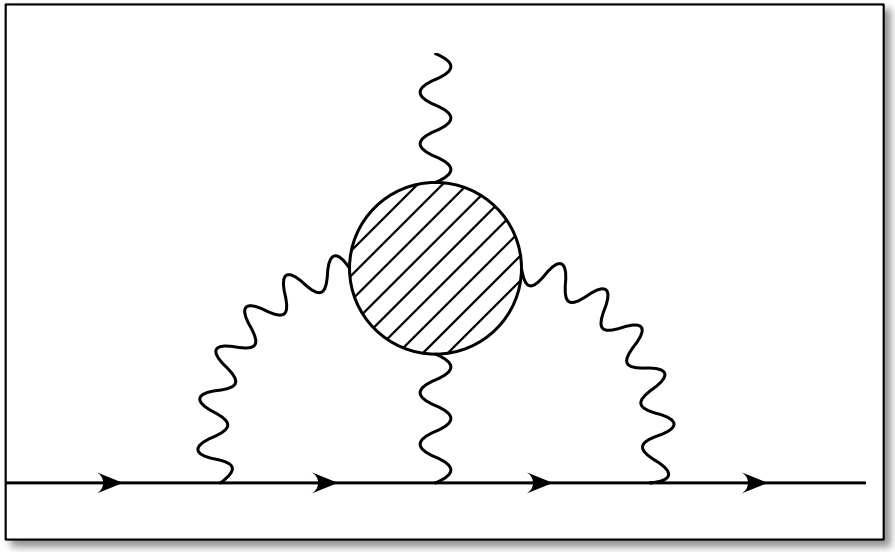

RBC/UKQCD 2018
-15 -10 -5 0 5 10 15
$a_{\mu,\ QED,\ disc} \times 10^{10}$



[Antonie Geradine's talk]

[Luchang Jin et al.
Phys.Rev. D96 (2017) no.3, 034515
Phys.Rev.Lett. 118 (2017) no.2, 022005]

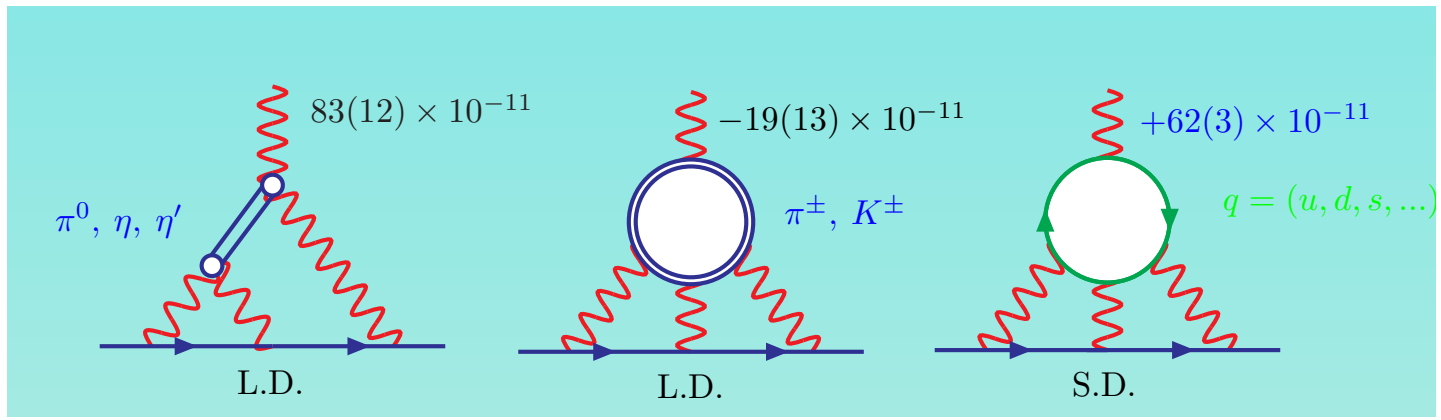
Hadronic Light-by-Light (HLbL) contributions



HLbL from Models

- Model estimate with non-perturbative constraints at the chiral / low energy limits using anomaly : (9—12) x 10^{-10} with 25-40% uncertainty

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 28.8(6.3)_{\rm exp}(4.9)_{\rm SM} \times 10^{-10} \quad [3.6\sigma]$$



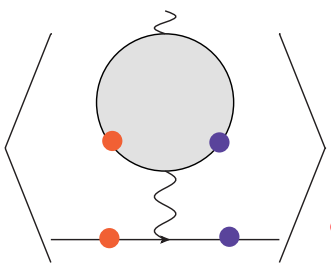
F. Jegerlehner , x 10^{11}

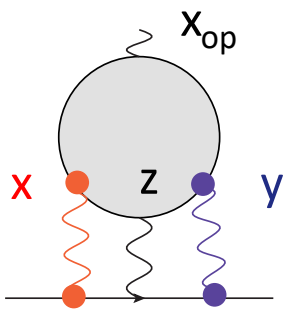
Contribution	BPP	HKS	KN	MV	PdRV	N/JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	0±10	−19±19	−19±13
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	15±10	22± 5
scalars	−6.8±2.0	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	105±26	116±39

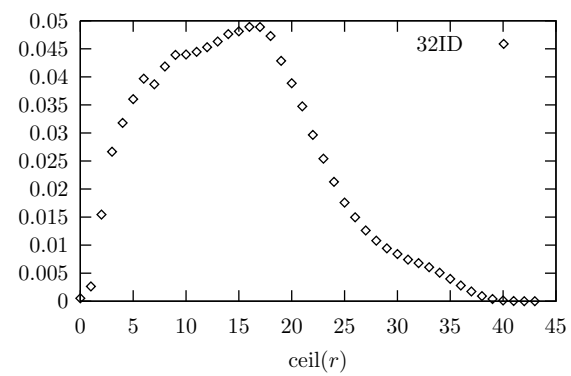
Coordinate space Point photon method

[Luchang Jin et al. , PRD93, 014503 (2016)]

- Treat all 3 photon propagators exactly (3 analytical photons) , which makes the quark loop and the lepton line connected :
  disconnected problem in Lattice QED+QCD -> connected problem with analytic photon
- QED 2-loop in coordinate space. Stochastically sample, two of quark-photon vertex location x,y, z and x_{op} is summed over space-time exactly







- Short separations, Min[|x-z|,|y-z|,|x-y|] < R ~ O(0.5) fm, which has a large contribution due to confinement, are summed for all pairs
- longer separations, Min[|x-z|,|y-z|,|x-y|] >= R, are done stochastically with a probability shown above (Adaptive Monte Carlo sampling)

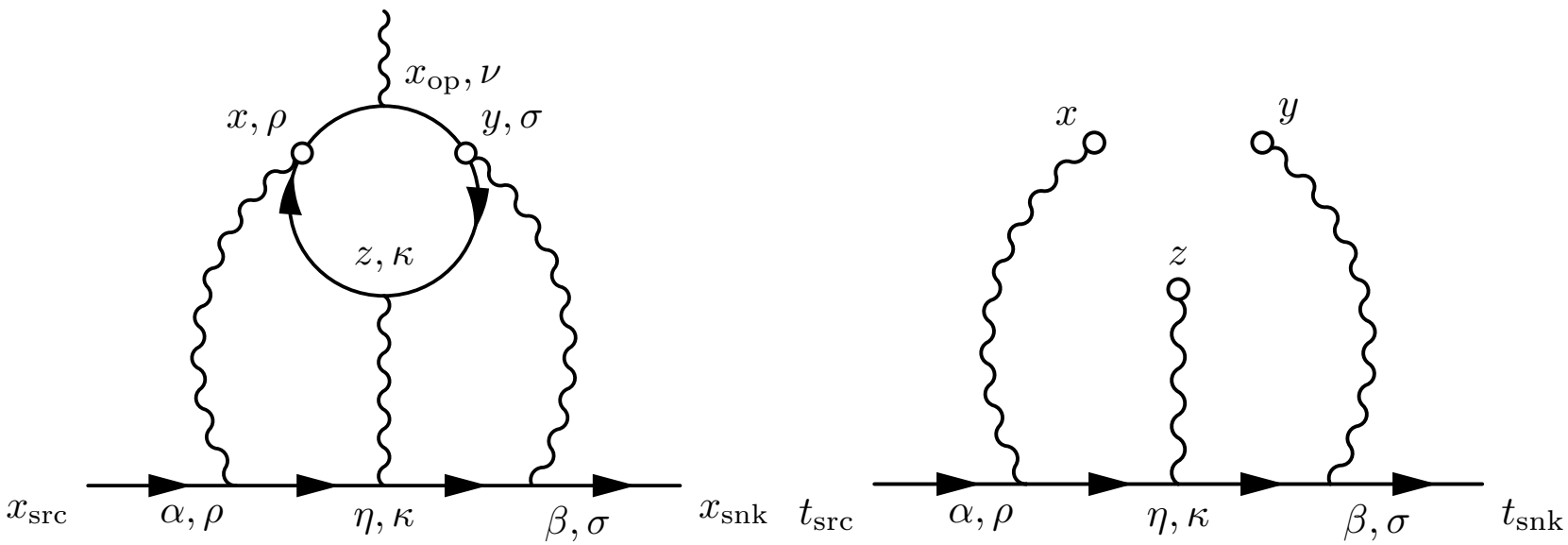
HLbL point source method [L. Jin et al. 1510.07100]

- Anomalous magnetic moment, $F_2(q^2)$ at $q^2 \to 0$ limit

$$\frac{F_2^{\mathrm{cHLbL}}(q^2=0)}{m}\frac{(\sigma_{s',s})_i}{2} = \frac{\sum_{x,y,z,x_{\mathsf{op}}}}{2VT}\epsilon_{i,j,k}\left(x_{\mathsf{op}}-x_{\mathrm{ref}}\right)_j \cdot i\bar{u}_{s'}(\vec{0})\mathcal{F}_k^C\left(x,y,z,x_{\mathsf{op}}\right)u_s(\vec{0})$$

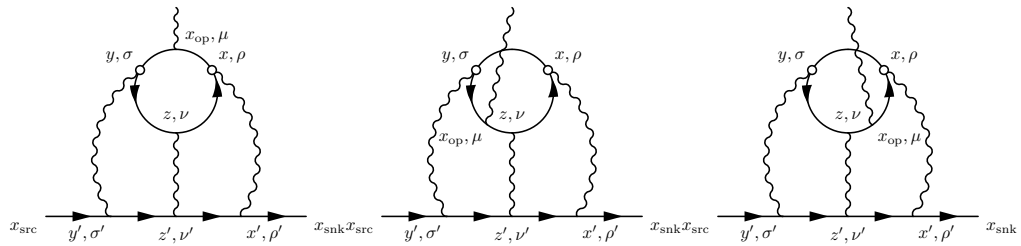
- Stochastic sampling of x and y point pairs. Sum over x and z.

$$\mathcal{F}_\nu^C\left(x,y,z,x_{\mathsf{op}}\right) = (-ie)^6\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}),$$



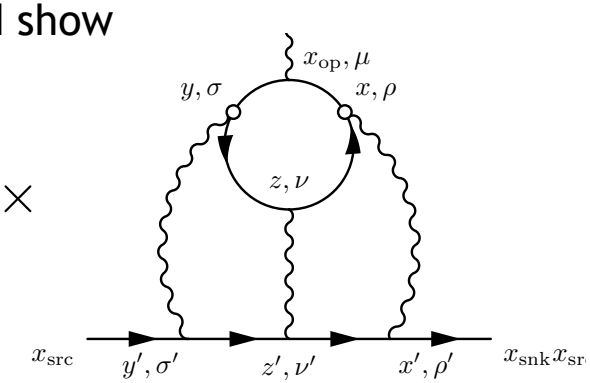
Conserved current & moment method

- [conserved current method at finite q2] To tame UV divergence, one of quark-photon vertex (external current) is set to be conserved current (other three are local currents). All possible insertion are made to realize conservation of external currents config-by-config.



- [moment method , q2→0] By exploiting the translational covariance for fixed external momentum of lepton and external EM field, q->0 limit value is directly computed via the first moment of the relative coordinate, xop – (x+y)/2, one could show

$$\frac{\partial}{\partial q_i}\mathcal{M}_\nu(\vec{q})|_{\vec{q}=0} = i \sum_{x,y,z,x_{\rm op}} (x_{\rm op} - (x+y)/2)_i \times$$



to directly get $F_2(0)$ without extrapolation.

$$\text{Form factor}: \ \Gamma_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2\, m_l} F_2(q^2)$$

HVP
Current conservation & subtractions

- conservation => transverse tensor

$$\Pi^{\mu\nu}(q) = (\hat{q}^2\delta^{\mu\nu} - \hat{q}^\mu\hat{q}^\nu)\Pi(\hat{q}^2)$$

- In infinite volume, q=0, $\Pi_{\mu\nu}(q) = 0$
- For finite volume, $\Pi_{\mu\nu}(0)$ is exponentially small (L.Jin, use also in HLbL)

$$\int_V dx^4 \langle V_\mu(x)\mathcal{O}(0)\rangle = \int_V dx^4\, \partial_x \left(x\langle V_\mu(x)\mathcal{O}(0)\rangle\right)$$
$$= \int_{\partial V} dx^3\, x\langle V_\mu(x)\mathcal{O}(0)\rangle \propto L^4 \exp(-ML/2) \to 0$$

- e.g. DWF L=2, 3, 5 fm $\Pi_{\mu\nu}(0)$ = 8(3)e-4, 2(13)e-5, -1(5)e-8
- Subtract $\Pi_{\mu\nu}(0)$ alternates FVE, and reduce stat error
  "-1" subtraction trick [Bernecker & Meyer, Maintz] :

$$\Pi^{\mu\nu}(q) - \Pi^{\mu\nu}(0) = \int d^4x (e^{iqx} - 1)\langle J^\mu(x) J^\nu(0)\rangle$$

cHLbL Subtraction using current conservation

- From current conservation, $\partial_\rho V_\rho(x) = 0$, and mass gap, $\langle x V_\rho(x) \mathcal{O}(0)\rangle \sim |x|^n \exp(-m_\pi |x|)$

$$\sum_x \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) = \sum_x \langle V_\rho(x) V_\sigma(y) V_\kappa(z) V_\nu(x_{\rm op})\rangle = 0$$

$$\sum_z \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) = 0$$

  at $V \to \infty$ and $a \to 0$ limit (we use local currents).

- We could further change QED weight

$$\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(x,y,z) = \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,y) + \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,y)$$

  without changing sum $\sum_{x,y,z} \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$.

- Subtraction changes discretization error and finite volume error.

- Similar subtraction is used for HVP case in TMR kernel, which makes FV error smaller.

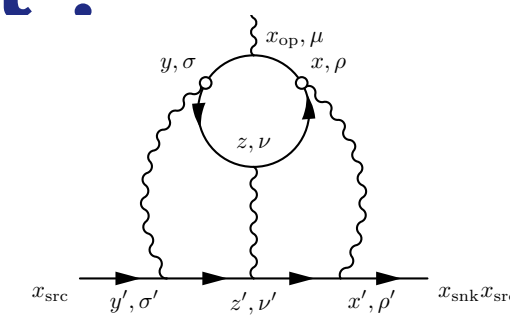
- Also now $\mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(z,z,x) = \mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(y,z,z) = 0$, so short distance $\mathcal{O}(a^2)$ is suppressed.

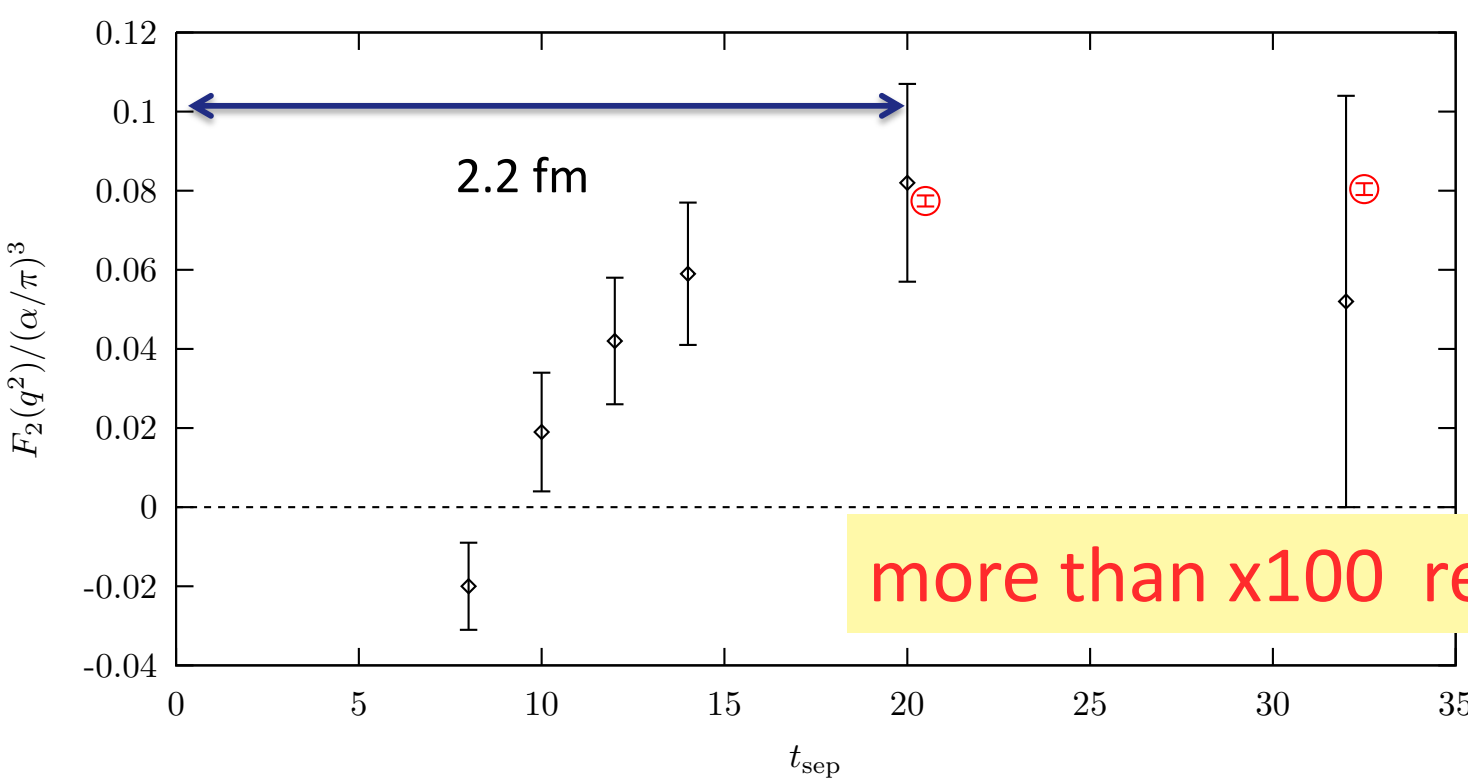
- The 4 dimensional integral is calculated numerically with the CUBA library cubature rules. (x,y,z) is represented by 5 parameters, compute on N^5 grid points and interpolates. ($|x-y| < 11$ fm).

Dramatic Improvement !

Luchang Jin

a=0.11 fm, 24^3x64 $(2.7\ \text{fm})^3$,
m_π = 329 MeV, m_μ =~ 190 MeV, e=1



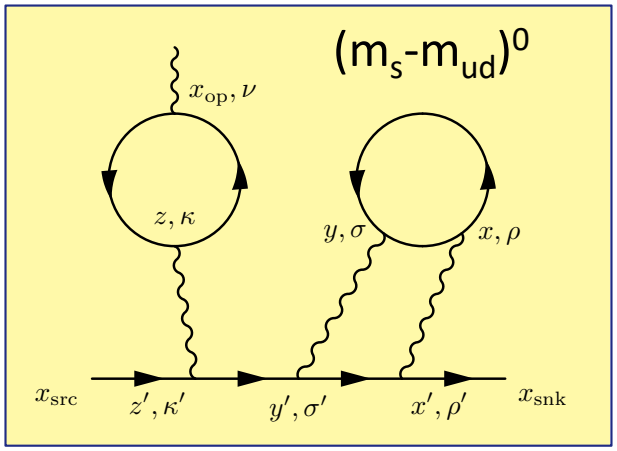


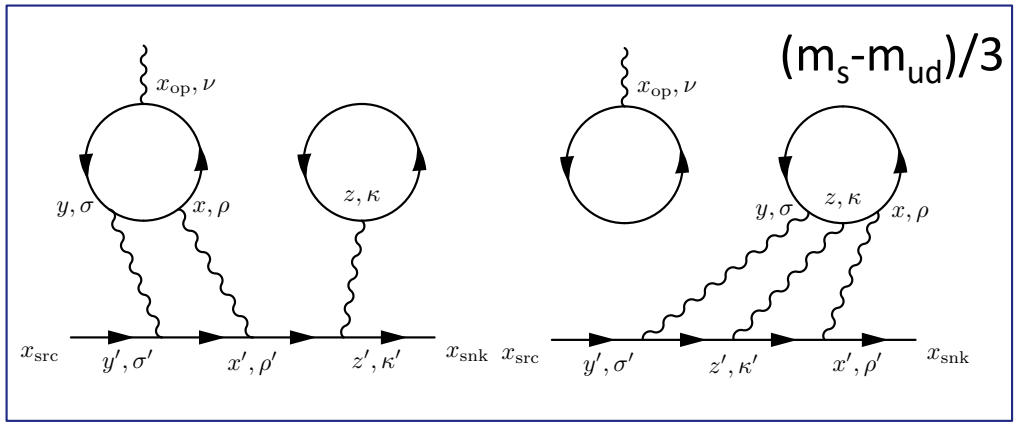
more than x100 reduced cost !

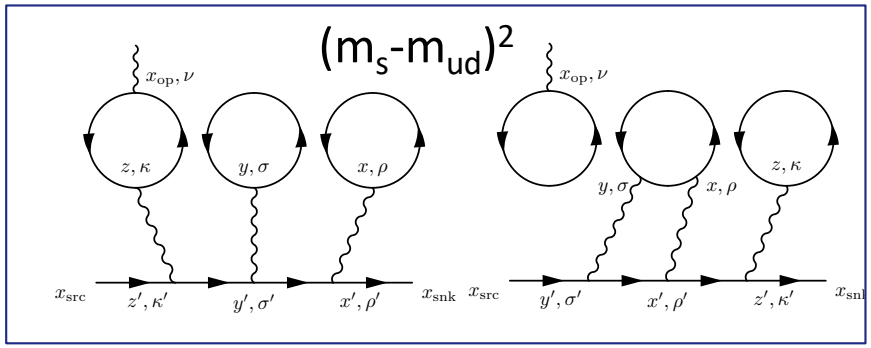
Method	$F_2/(\alpha/\pi)^3$	$N_{\rm conf}$	$N_{\rm prop}$	$\sqrt{\rm Var}$
Conserved	0.0825(32)	12	$(118+128)\times 2\times 7$	0.65
Mom.	0.0804(15)	18	$(118+128)\times 2\times 3$	0.24

SU(3) hierarchies for d-HLbL

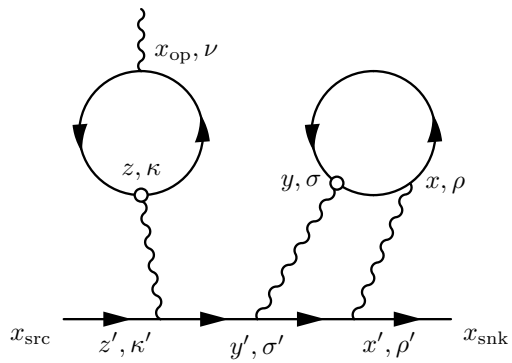
- At $m_s = m_{ud}$ limit, following type of disconnected HLbL diagrams survive $Q_u + Q_d + Q_s = 0$
- Physical point run using similar techniques to c-HLbL.
- other diagrams suppressed by $O(m_s - m_{ud})/3$ and $O((m_s - m_{ud})^2)$







Disconnected calculation



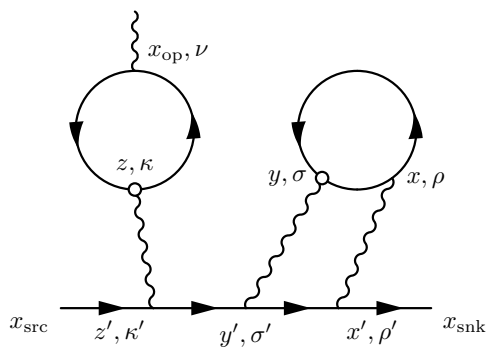
- We can use two point source photons at y and z, which are chosen randomly. The points $x_{\rm op}$ and x are summed over exactly on lattice.
- Only point source quark propagators are needed. We compute M point source propagators and all M^2 combinations of them are used to perform the stochastic sum over $r = z - y$.

$$\mathcal{F}^D_\nu(x, y, z, x_{\rm op}) \;=\; (-ie)^6\, \mathcal{G}_{\rho,\sigma,\kappa}(x, y, z)\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\rm op}) \tag{13}$$

$$\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\rm op}) \;=\; \left\langle \frac{1}{2}\,\Pi_{\nu,\kappa}(x_{\rm op}, z)\,[\Pi_{\rho,\sigma}(x, y) - \Pi^{\rm avg}_{\rho,\sigma}(x - y)] \right\rangle_{\rm QCD} \tag{14}$$

$$\Pi_{\rho,\sigma}(x, y) \;=\; -\sum_q\, (e_q/e)^2\, {\rm Tr}[\gamma_\rho\, S_q(x, y)\, \gamma_\sigma\, S_q(y, x)]. \tag{15}$$

Disconnected claculation



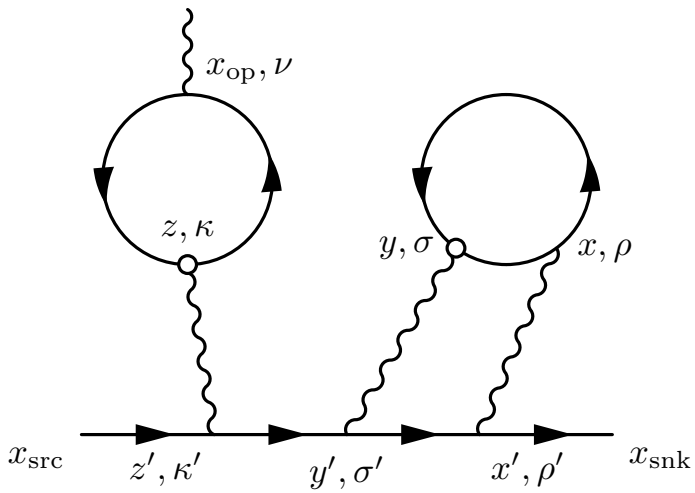
$$\frac{F_2^{\rm dHLbL}(0)}{m}\frac{(\sigma_{s',s})_i}{2} = \sum_{r,x}\sum_{x_{\rm op}}\frac{1}{2}\epsilon_{i,j,k}(\tilde{x}_{\rm op})_j \cdot i\,\bar{u}_{s'}(\vec{0})\,\mathcal{F}_k^D(x, y=r, z=0, x_{\rm op})u_s(\vec{0}) \quad (16)$$

$$\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) = \left\langle \frac{1}{2}\Pi_{\nu,\kappa}(x_{\rm op},z)\left[\Pi_{\rho,\sigma}(x,y) - \Pi^{\rm avg}_{\rho,\sigma}(x-y)\right]\right\rangle_{\rm QCD} \quad (17)$$

$$\sum_{x_{\rm op}}\frac{1}{2}\epsilon_{i,j,k}(x_{\rm op})_j\,\langle\Pi_{\rho,\sigma}(x_{\rm op},0)\rangle_{\rm QCD} = \sum_{x_{\rm op}}\frac{1}{2}\epsilon_{i,j,k}(-x_{\rm op})_j\,\langle\Pi_{\rho,\sigma}(-x_{\rm op},0)\rangle_{\rm QCD} = 0$$

- Because of the parity symmetry, the expectation value for the left loop average to zero.
- $[\Pi_{\rho,\sigma}(x,y) - \Pi^{\rm avg}_{\rho,\sigma}(x-y)]$ is only a noise reduction technique. $\Pi^{\rm avg}_{\rho,\sigma}(x-y)$ should remain constant through out the entire calculation.

$\mathbf{M^2}$ trick

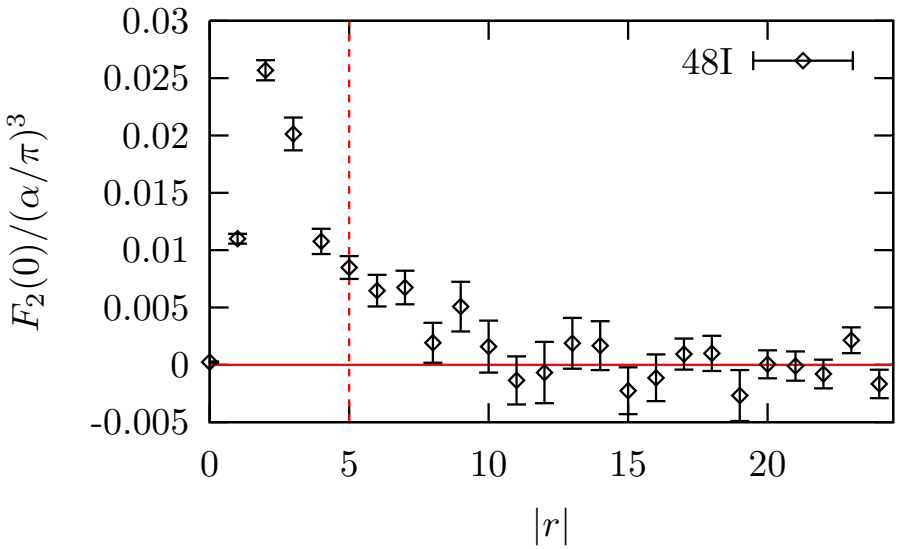


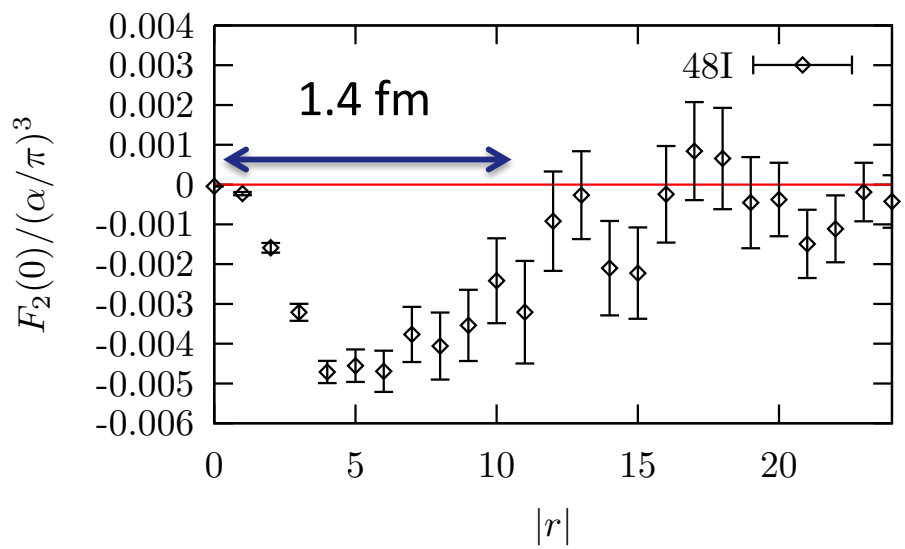
- For QED_L, we can compute the QED function for all z given the y location fixed and x summed over. Allow us to compute all combination of y, z with little efforts.
- For QED_∞, although we can compute all the function $\mathcal{G}_{\rho,\sigma,\kappa}(x, y, z)$ simply by interpolate, we cannot easily compute this function (even after fixing y) for all x and z, simply because of its cost is proportion to Volume^2.
- However, we with QED_∞ and interpolation, we can freely choose which coordinates we compute. For example, we may compute all z for $|x - y| \leqslant 5$, and sample z for $|x - y| > 5$.

140 MeV Pion, connected and disconnected LbL results

[Luchang Jin et al. , Phys.Rev.Lett. 118 (2017) 022005]

- left: connected, right : leading disconnected





- Using AMA with 2,000 zMobius low modes, AMA

(statistical error only)

$$\left.\frac{g_\mu-2}{2}\right|_{\rm cHLbL} = (0.0926 \pm 0.0077) \times \left(\frac{\alpha}{\pi}\right)^3 = (11.60 \pm 0.96) \times 10^{-10}$$

$$\left.\frac{g_\mu-2}{2}\right|_{\rm dHLbL} = (-0.0498 \pm 0.0064) \times \left(\frac{\alpha}{\pi}\right)^3 = (-6.25 \pm 0.80) \times 10^{-10}$$

$$\left.\frac{g_\mu-2}{2}\right|_{\rm HLbL} = (0.0427 \pm 0.0108) \times \left(\frac{\alpha}{\pi}\right)^3 = (5.35 \pm 1.35) \times 10^{-10}$$

Updates from PRL (2017)

[Tom Blum, C. Lehner, TI, Luchang Jin]

- Discretization error

  → a scaling study for 1/a = 2.7 GeV, 64 cube lattice at physical quark mass for both connected and disconnected is proposed to ALCC at Argonne

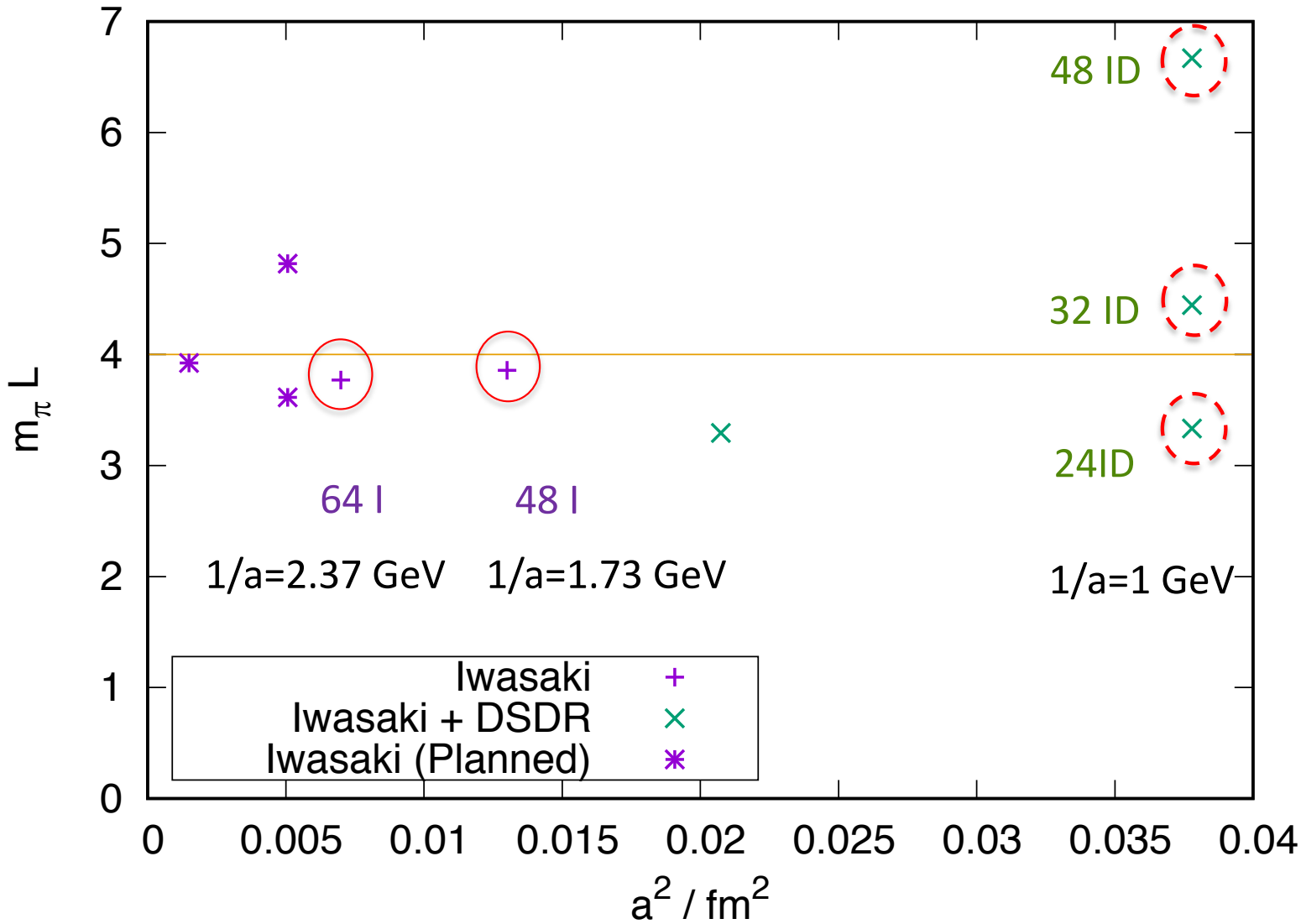
  [Tom Blum Lat17]

- Finite volume

  Using Infinite Volume and continuum lepton + photon diagrams using L~ 5, 6, 10 fm box

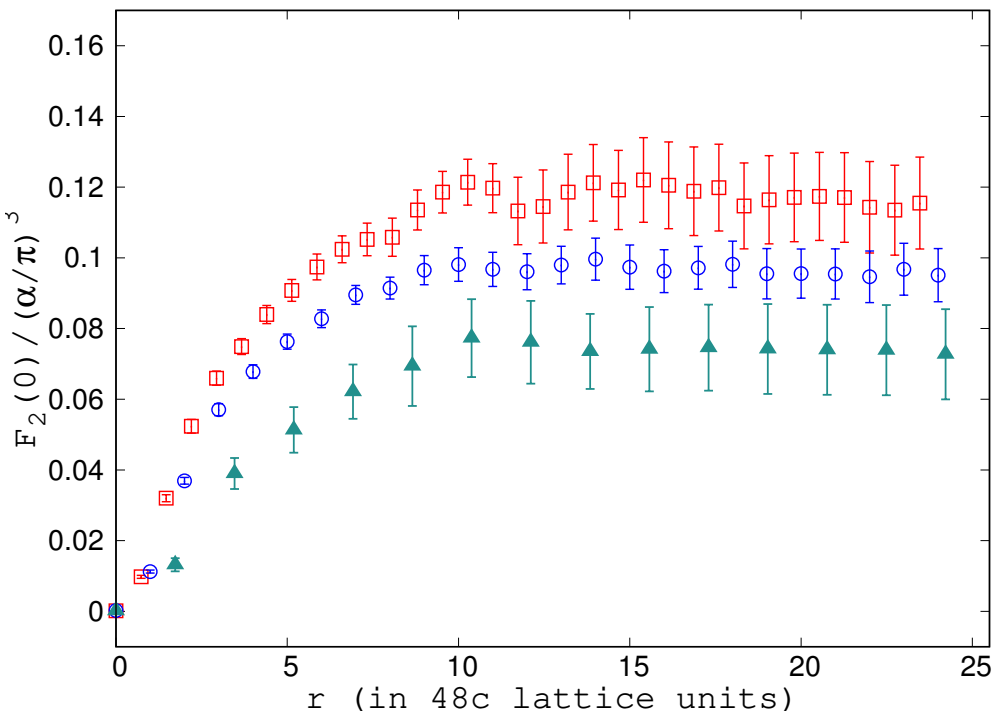
  [C.Lehner Uconn g-2 Theory Initiative] [TI Lat17]

Nf=2+1 DWF QCD ensemble at physical quark mass



cHLbL Different lattice spacings

cHLbL: lattice spacing effect (preliminary)



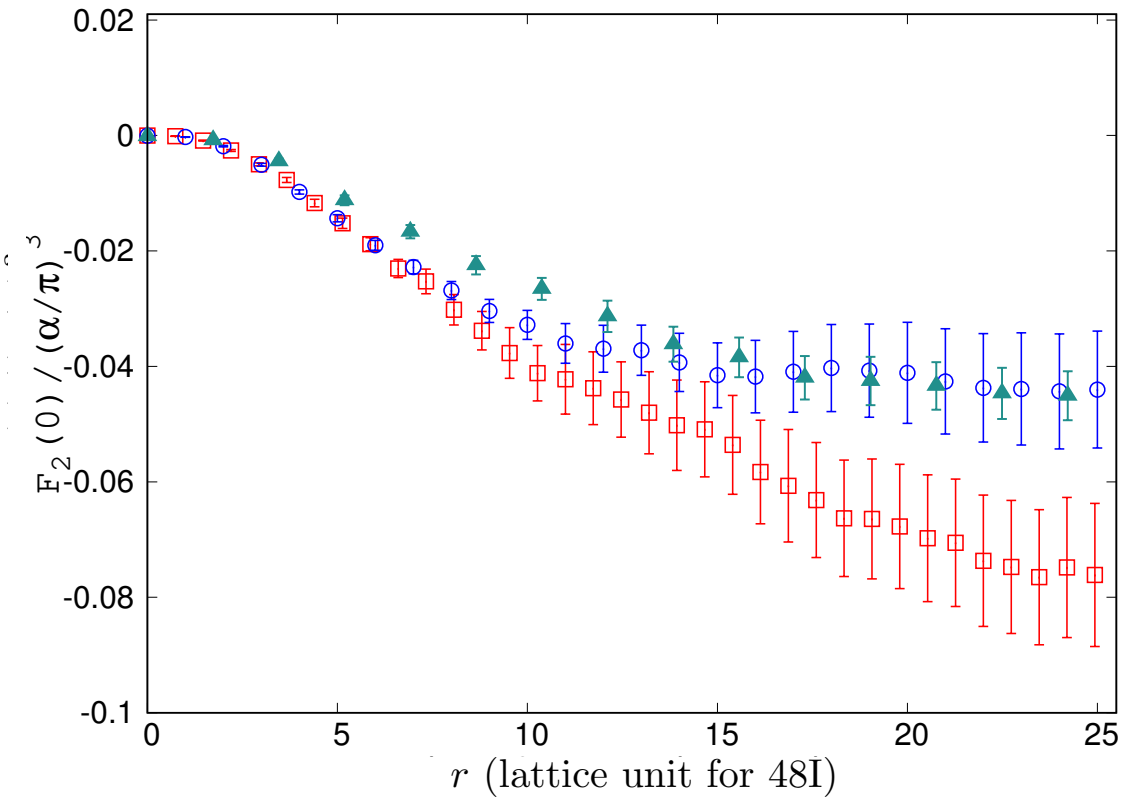
1/a = 2.37 GeV, 1.73 GeV, 1.0 GeV

- Add new 24^3, 1 GeV, ID ensemble (green)
- I and ID slightly different, but disc. errors similar
- Collecting more statistics (9 configs)

- Significant increase as $a \to 0$

dHLbL Different lattice spacings

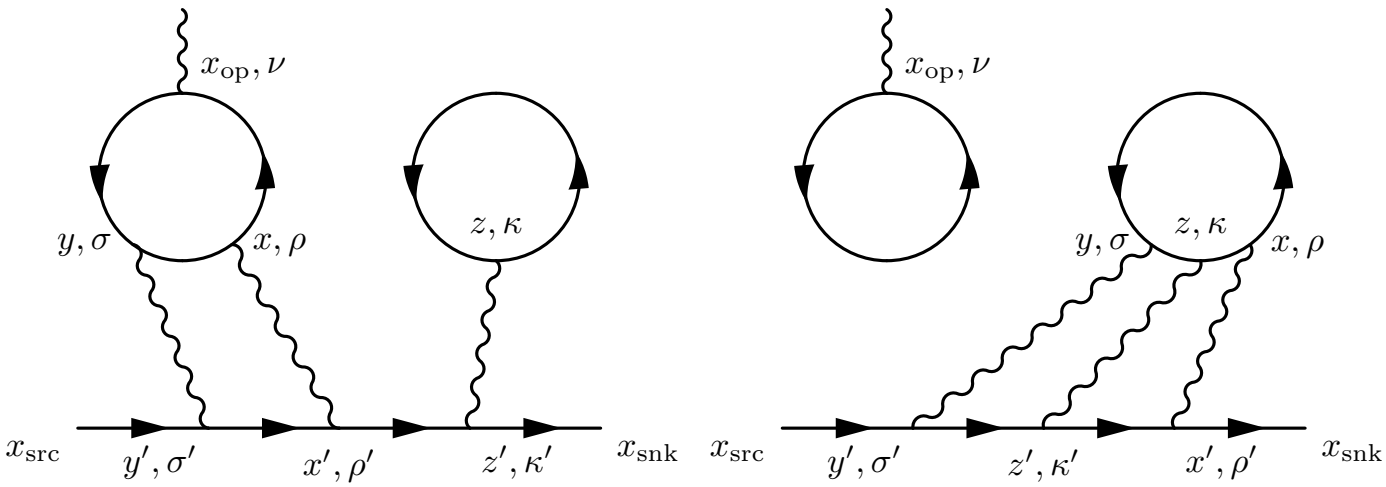
dHLbL contribution: lattice spacing effect (preliminary)



1/a = 2.37 GeV, 1.73 GeV, 1.0 GeV

- Large negative increase tends to cancel connected one
- Collecting more statistics!

Remaining dHLbL



- These are the subleading disconnected diagrams in the SU(3) limit.
- The right diagram has a factor of $1/3$ suppression from the multiplicity of the diagram compare with the left diagram, i.e. the external photon is more likely to be on the loop with three photons.
- For the left diagram, the moment method works just like the connected case. With both QED_L or QED_∞, we can sample x, y and sum over z. We can use the M^2 trick for the x, y sampling. Low-modes-averaging for the loop with z.
- For the right diagram, The moment method still works, however, we have to use a point on the other loop as the reference point, which may be more noisy. But as mentioned above, the right diagram is more suppressed.

Infinite Volume Photon and Lepton QED$_\infty$

[Feynman, Schwinger, Tomonaga] [Mainz]

- Instead of, or, in addition to, larger QED box, one could use infinite volume QED to compute $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$.

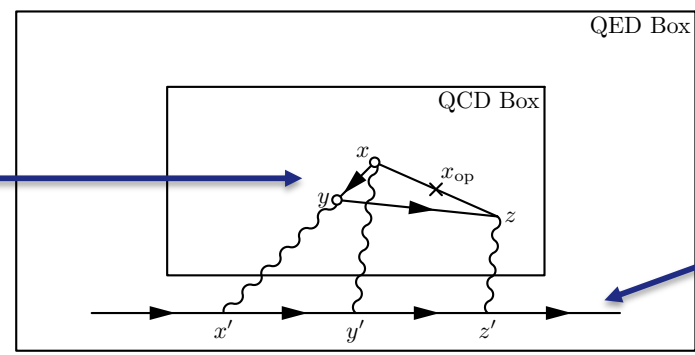
- Hadron part $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$ has following features due to the mass gap :
  - ▷ *For large distance separation, the 4pt Green function is exponentially suppressed:* $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \sim \exp[-m_\pi \times \mathit{dist}(x,y,z,x_{\rm op})]$
  - ▷ *For fixed* $(x,y,z,x_{\rm op})$*, FV error (wraparound effect etc.) is exponentially suppressed:* $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}|_V - \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}|_\infty \sim \exp[-m_\pi \times L]$

- By using QED$_\infty$ weight function $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$, which is not exponentially growing, asymptotic FV correction is exponentially suppressed

$$\Delta_V \left[\sum_{x,y,z,x_{\rm op}} \mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})] \right] \sim \exp[-m_\pi L]$$

($x_{\rm ref} = (x+y)/2$ is at middle of QCD box using transnational invariance)

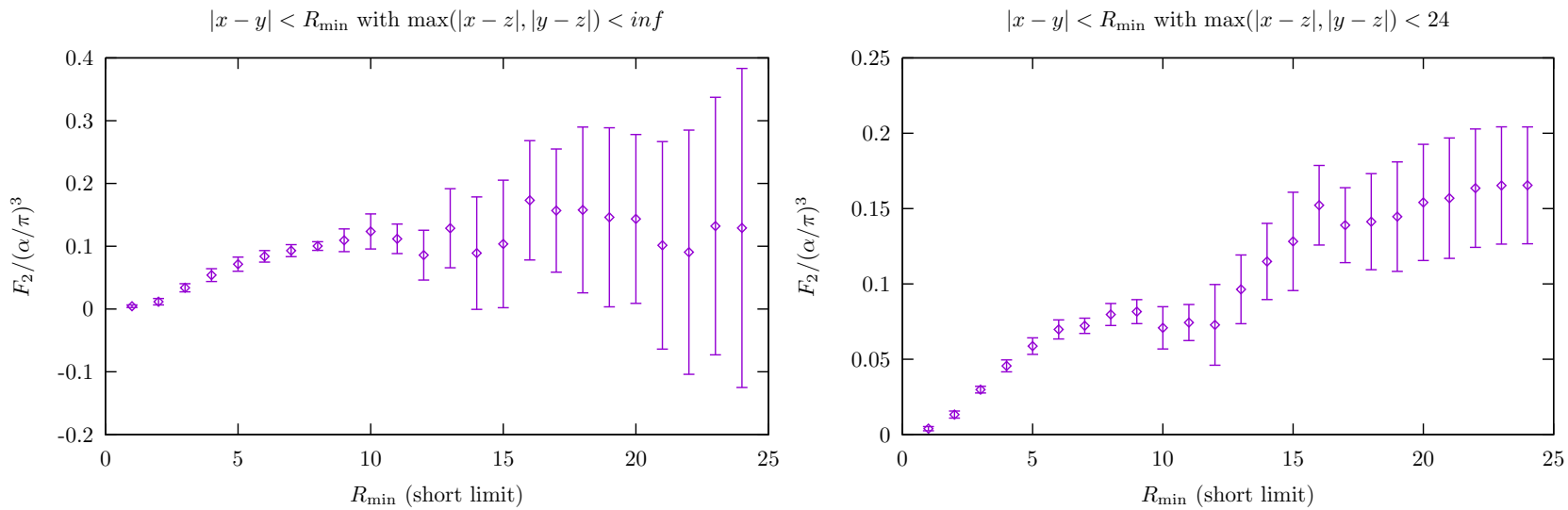
$\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$



$\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$

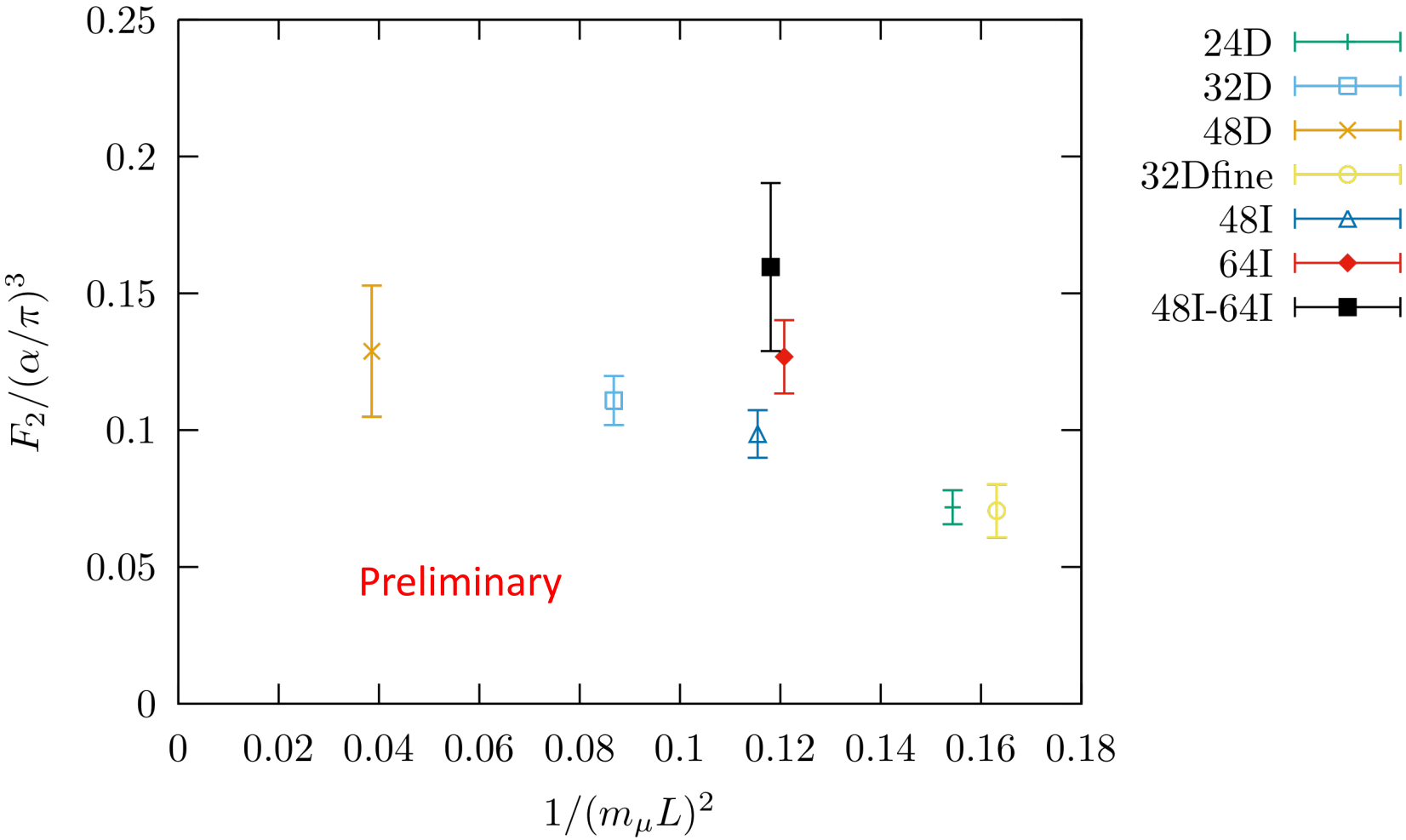
Preliminary results, QCD case

- QCD case with physical point quark mass,
- $48^3 \times 96$ lattice, with $a^{-1} = 1.73\,\mathrm{GeV}$, $m_\pi = 139\,\mathrm{MeV}$, $m_\mu = 106\,\mathrm{MeV}$.



- c.f. QED$_L$ case, $\left.\frac{g\mu-2}{2}\right|_{\text{cHLbL}} = (0.0926 \pm 0.0077)\left(\frac{\alpha}{\pi}\right)^3$

Discretization error & QED_L FV error summary (preliminary)

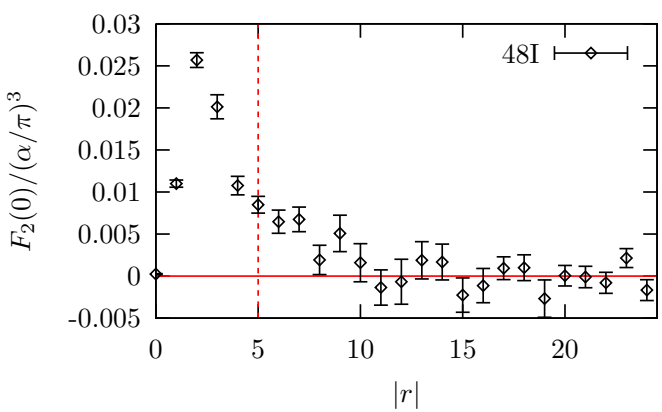


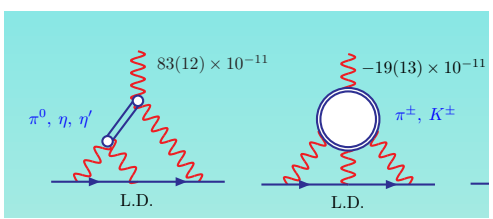
HLbL (near) future plans

- c-HLbL, Leading d-HLbL :
  - Finalize
    QED_L Statistical, FV, discretization analysis
  - Same for QED_Inf (Noisier)

- Higher order d-HLbL

- Comparing with Long distance LQCD calculation with Model/dispersive Hadron contributions (pi0 exchange, …), and perhaps combine LQCD+Model/dispersive





Summary

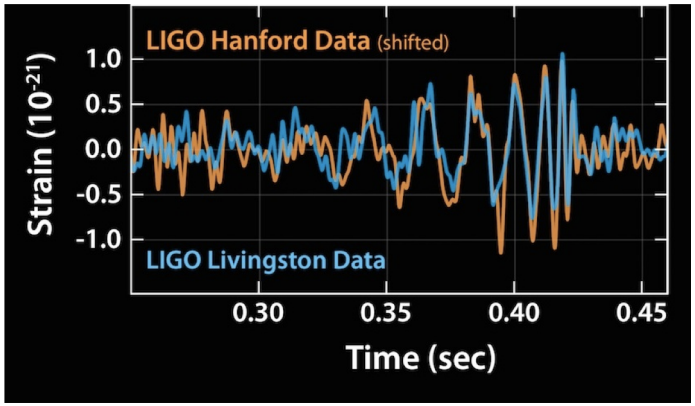
- Lattice calculation for g-2 calculation is improved very rapidly

- **HVP** [Christoph Lehner et al.]
  - New methods using low mode for connected at physical quark mass,
  - disconnected quark loop at physical quark mass,
  - Combining with R-ratio experiment data for cross-check and improvement => 1% error
  - Eventually the window will be enlarged for a pure LQCD prediction
  - QED and IB studies are included. [V. Gulper's talk]
  - Long distance 2 pi contribution from a separate analysis (distillation, GEVP) [A. Meyer et al]
  - Tau input for g-2 and Lattice interplay [M. Bruno's talk]

- **HLbL** [Luchang Jin et al]
  - computing leading disconnected diagrams :
    -> 8 % stat error in connected, 13 % stat error in leading disconnected
  - coordinate-space integral using analytic photon propagator with adaptive probability (point photon method), config-by-config conserved external current
  - take moment of relative coordinate to directly take $q \to 0$
  - AMA, zMobius, 2000 low modes
  - Infinite volume / continuum QED weight function to avoid power-like FV

- Goal : HVP sub 1% (then 0.25%) ,
  HLbL 10% error

Can we see the next physics Revolution
(c.f GW) ?



[Eigo Shintani Lat17]

Studies of finite volume

- ChPT

  Aubin et al., PRD93(2016)

  - Lowest-order SChPT gives VPF tensor: $\Pi_{\mu\nu}(q)$
  - 10% -- 15% discrepancy between $a_\mu^{HLO}[A_1]$ and $a_\mu^{HLO}[A_1^{44}]$

    consistent with lattice calculation (L=3.8 fm, 0.22 GeV pion, $m_\pi L$=4.2)

- Gounaris-Sakurai model

  Wittig (2016,2017), Mainz 1705.01775

  - By using time-like pion form factor, g-2 can be described in infinite volume.
  - 3% FV effect in L=4 fm, 0.19 GeV pion, $m_\pi L$=4

- Anisotropic study

  Lehner (2016)

  - Coordinate space integral along temporal or spatial direction.
  - Discrepancy is a_μ^{HLO} [spatial] – a_μ^{HLO} [temporal] ~ 3%.

- **Direct lattice study (PACS)**
  - Comparison between two volumes in physical pion at fixed *a*
  - L > 5 fm, $m_\pi L \gtrsim 3.8$
  - Compare the different boundary

[Eigo Shintani Lat17]

PACS 96^4 and 64^4 at a=0.08 fm

PACS group recently generates two gauge ensembles:

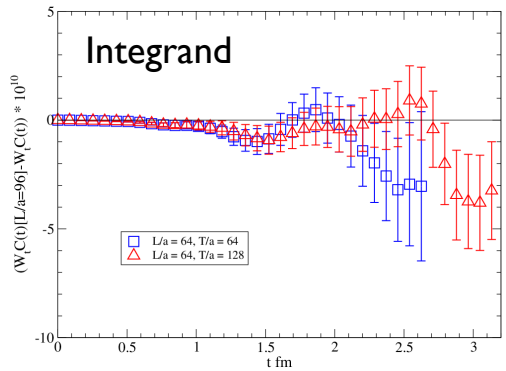
- Nf=2+1 O(a) improved clover fermion + Stout smearing
- a=0.083 fm, and two lattice sizes 64^4 and 96^4
- (almost) physical pion,

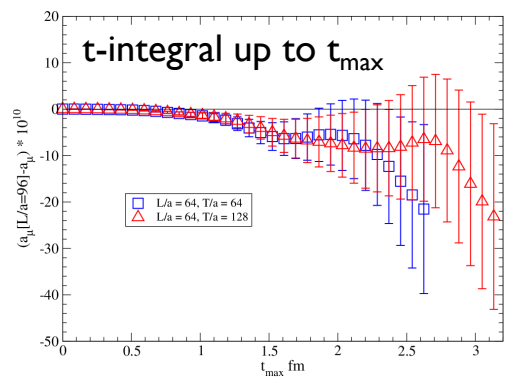
L=5.4 fm, 0.140 GeV ($m_\pi L$=3.8),

with K_{ud}=0.126117, K_s=*0.124790*

L=8.1 fm, 0.145 GeV ($m_\pi L$=6.0)

with K_{ud}=0.126117, K_s=*0.124902*





~5 MeV difference in pion mass

- Slightly negative for t_{max} > 1.3 fm → Δ_{FV}[(L/a=96)-(L/a=64)]~-10, opposite sign from expectation (ChPT etc) Aubin et al., PRD93(2016)

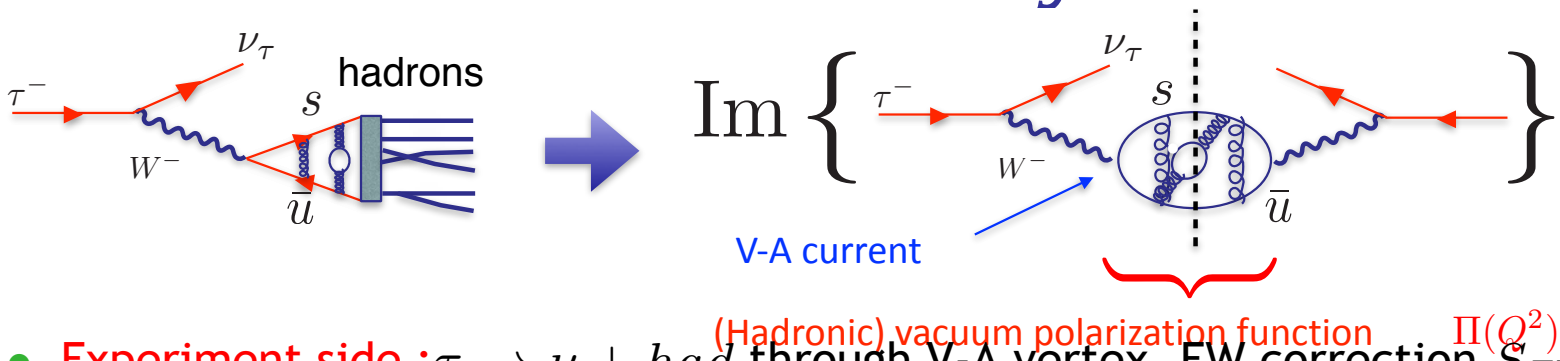
  However pion mass difference, m_π[(L/a=96)–(L/a=64)] = +5 MeV, due to slightly different K_s in two ensembles. For same m_π such a difference would have been reduced by $\Delta a_\mu = +3$ under assumption from ansatz in HPQCD(2016), Mainz (2017)
  ⇒ conservatively ~±2(2)% FV correction in L/a=64 lattice at finite t_{max} ~ 2.5 fm including mass correction.

CKM V_{us} from Inclusive tau decay

Yet another by-product of muon g-2 HVP

[Hiroshi Ohki et al. arXiv:1803.07228]

Tau decay



- Experiment side : $\tau \to \nu + had$ through V-A vertex. EW correction S_{EW}

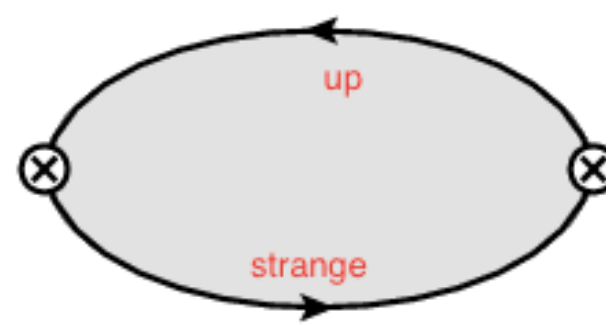
$$R_{ij} = \frac{\Gamma(\tau^- \to \text{hadrons}_{ij}\ \nu_\tau)}{\Gamma(\tau^- \to e^-\bar{\nu}_e\nu_\tau)}$$

$$= \frac{12\pi|V_{ij}|^2 S_{EW}}{m_\tau^2}\int_0^{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)\underbrace{\left[\left(1+2\frac{s}{m_\tau^2}\right)\text{Im}\Pi^{(1)}(s)+\text{Im}\Pi^{(0)}(s)\right]}_{\equiv\ \text{Im}\,\Pi(s)}$$

- Lattice side : The Spin=0 and 1, vacuum polarization, Vector(V) or Axial (A) current-current two point

$$\Pi^{\mu\nu}_{ij;V/A}(q^2) = i\int d^4x e^{iqx}\left\langle 0|T J^\mu_{ij;V/A}(x) J^{\dagger\mu}_{ij;V/A}(0)|0\right\rangle$$

$$= (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi^{(1)}_{ij;V/A}(q^2) + q^\mu q^\nu \Pi^{(0)}_{ij;V/A}$$



Finite Energy Sum Rule (FESR)

[Shifman, Vainshtein, and Zakharov '79]

The finite energy sum rule (FESR)

$$\int_0^{s_0} \omega(s)\rho(s)ds = -\frac{1}{2\pi i}\oint_{|s|=s_0} \omega(s)\Pi(s)ds, \quad (s_0 : \text{ finite energy})$$

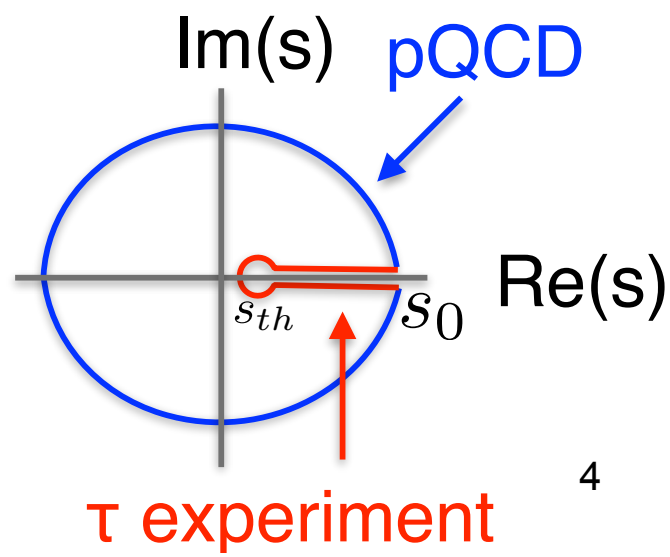
w(s) is an arbitrary regular function such as polynomial in s.

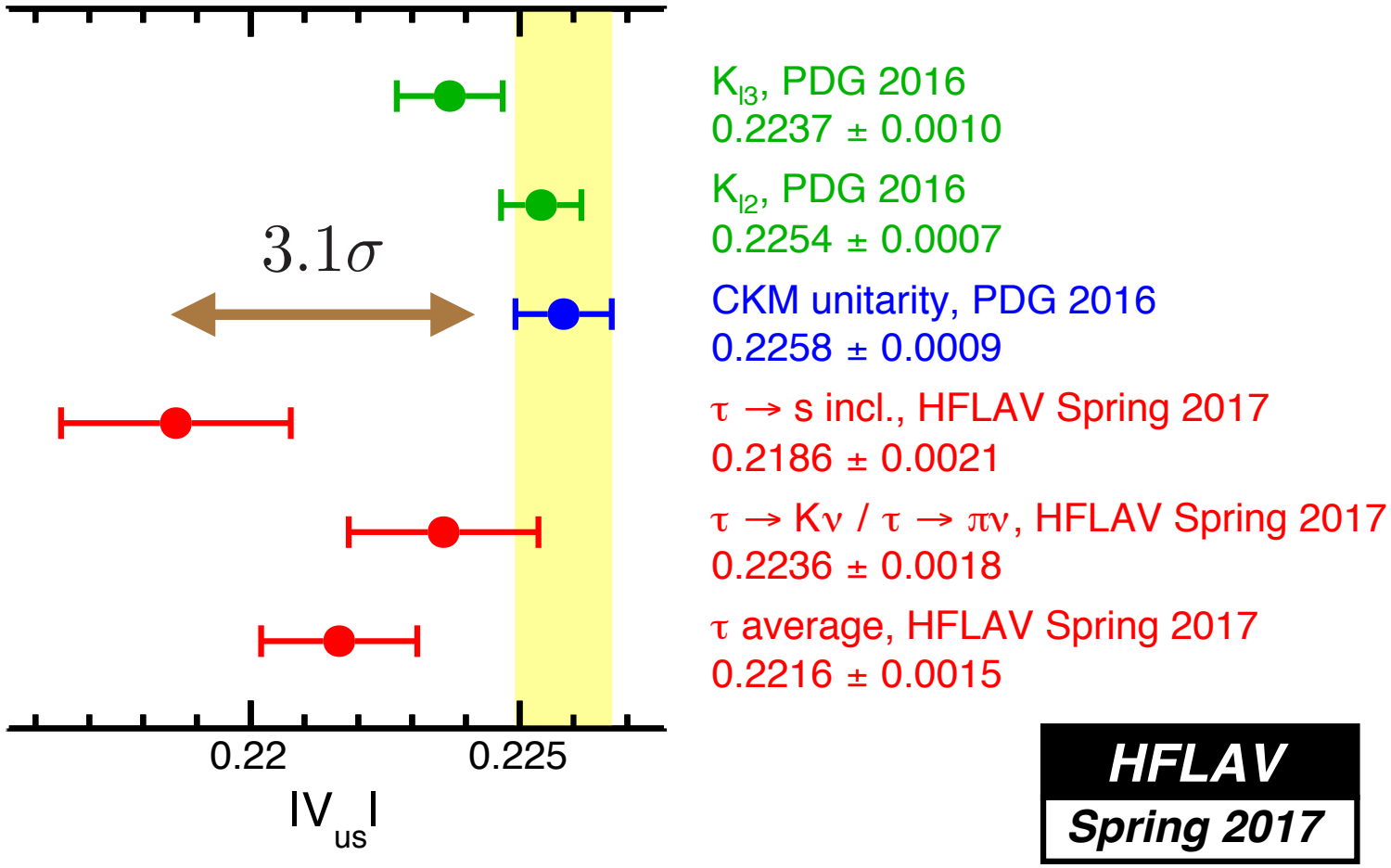
- LHS : spectral function ρ(s) is related to the experimental τ inclusive decays

$$\frac{dR_{us;V/A}}{ds} = \frac{12\pi^2|V_{us}|^2 S_{EW}}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\mathrm{Im}\Pi^1(s)+\mathrm{Im}\Pi^0(s)\right]$$

$$\tilde{\rho}(s) \equiv |V_{us}|^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\mathrm{Im}\Pi^1(s)+\mathrm{Im}\Pi^0(s)\right]$$

- RHS ⋯ Analytic calculation
  with perturbative QCD (pQCD) and OPE





- τ result v.s. non-τ result : more than 3 σ deviation : |Vusl puzzle
- new physics effect?
- incl. analysis uses Finite energy sum rule (FESR)
- pQCD and higher order OPE for FESR:
  underestimation of truncation error and/or non-perturbative effects ?
  (c.f. alternative FESR approach, R. Hudspith et. al arXiv:1702.01767)

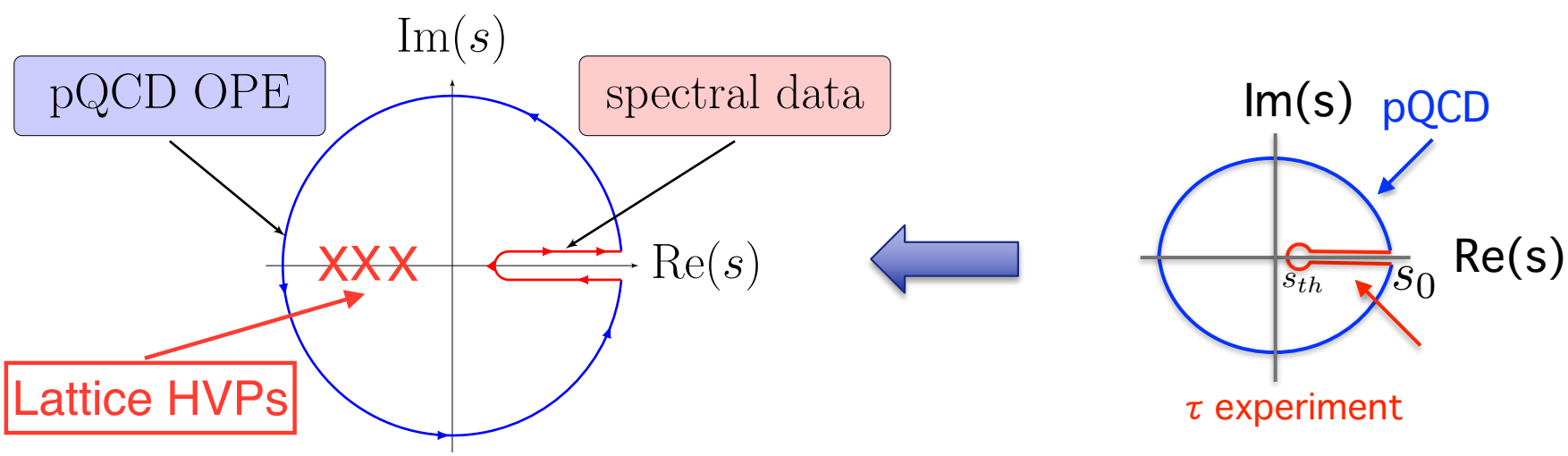
Our new method : Combining FESR and Lattice

- If we have a reliable estimate for $\Pi(s)$ in Euclidean (space-like) points, $s = -Q_k^2 < 0$, we could extend the FESR with weight function $w(s)$ to have poles there,

$$\int_{s_{th}}^{\infty} w(s)\mathrm{Im}\Pi(s) = \pi \sum_k^{N_p} \mathrm{Res}_k[w(s)\Pi(s)]_{s=-Q_k^2}$$

$$\Pi(s) = \left(1 + 2\frac{s}{m_\tau^2}\right)\mathrm{Im}\Pi^{(1)}(s) + \mathrm{Im}\Pi^{(0)}(s) \propto s \quad (|s| \to \infty)$$

- For $N_p \geq 3$, the $|s| \to \infty$ circle integral vanishes.



weight function $w(s)$

- Choice of weight function

$$w(s) = \prod_k^{N_p} \frac{1}{(s+Q_k^2)} = \sum_k a_k \frac{1}{s+Q_k^2}, \quad a_k = \sum_{j\neq k} \frac{1}{Q_k^2 - Q_j^2}$$

$$\Longrightarrow \sum_k (Q_k)^M a_k = 0 \;\; (M = 0, 1, \cdots, N_p - 2)$$

- The residue constraints automatically subtracts $\Pi^{(0,1)}(0)$ and $s\Pi^{(1)}(0)$ terms.
- For experimental data, $w(s) \sim 1/s^n, n \geq 3$ suppresses
  - ▷ *larger error from higher multiplicity final states at larger* $s < m_\tau^2$
  - ▷ *uncertanties due to pQCD+OPE at* $m_\tau^2 < s$
- For lattice, Q_k^2 should be not too small to avoid large stat. error, $Q^2 \to 0$ extrapolation, Finite Volume error. Also not too larger than m_τ^2 to make the suppression in time-like, higher energy, higher multiplicity, region enhanced.
- Comparison of different C, N values provides a self-consistency check for reliable error.

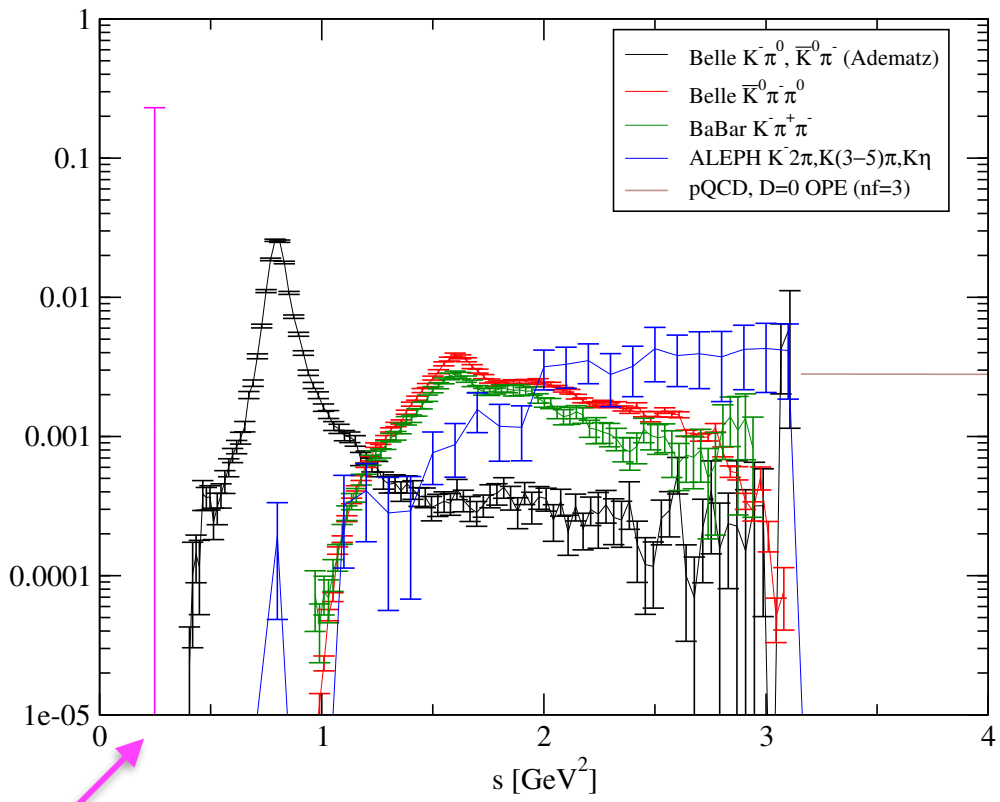
τ inclusive decay experiments

$$\tilde{\rho}(s) \equiv |V_{us}|^2 \left[\left(1 + 2\frac{s}{m_\tau^2} \right) \mathrm{Im}\Pi^1(s) + \mathrm{Im}\Pi^0(s) \right]$$

To compare with experiments,

a conventional value of |Vus|=0.2253 is used

$\tilde{\rho}(s)$



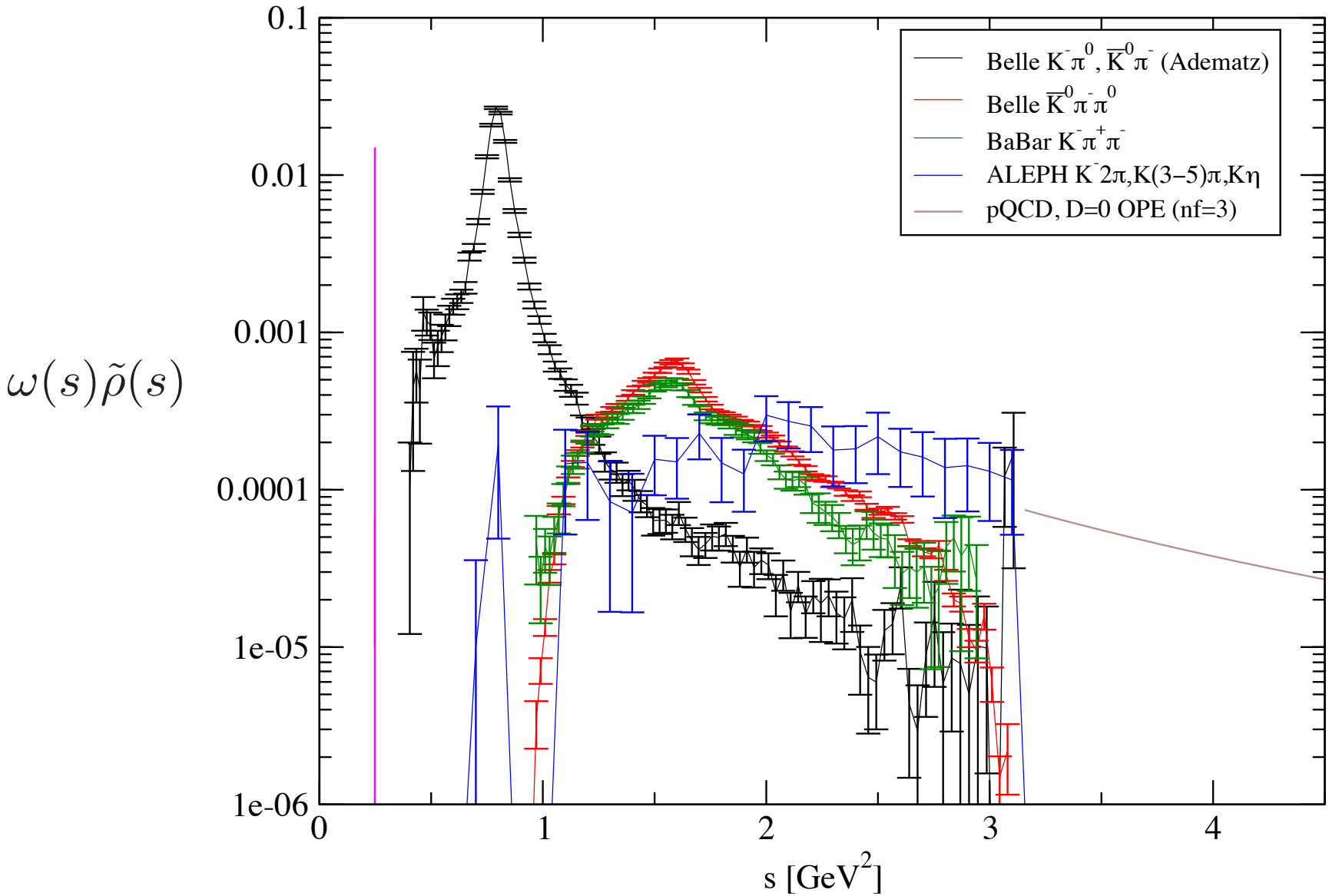
For K pole, we assume a delta function form $\gamma_K \omega(m_K^2)$

$\gamma_K \sim 2|V_{us}|^2 f_K^2$ obtained from either experimental value of K→μ or τ→k decay width.

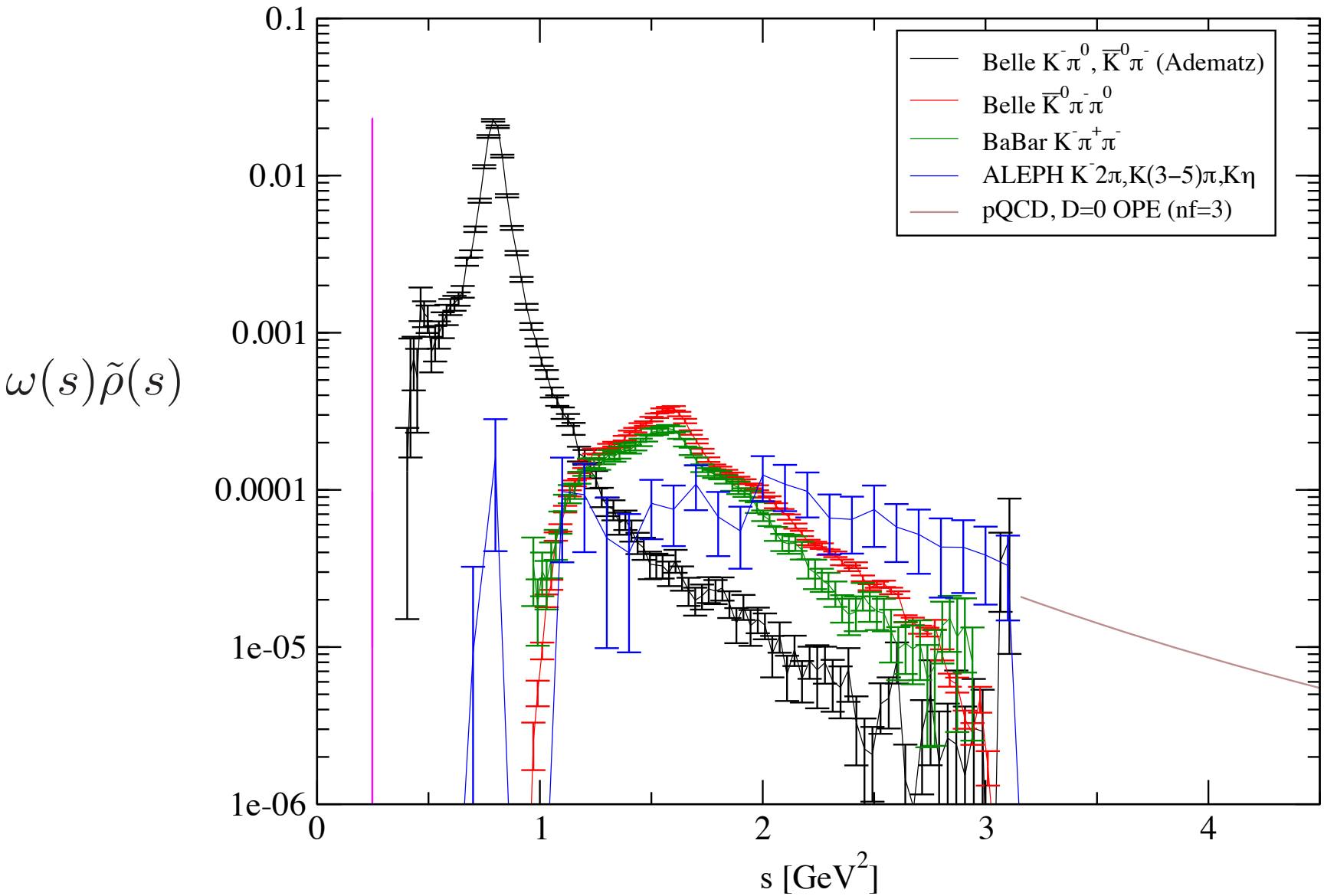
$$\gamma_K[\tau \to K\nu_\tau] = 0.0012061(167)_{exp}(13)_{IB} \ [\mathrm{HFAG16}]$$

$$\gamma_K[K_{\mu 2}] = 0.0012347(\ 29)_{exp}(22)_{IB} \ [\mathrm{PDG16}]$$

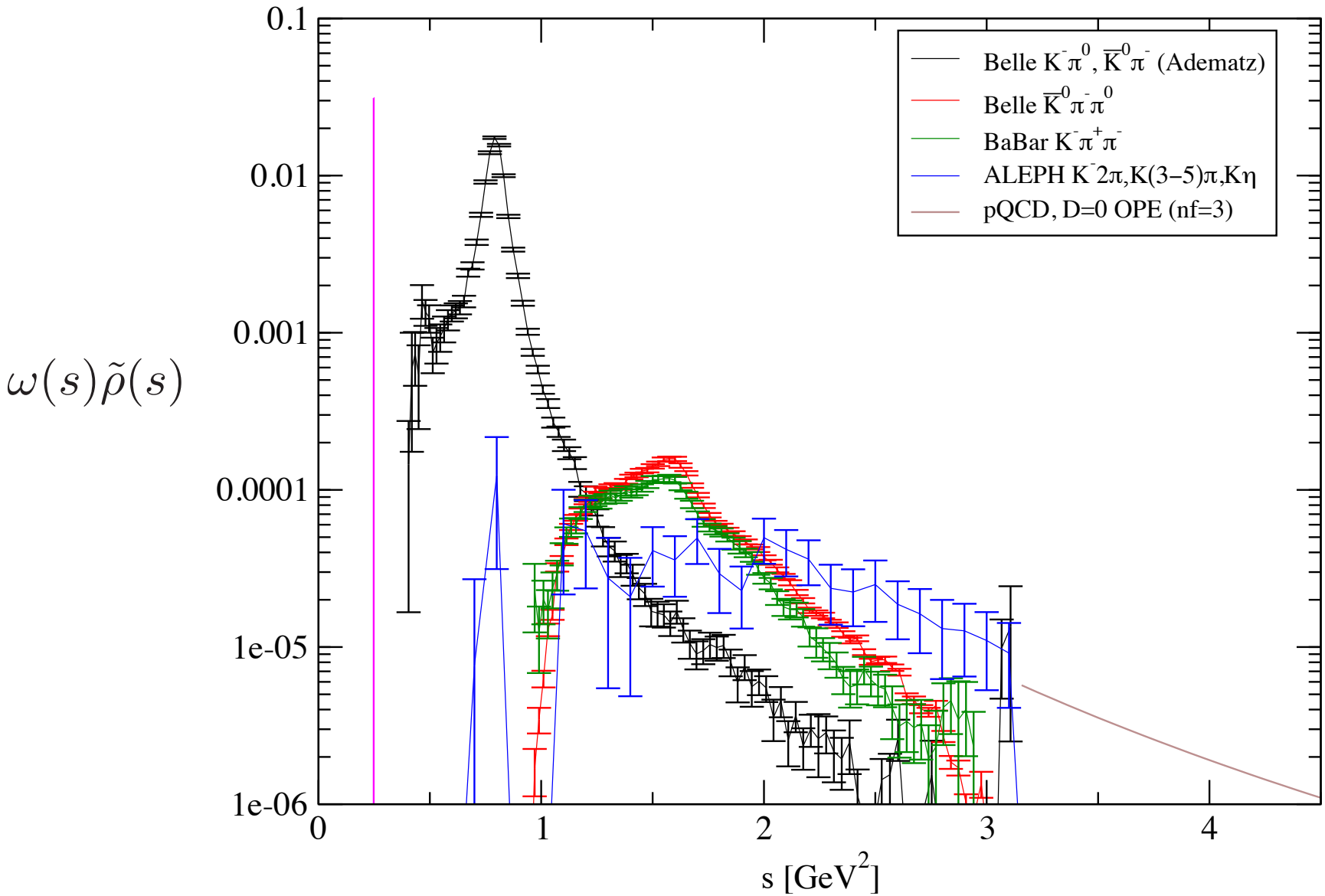
- example: N=3, $\{Q_1^2, Q_2^2, Q_3^2\} = \{0.1, 0.2, 0.3\}$ $[\mathrm{GeV}^2]$



- example: N=4, $\{Q_1^2, Q_2^2, Q_3^2, Q_4^2\} = \{0.1, 0.2, 0.3, 0.4\}$ $[\mathrm{GeV}^2]$



- example: N=5, $\{Q_1^2, Q_2^2, Q_3^2, Q_4^2, Q_5^2\} = \{0.1, 0.2, 0.3, 0.4, 0.5\}$ $[\mathrm{GeV}^2]$



QCD ensemble and statistics

- Main analysis is on two ensemble, at almost physical quark masses ($M_\pi \approx 140$ MeV, $M_K \approx 499$ MeV), V=(5 fm)^3.

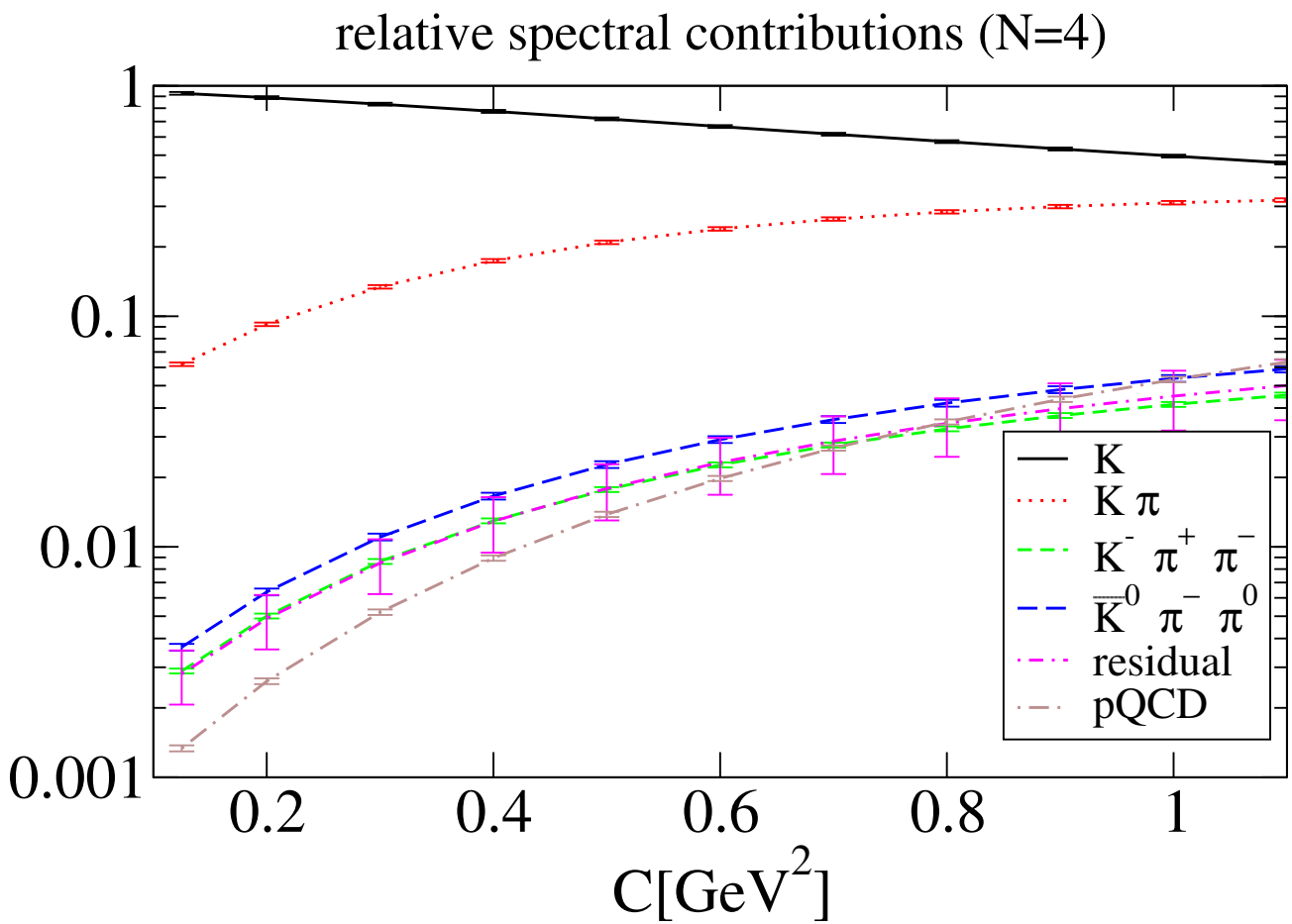
- Correct the residual up and strange quark mass error by partially quenched calculation.

- Consistent with other heavier / smaller ensemble are used to estimate size and direction of discretization errors.

Vol	a^{-1} [GeV]	M_π [MeV]	M_K [MeV]	conf
$48^3 \times 96$	1.7295(38)	139	499	88
		135	496	5 (PQ-correction)
$64^3 \times 128$	2.359(7)	139	508	80
		135	496	5 (PQ-correction)

Tuning of the "inclusiveness" of experimental spectral integral

$$(N = 4,\ \Delta = 0.067\ \mathrm{GeV}^2)$$

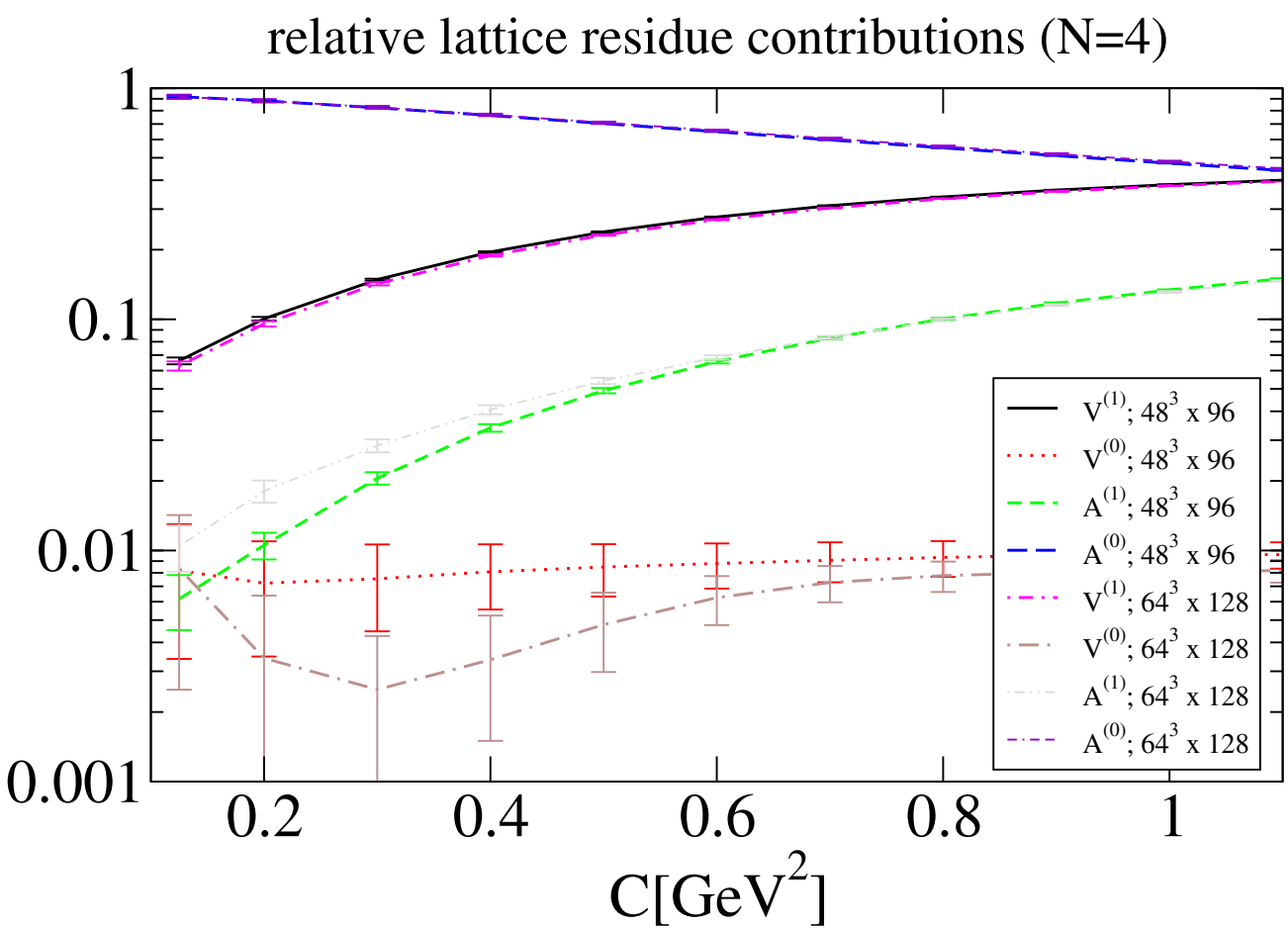


K, Kπ dominates spectral integrals,

high multiplicity modes and pQCD$(s > m_\tau^2)$ strongly suppressed

Lattice residue contributions

$(N = 4,\ \Delta = 0.067\ \mathrm{GeV}^2)$



Ratios of each contribution of V/A with spin=0, 1 to the total residue (Lattice)

$A^{(0)}$ dominance (K-pole)

|Vus| from inclusive decays

- 4 channels: Vector or Axial (V or A), spin 0 and 1
- A0 channel is dominated by K pole.

  → For the K pole contribution we use

  $f_K^{phys} = 0.15551(83)[\mathrm{GeV}]$ [RBC/UKQCD, 2014] instead of $A^{(0)}$
- Other channels :

A1, V1, V0 (& residual A0) → multi hadron states & pQCD ("other")

- We take the continuum limit using the data L=48 and 64

$$V_1 + V_0 + A_1 + A_0 \ : |V_{us}^{V_1+V_0+A_1+A_0}| = \sqrt{\frac{\rho_{exp}^{\mathrm{K-pole}} + \rho_{exp}^{others}}{(f_K^{phys})^2 \omega(m_K^2) + F_{lat}(\Pi_{\mathrm{others}}) - \rho_{\mathrm{pQCD}}}},$$

$$\rho_{exp}^{others} = |V_{us}|^2 \int_{s_{th}}^{m_\tau^2} ds \omega(s) \mathrm{Im}\Pi(\mathrm{s}) \qquad \rho_{pQCD} = \int_{m_\tau^2}^{\infty} ds \omega(s) \Pi_{OPE}(s)$$

$$F_{lat} = \sum_{k=1}^{N} \mathrm{Res}(\omega(-\mathrm{Q}_\mathrm{k}^2)) \Pi_{\mathrm{lat}}(-\mathrm{Q}_\mathrm{k}^2)$$

Systematic error estimate

- Higher order (a^4) discretization error for V1+V0+A1+(residual A0)

$$\mathcal{O}(C^2a^4) \sim 0.1Ca^2, \quad (a^{-1} = 2.37[\mathrm{GeV}])$$

Two lattice ensembles yield (less than)10% difference

→ We estimate 10% reduction of O(a^4) relative to O(a^2)

- Finite volume correction

1 loop ChPT analysis of current-current correlation function on finite volume for Kπ channel (V1).

- Isospin breaking effects

s-dependent strong isospin breaking corrected Kπ experimental data used.

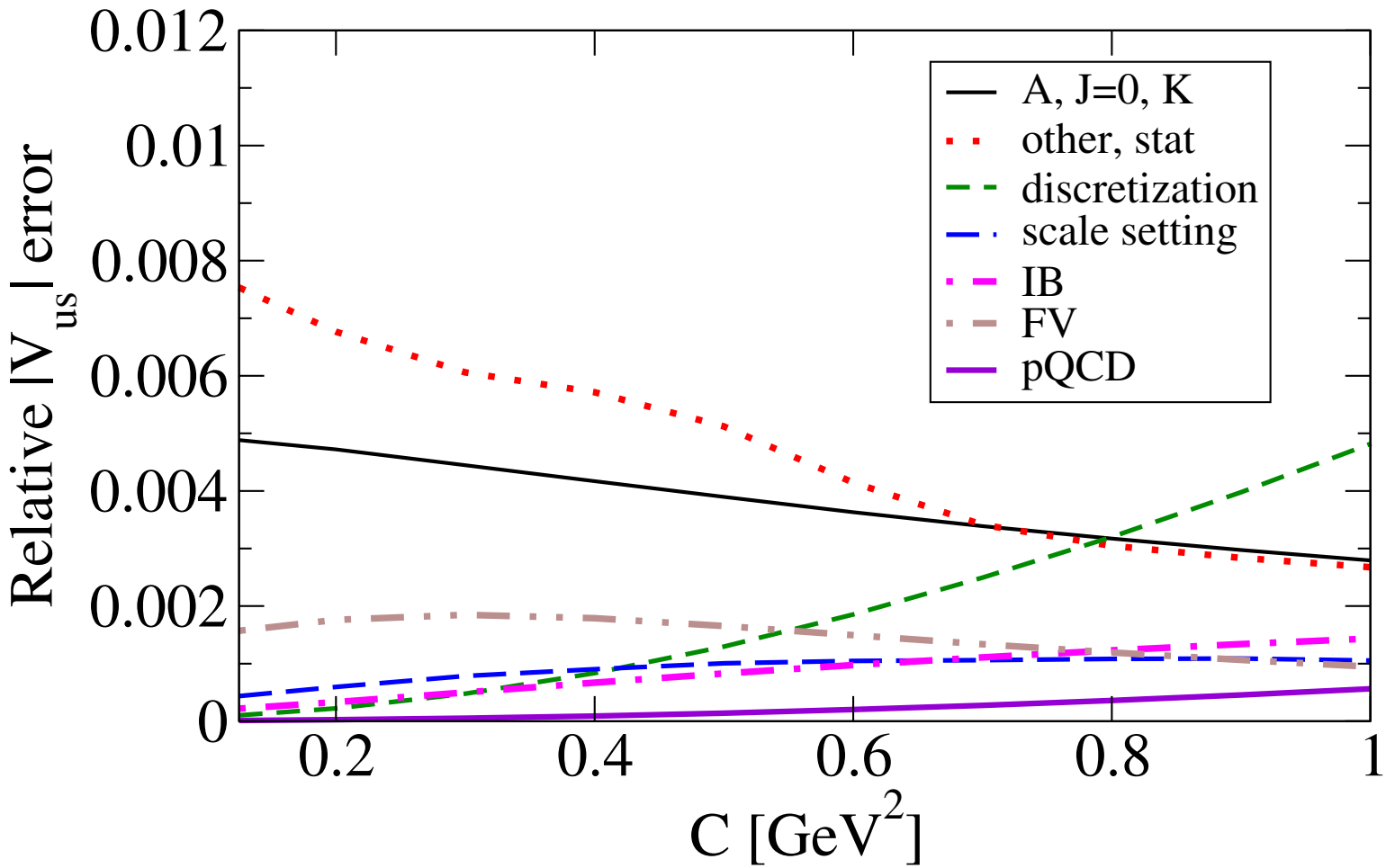
Theory error for dominant Kπ channels: 0.2 % for electromagnetic effects and ~ 1% strong isospin breaking effect on V1. [Ref: Antonelli, et al., JHEP10(2013)070]

- pQCD (OPE) uncertainty

2% for possible quark hadron duality-violation effect

$|V_{us}|$ systematic error of lattice residue contributions

$(N = 4,\ \Delta = 0.067\ \mathrm{GeV}^2)$

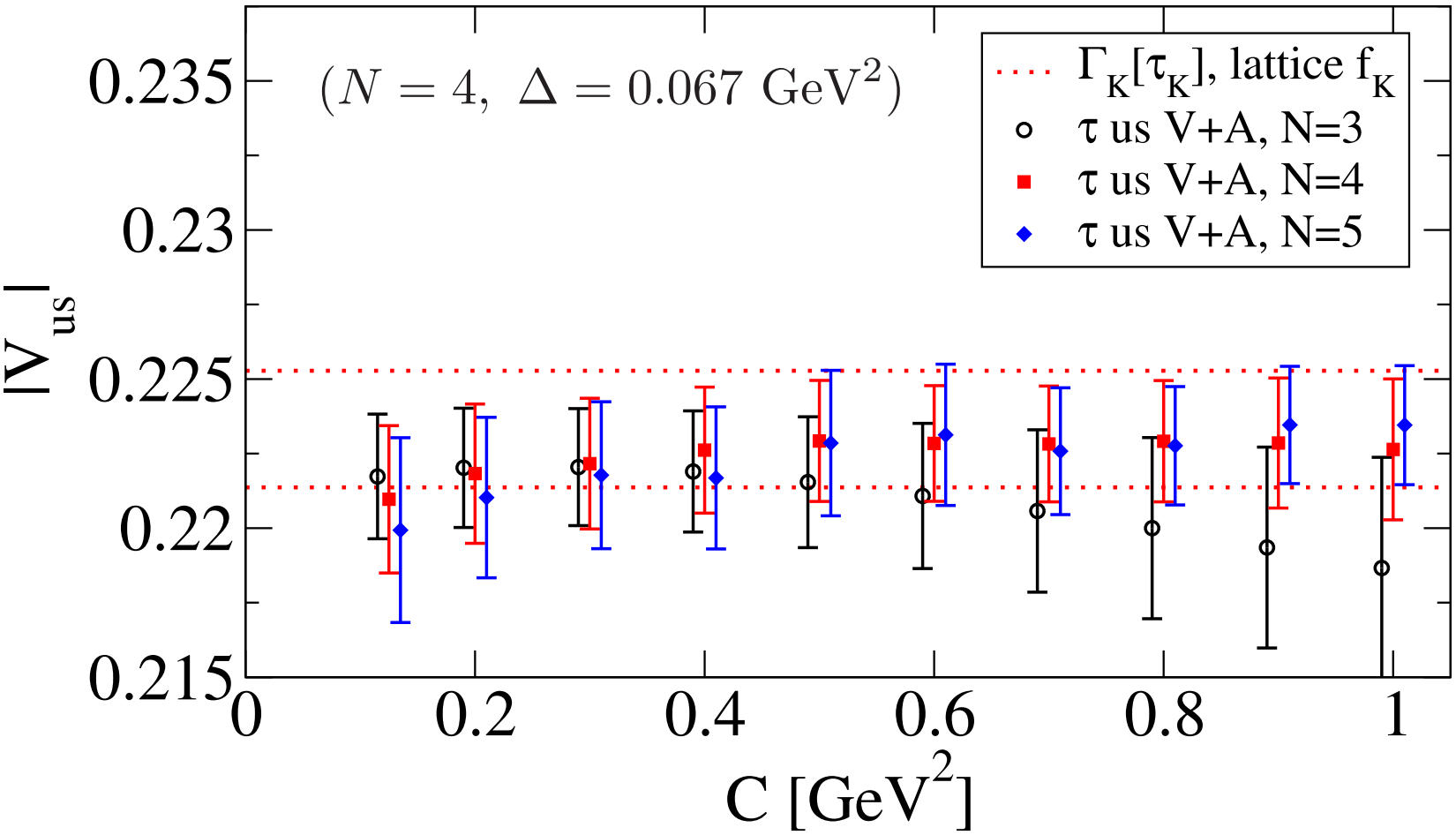


For small C, statistical error dominates.

For large C, discretization error becomes large.

We obtain optimal inclusive determinations around C=0.7.

Lattice Inclusive $|V_{us}|$ determinations

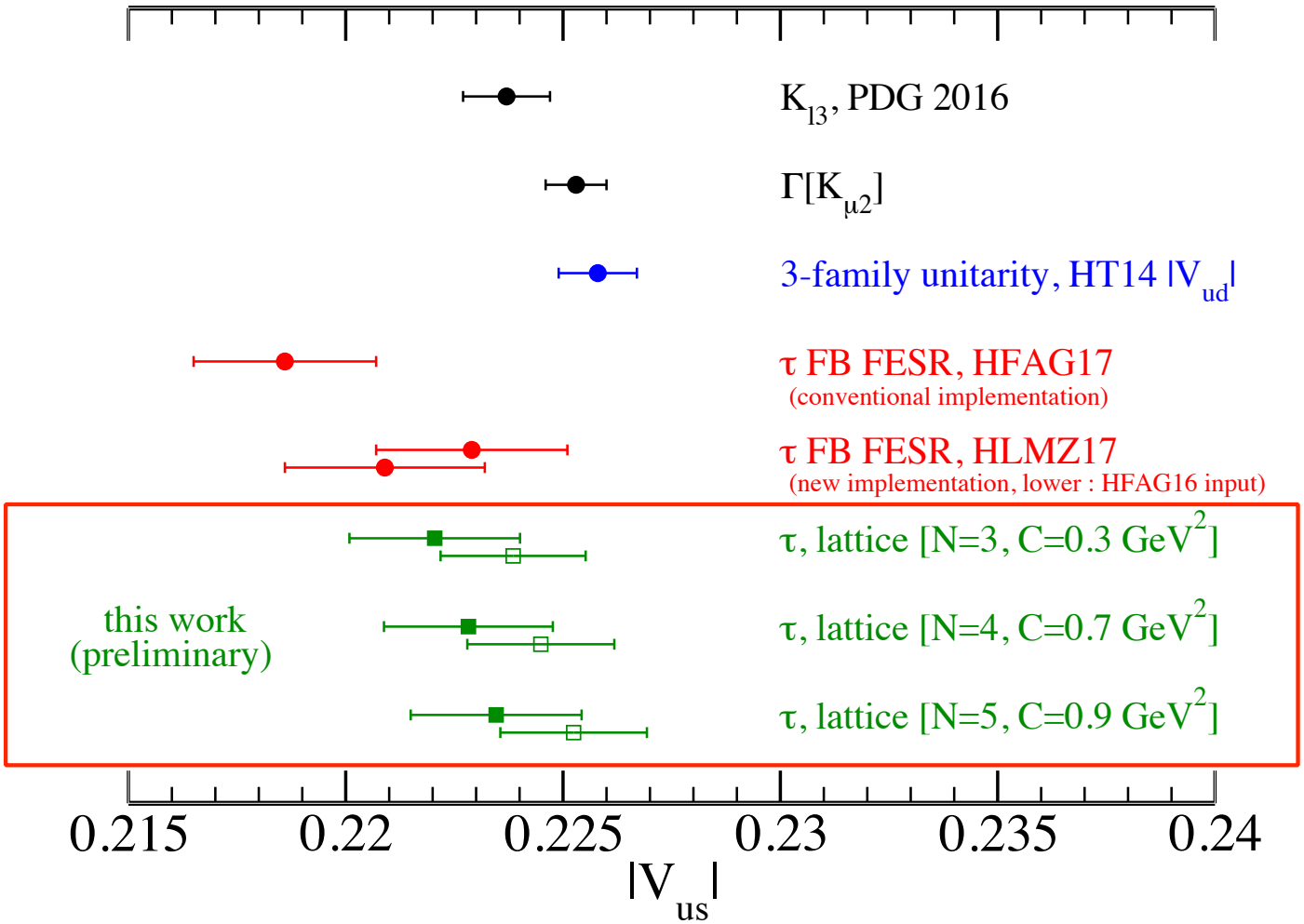


Theory and experimental errors are included.

The result is stable against changes of C and N.

$N = 4, C = 0.7[\mathrm{GeV}^2] : |V_{us}| = 0.2228(15)_{exp}(13)_{th}$ (0.87% total error)

Comparison to $|V_{us}|$ from others



$\gamma_K[\tau \to K\nu_\tau]$: (filled square)

$\gamma_K[K_{\mu 2}]$: (empty square)

All our results (C<1, N=3, 4, 5) are consistent with each other within 1 σ error, as well as to CKM unitarity.

Infinite Volume Photon and Lepton QED$_\infty$

[Feynman, Schwinger, Tomonaga]

- Instead of, or, in addition to, larger QED box, one could use infinite volume QED to compute $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$.

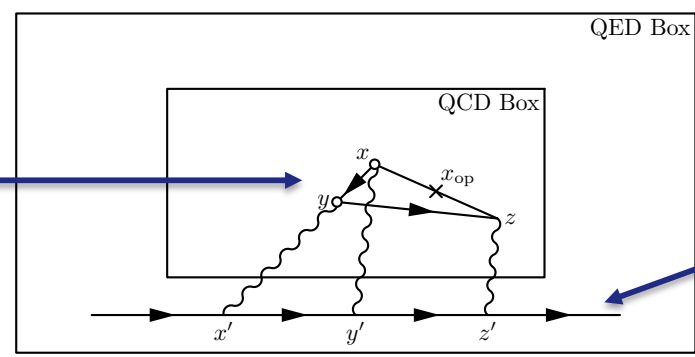
- Hadron part $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$ has following features due to the mass gap :
  - ▷ *For large distance separation, the 4pt Green function is exponentially suppressed:* $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \sim \exp[-m_\pi \times \mathit{dist}(x,y,z,x_{\rm op})]$
  - ▷ *For fixed* $(x,y,z,x_{\rm op})$*, FV error (wraparound effect etc.) is exponentially suppressed:* $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}|_V - \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}|_\infty \sim \exp[-m_\pi \times L]$

- By using QED$_\infty$ weight function $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$, which is not exponentially growing, asymptotic FV correction is exponentially suppressed

$$\Delta_V \left[\sum_{x,y,z,x_{\rm op}} \mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})] \right] \sim \exp[-m_\pi L]$$

($x_{\rm ref} = (x+y)/2$ is at middle of QCD box using transnational invariance)

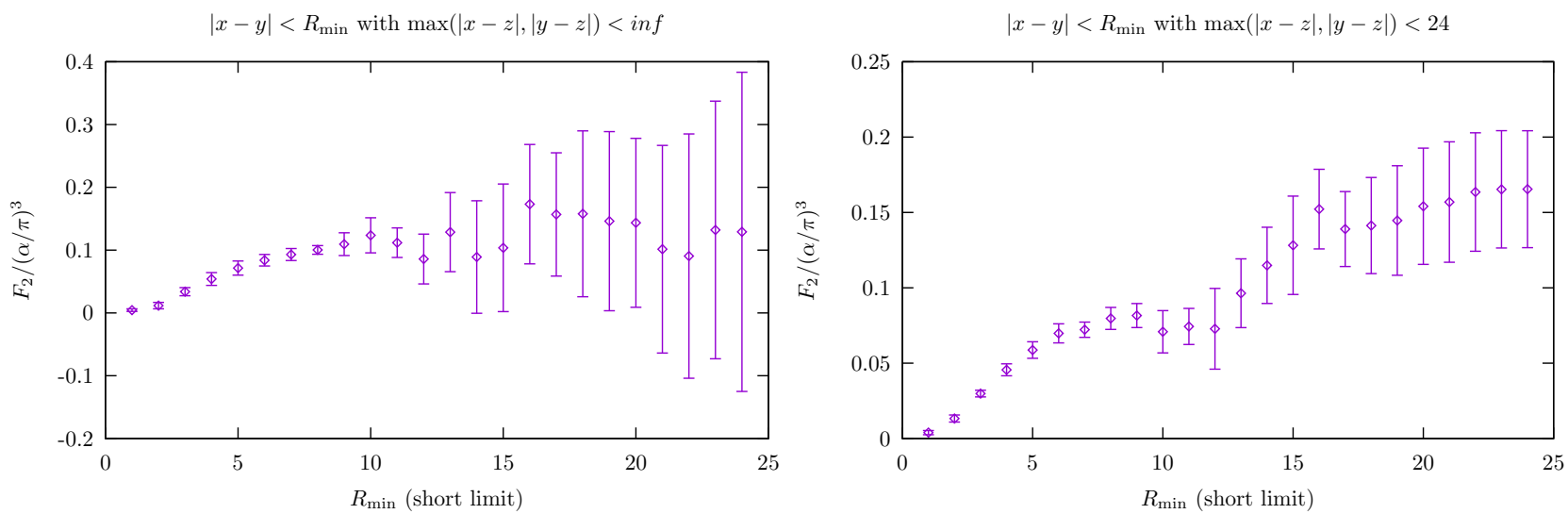
$\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$



$\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$

Preliminary results, QCD case

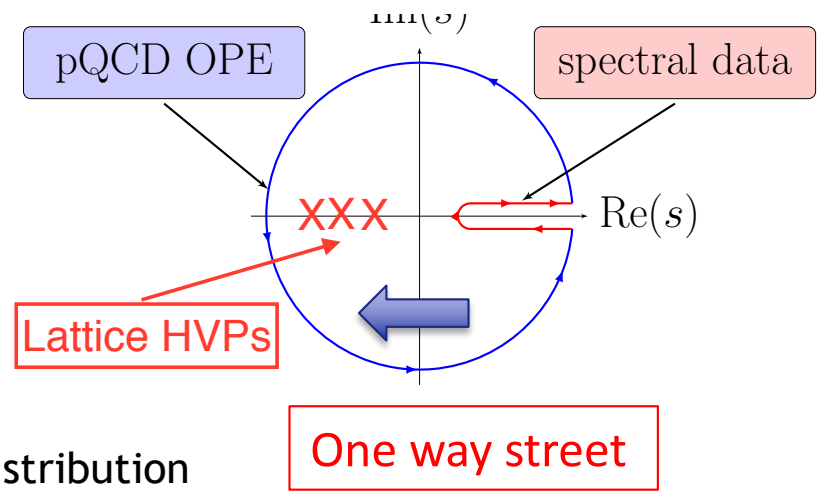
- QCD case with physical point quark mass,
- $48^3 \times 96$ lattice, with $a^{-1} = 1.73\,\mathrm{GeV}$, $m_\pi = 139\,\mathrm{MeV}$, $m_\mu = 106\,\mathrm{MeV}$.



- c.f. QED$_L$ case, $\left.\frac{g\mu-2}{2}\right|_{\text{cHLbL}} = (0.0926 \pm 0.0077)\left(\frac{\alpha}{\pi}\right)^3$

Dispersive + Lattice

- There are wide variety of application for dispersive analysis using both inclusive decay data (real world!) + non-perturbative Lattice QCD
- Quark hadron duality-violation is suppressed by non-perturbative LQCD
- Lattice point of view : good use of non-plateau region data, which otherwise is wasted !



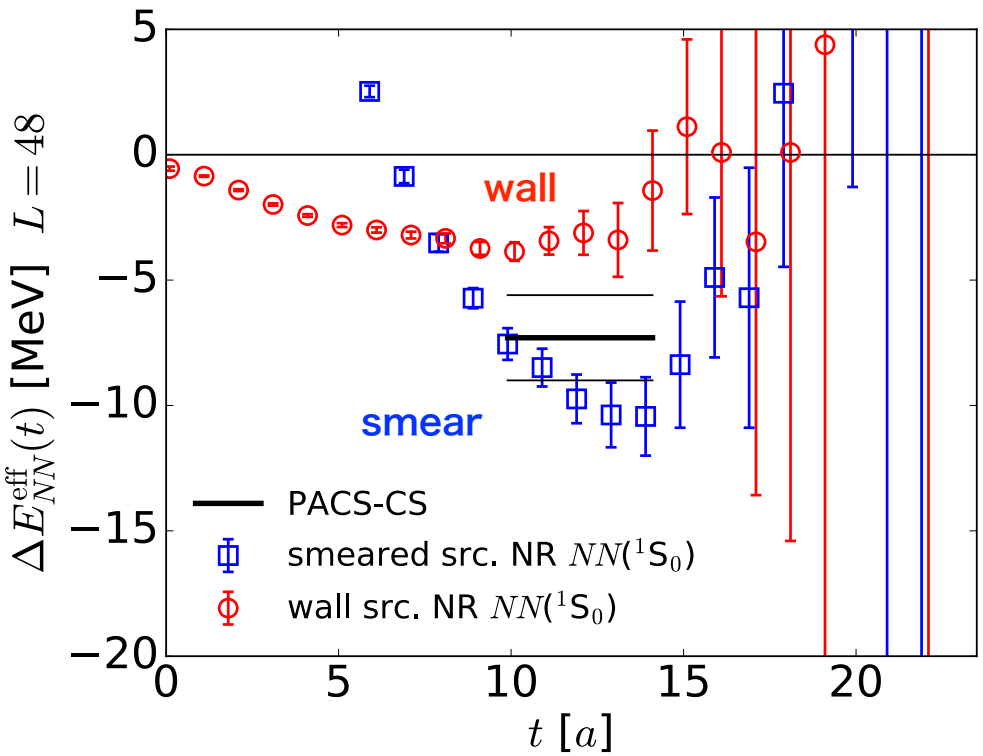
- Other interesting applications :
  - Total decay and transition rate
    [Daniel Robaina Lat17] [Max Hansen Lat17]
  - B meson inclusive semileptonic decay
    [JLQCD, Shoji Hashimoto Lat17]
  - Nucleon deep in elastic scattering and Parton Distribution
    [QCDSF, Ross Young Lat17]

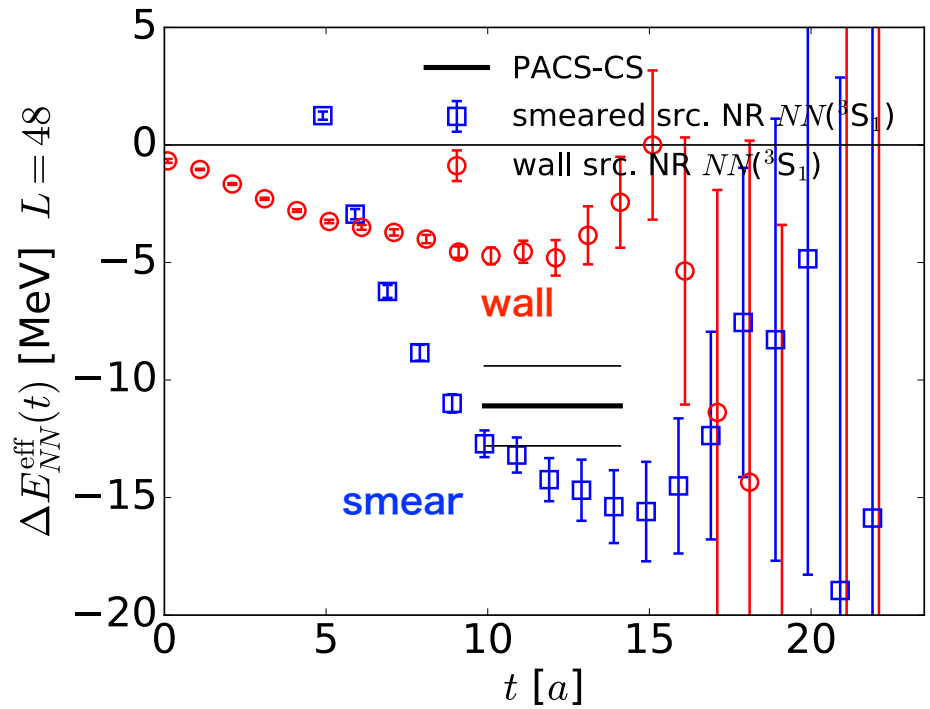
Must be many more interesting applications

source operator independence

Data	$NN(^1S_0)$				$NN(^3S_1)$			
	Operator independence	Sanity check (i)	(ii)	(iii)	Operator independence	Sanity check (i)	(ii)	(iii)
YIKU2012	No	†	No		No	†	No	

[Sinya Aoki Lat17]



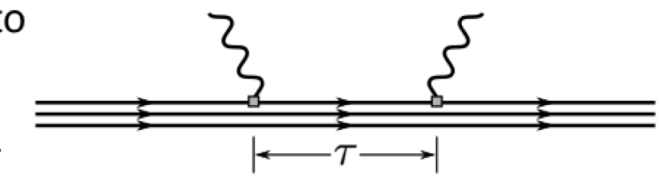


[Max Hansen Lat17]

Total rates from LQCD via Backus-Gilbert

Begin with a **four-point function** designed to give a particular spectral decomposition

$$G(\tau) = \sum_n |\langle n, L|\mathcal{J}|N\rangle|^2 \, e^{-E_n(L)\tau}$$

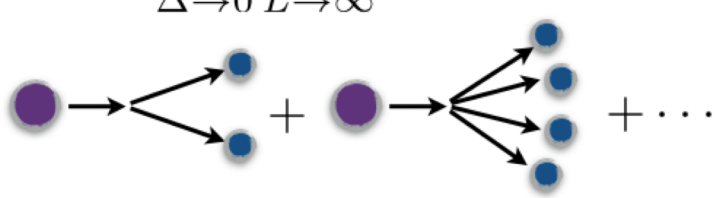


Apply the **Backus-Gilbert method** to the inverse Laplace problem

$$G(\tau) = \int_0^\infty \frac{d\omega}{2\pi}\, \rho(\omega, L) \xrightarrow[\text{Backus-Gilbert}]{} \widehat{\rho}(\overline{\omega}, L, \Delta) = \int d\omega\, \delta_\Delta(\overline{\omega}, \omega)\, \rho(\omega, L)$$

Estimate the **ordered double limit** to extract total transition rates

$$\rho(\omega) = \lim_{\Delta\to 0} \lim_{L\to\infty} \widehat{\rho}(\omega, L, \Delta) \longrightarrow$$ **Total decay and transition rates**



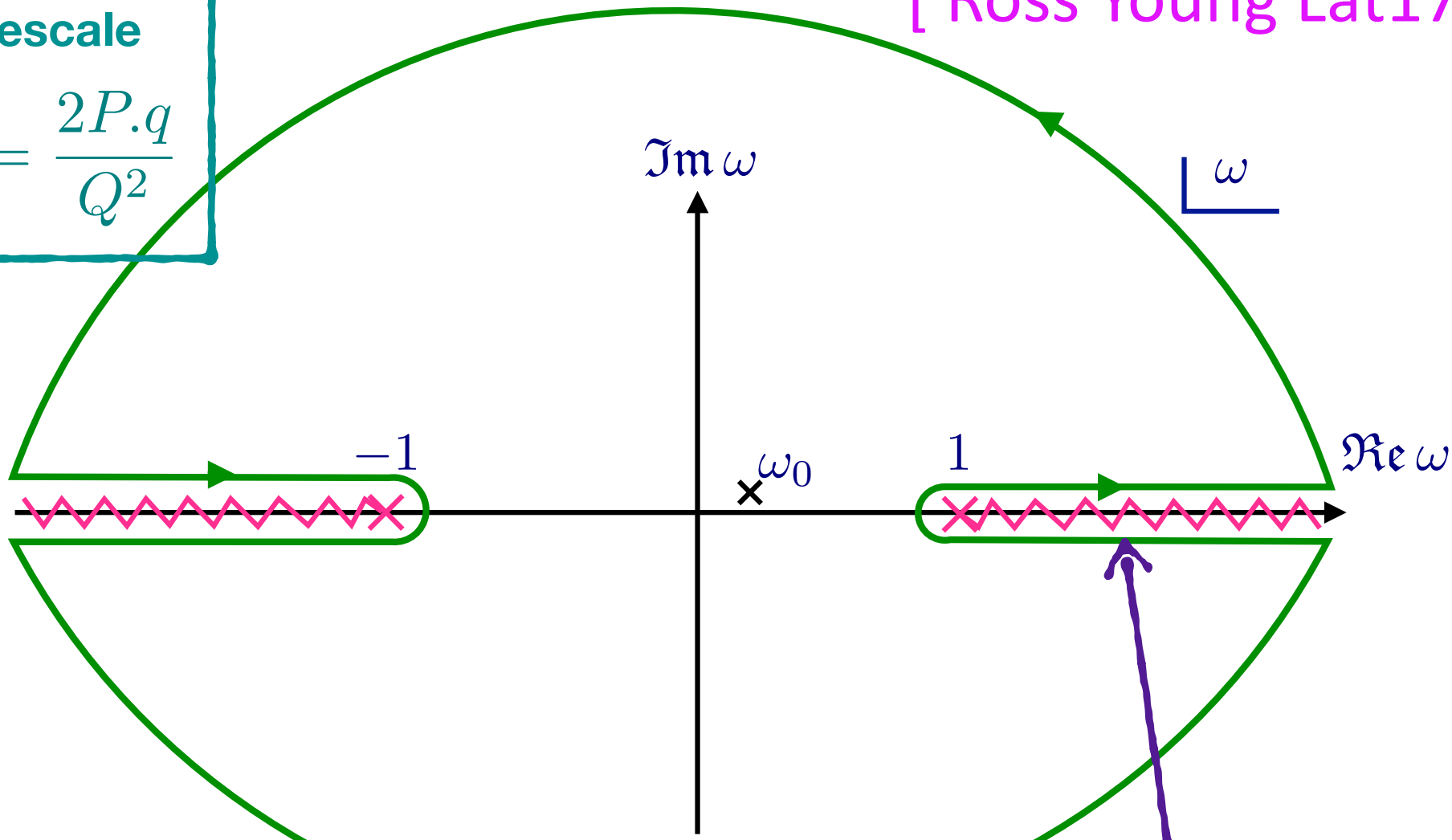
e.g. $\Gamma_{B\to s} = \dfrac{\rho_{B\to s}(M_B)}{2M_B}$

Analytic structure of Compton amplitude

[Ross Young Lat17]

Rescale

$$\omega = \frac{2P.q}{Q^2}$$

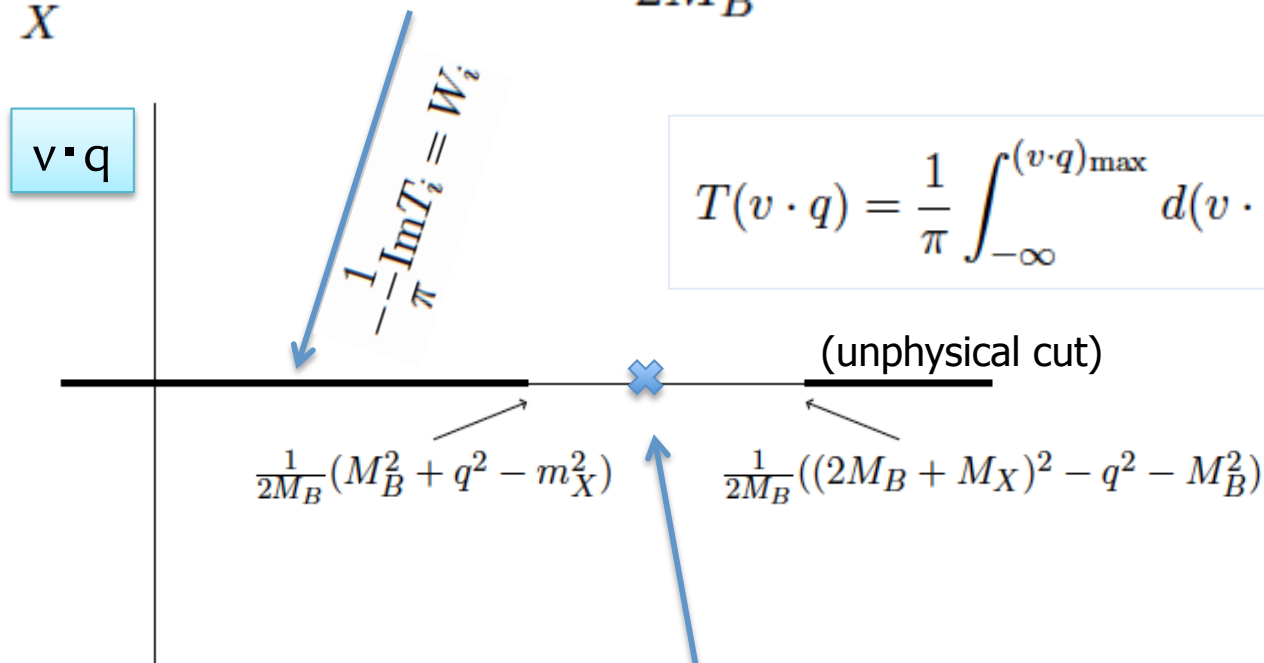


$$T_i(\omega_0, Q^2) = \frac{1}{2\pi i}\oint_{\mathcal{C}} d\omega' \frac{T_i(\omega', Q^2)}{\omega' - \omega_0} \sim \frac{1}{2\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}\, T_i(\omega', Q^2)}{\omega' - \omega_0}$$

Decay amplitude: $|\mathcal{M}|^2 = |V_{qQ}|^2 G_F^2 M_B l^{\mu\nu} W_{\mu\nu}$ (function of v·q and q^2)

Structure function:

$$W_{\mu\nu} = \sum_X (2\pi)^3 \delta^4(p_B - q - p_X) \frac{1}{2M_B} \langle B(p_B)|J_\mu^\dagger(0)|X\rangle\langle X|J_\nu(0)|B(p_B)\rangle$$



$$T(v\cdot q) = \frac{1}{\pi}\int_{-\infty}^{(v\cdot q)_{\max}} d(v\cdot q')\frac{\mathrm{Im}T(v\cdot q')}{v\cdot q' - v\cdot q}$$

calculable on the lattice
in the unphysical kinematical regime

Matrix element:

$$T_{\mu\nu} = i\int d^4x\, e^{-iqx}\frac{1}{2M_B}\langle B|T\{J_\mu^\dagger(x)J_\nu(0)\}B\rangle$$

[Shoji Hashimoto Lat17]

Future plans

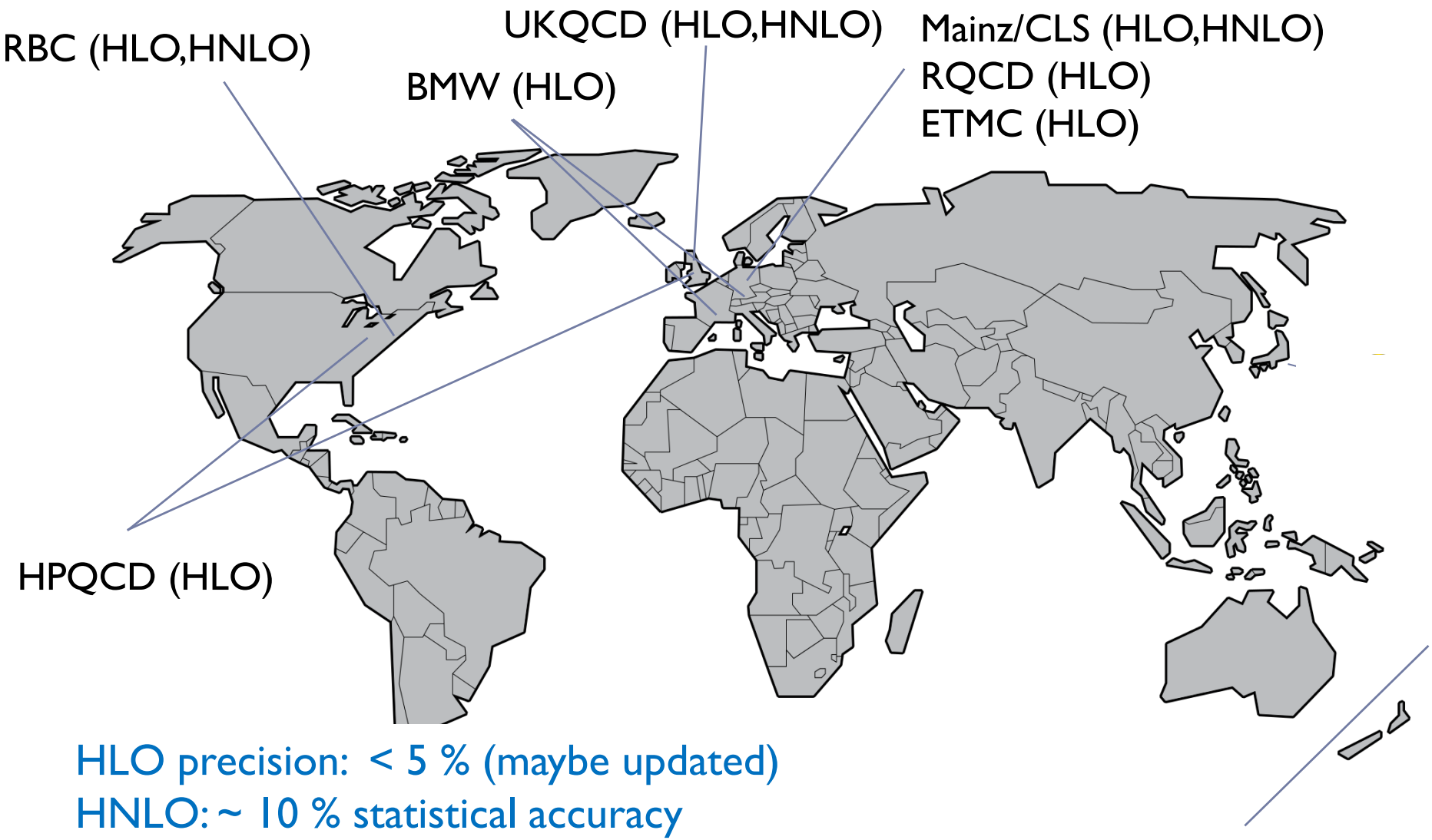
- HVP : complete QED and Isospin study, improve, tau
- HVP: FV error study on ~ $(10\ \mathrm{fm})^3$ box
- HLbL: (discretization error) Nf=2+1 DWF/ Mobius ensemble at physical point, L=5.5 fm, a=0.083 fm, $(64)^3$ at Mira, ALCC @Argonne started to run
- HLbL: FV error study on ~ $(10\ \mathrm{fm})^3$ box
- HLbL: Subleading Disconnected diagrams

Backup slides

1. Introduction

Lattice works

[Slide from Eigo Shintani]



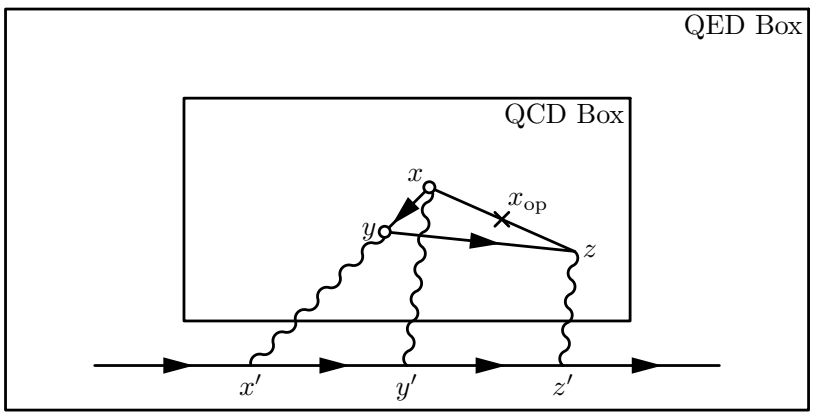
HLO precision: < 5 % (maybe updated)
HNLO: ~ 10 % statistical accuracy

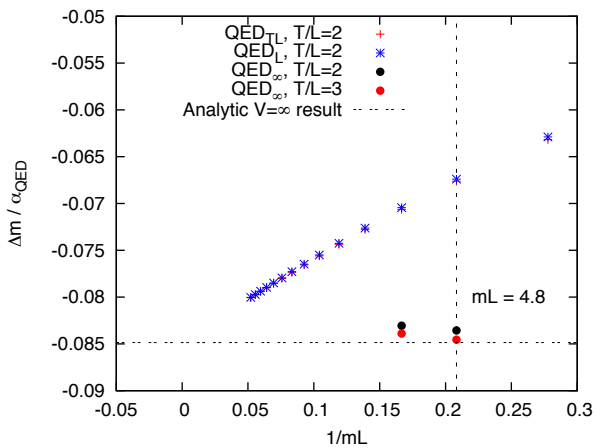
Approaches to determination of |Vus| from inclusive τ decays

Method	pQCD (OPE)	issues	Precision limit for |Vus|
Conventional FESR	higher order OPE: vacuum saturation approximation	inconsistent OPE treatment ([Ref:HLMZ 17]) large contributions from high-s region contribution	3+σ discrepancy from CKM unitarity (uncontrolled QCD systematic errors?)
Alternative FESR [HLMZ 17]	higher order OPE: fit by experimental data, checked with lattice QCD data	large contributions from high-s region	dominant high multiplicity experimental data (residual modes : 25% error to the total contribution) [1.1% total error]
Our method (lattice-based inclusive analysis)	systematically suppressed uncertainties via first principle lattice QCD data		currently lattice and experimental errors are comparable (<1%) pQCD error is negligible. [0.87 % total error]

QCD box in QED box

- FV from quark is exponentially suppressed ~ exp(- $M_\pi L_{QCD}$)
- Dominant FV effects would be from photon
- Let photon and muon propagate in larger (or infinite) box than that of quark

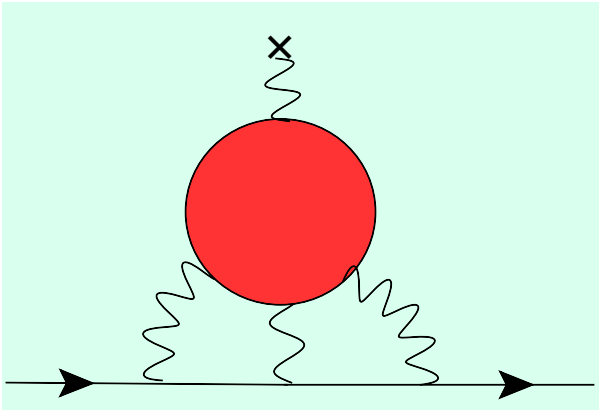




- We could examine different lepton/photon in the off-line manner e.g. QED_L (Hayakwa-Uno 2008) with larger box,
  Twisting Averaging [Lehner TI LATTICE14] or
  Infinite Vol. Photon propagators [C. Lehner, L.Jin, TI LATTICE15]
  [Maintz group, LATTICE16]

Hadronic Light-by-Light

- 4pt function of EM currents
- No direct experimental data available
- Dispersive approach



$$\Gamma_\mu^{(\mathrm{Hlbl})}(p_2,p_1) = ie^6\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{\Pi^{(4)}_{\mu\nu\rho\sigma}(q,k_1,k_3,k_2)}{k_1^2\,k_2^2\,k_3^2} \times\gamma_\nu S^{(\mu)}(\not p_2+\not k_2)\gamma_\rho S^{(\mu)}(\not p_1+\not k_1)\gamma_\sigma$$

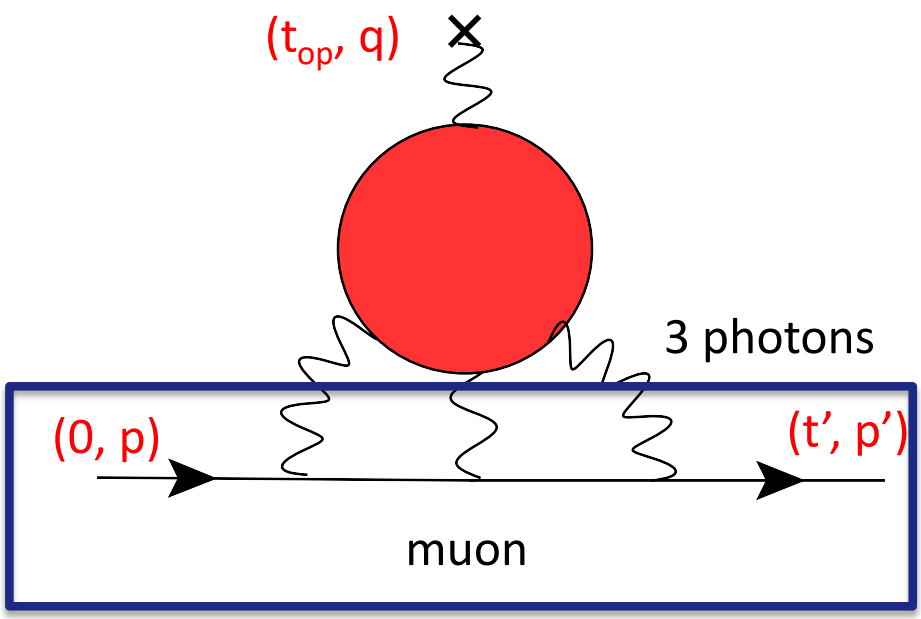
$$\Pi^{(4)}_{\mu\nu\rho\sigma}(q,k_1,k_3,k_2) = \int d^4x_1\,d^4x_2\,d^4x_3\,\exp[-i(k_1\cdot x_1+k_2\cdot x_2+k_3\cdot x_3)] \times\langle 0|T[j_\mu(0)j_\nu(x_1)j_\rho(x_2)j_\sigma(x_3)]|0\rangle$$

$$\text{Form factor}: \ \Gamma_\mu(q) = \gamma_\mu\,F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2\,m_l}\,F_2(q^2)$$

Our Basic strategy : Lattice QCD+QED system

- 4pt function has too much information to parameterize (?)
- Do Monte Carlo integration for QED two-loop with 4 pt function $\pi^{(4)}$ which is sampled in lattice QCD with chiral quark (Domain-Wall fermion)
- Photon & lepton part of diagram is derived either in lattice QED+QCD [Blum et al 2014] (stat noise from QED), or exactly derive for given loop momenta [L. Jin et al 2015] (no noise from QED+lepton).

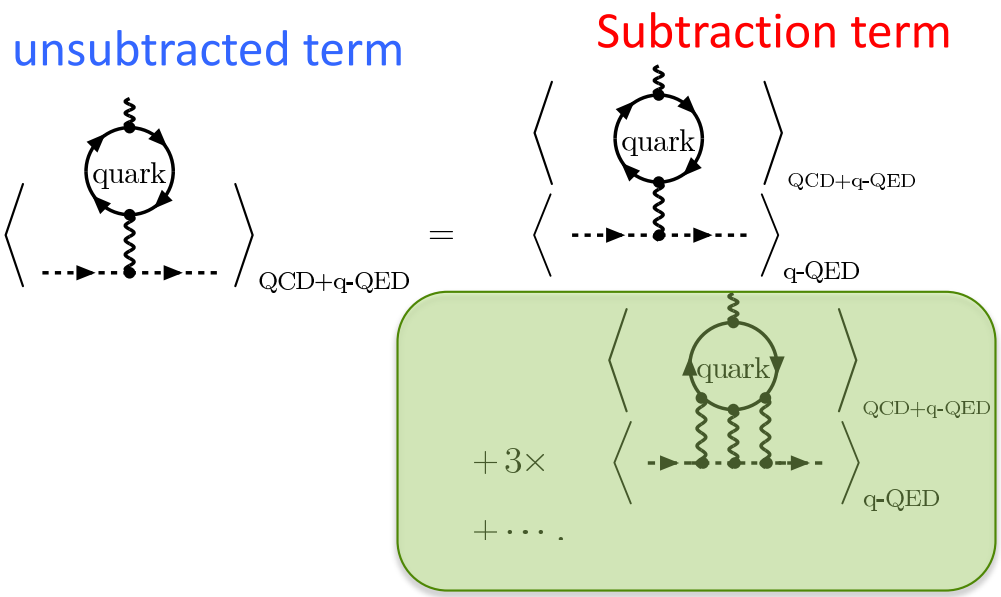
$$\Gamma_\mu^{(\mathrm{Hlbl})}(p_2,p_1) = ie^6 \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\Pi^{(4)}_{\mu\nu\rho\sigma}(q,k_1,k_2,k_3)$$
$$\times\,[S(p_2)\gamma_\nu S(p_2+k_2)\gamma_\rho S(p_1+k_1)\gamma_\sigma S(p_1) + (\mathrm{perm.})]$$

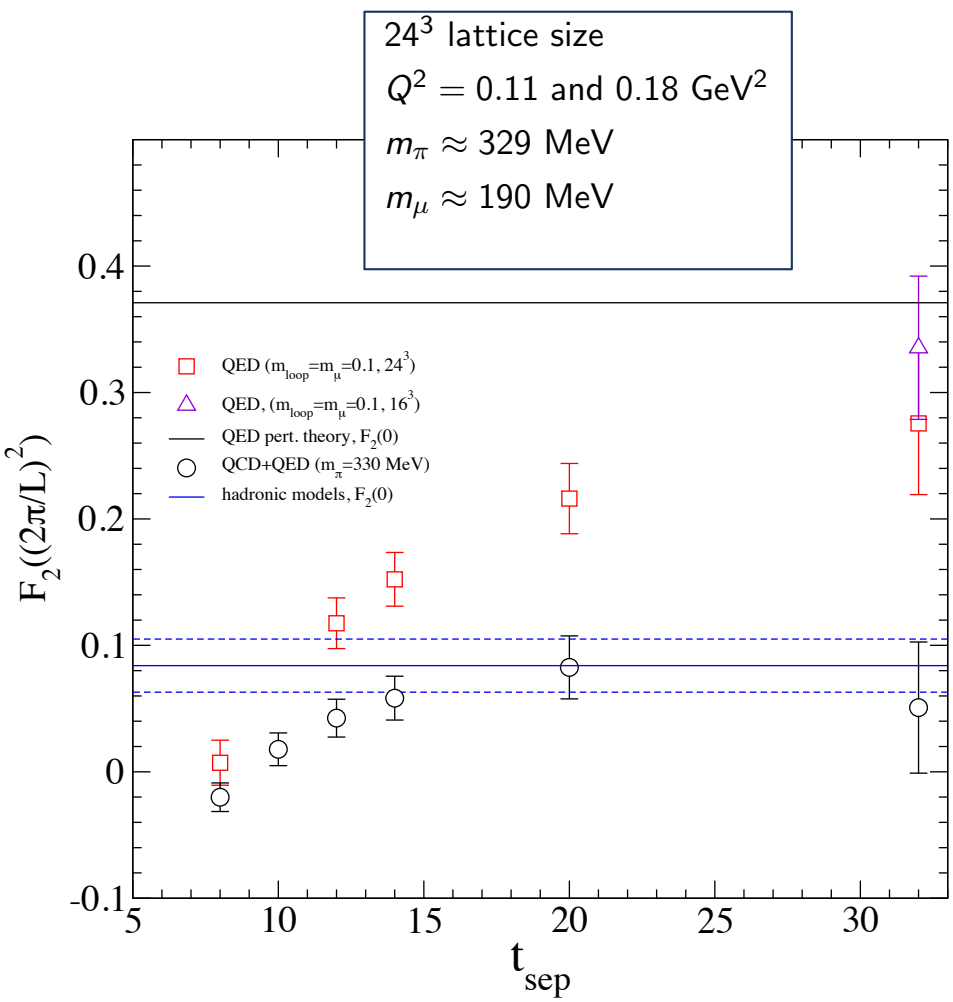


- set spacial momentum for
  - external EM vertex q
  - in- and out- muon p, p'
  q = p-p'
- set time slice of muon source(t=0), sink(t') and operator (t_{op})
- take large time separation for ground state matrix element

QCD+QED method [Blum et al 2015]

- One photon is treated analytically
- other two sampled stochastically
- needs subtraction
- use AMA for error reduction
- use Furry's theoretm to reduce α^2 noise



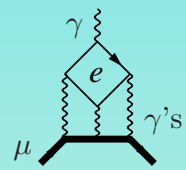


- Connected part only

- QED only calculation consistent with QED loop calculation for larger volume

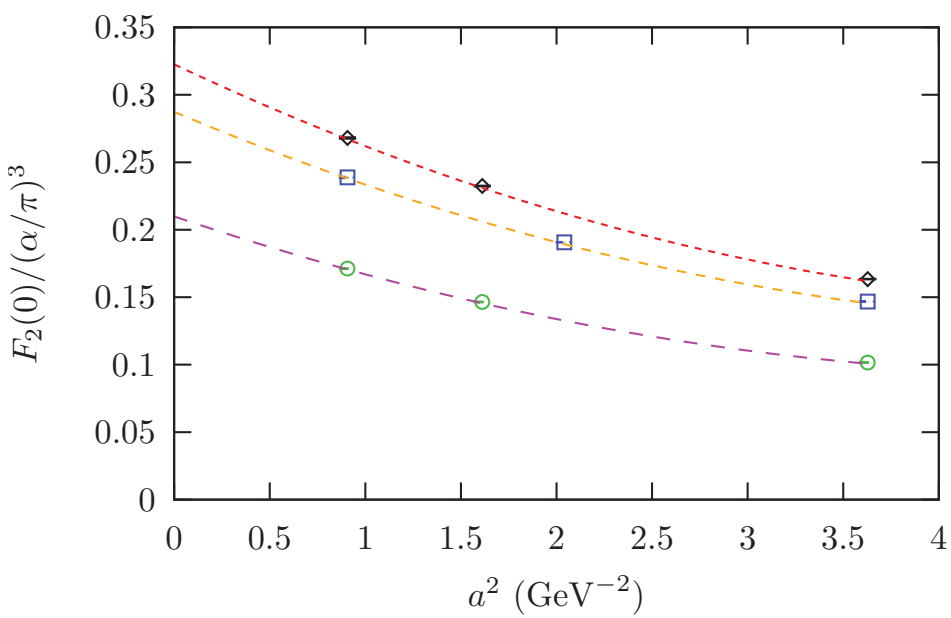
- QED+QCD
  - ball park of model values
  - significant exited state effects ?

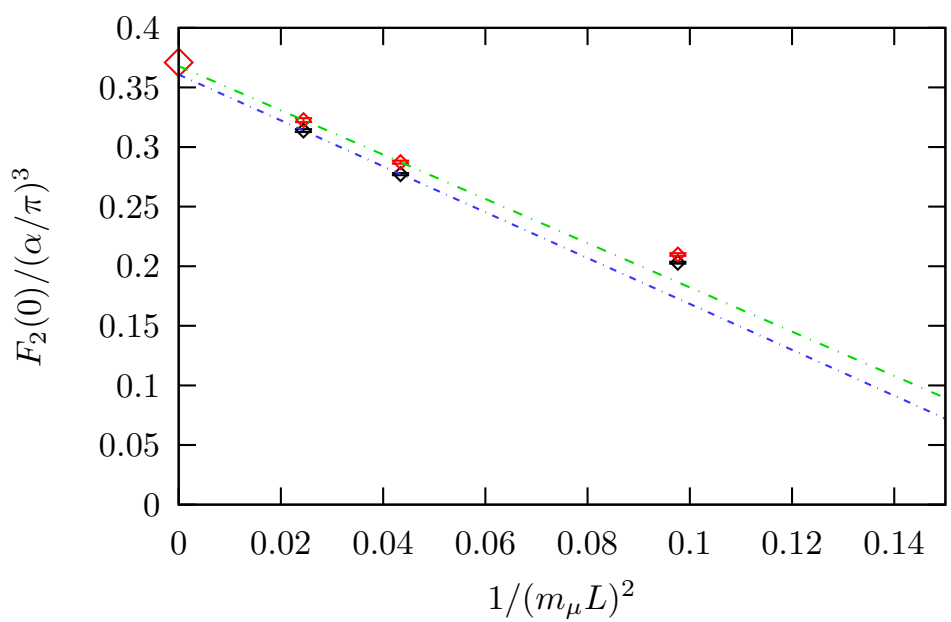
Systematic effects in QED only study



$$a_\mu^{(6)}(\mathrm{lbl}, e) = \left[\frac{2}{3}\pi^2 \ln\frac{m_\mu}{m_e} + \frac{59}{270}\pi^4 - 3\,\zeta(3) - \frac{10}{3}\pi^2 + \frac{2}{3} + O\left(\frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^3$$

- muon loop, muon line
- a = a m_μ / (106 MeV)
- L= 11.9, 8.9, 5.9 fm
- known result : F2 = 0.371 (diamond) correctly reproduced (good check)



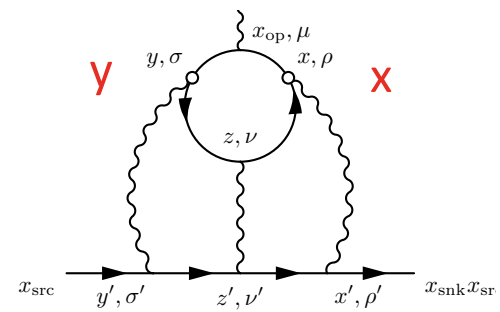


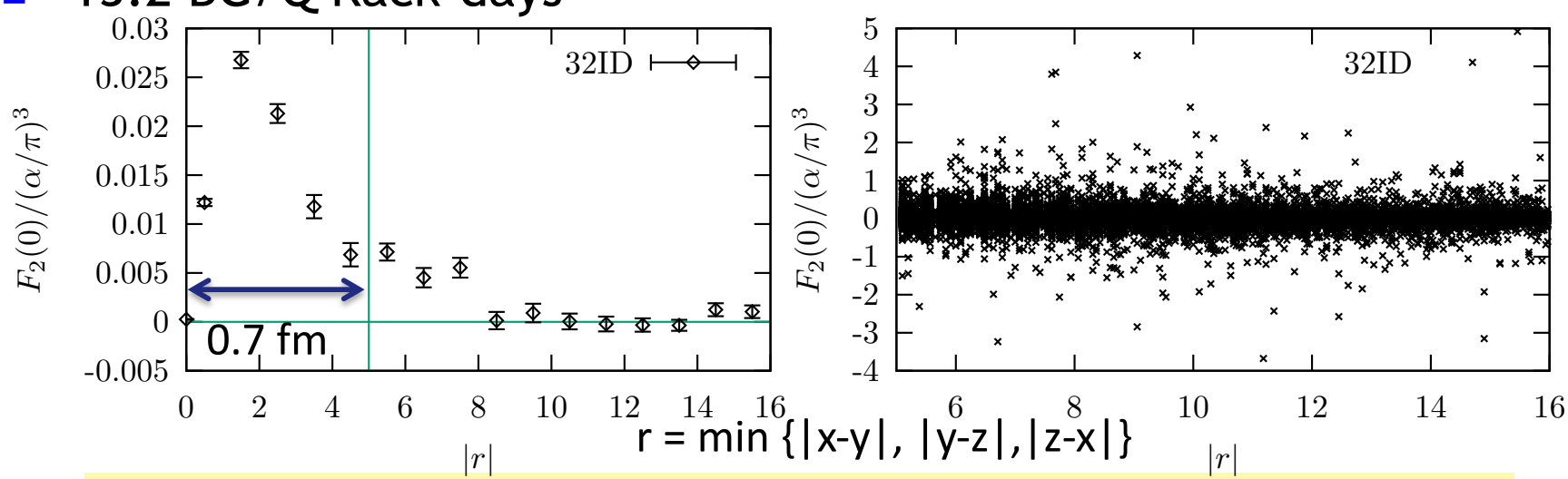
FV and discretization error could be as large as 20-30 %, similar discretization error seen from QCD+QED study

M_π=170 MeV cHLbL result

[Luchang Jin et al.,PRD93, 014503 (2016)]

- V=(4.6 fm)^3, a = 0.14 fm, m_μ=130 MeV, 23 conf
- pair-point sampling with AMA (1000 eigV, 100CG) ; > 6000 meas/conf
  - |x-y| <= 0.7fm, all pairs, x2-5 samples
    217 pairs (10 AMA-exact)
  - |x-y| > 0.7fm, 512 pairs (48 AMA-exact)
- 13.2 BG/Q Rack-days





r = min {|x-y|, |y-z|,|z-x|}

$$\frac{(g_\mu - 2)_{\rm cHLbL}}{2} = (0.1054 \pm 0.0054)(\alpha/\pi)^3 = (132.1 \pm 6.8) \times 10^{-11}.$$

Strange contribution : (0.0011± 0.005) (α/π)^3

Infinite Volume Photon and Lepton QED$_\infty$

[Feynman, Schwinger, Tomonaga]

- Instead of, or, in addition to, larger QED box, one could use infinite volume QED to compute $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$.

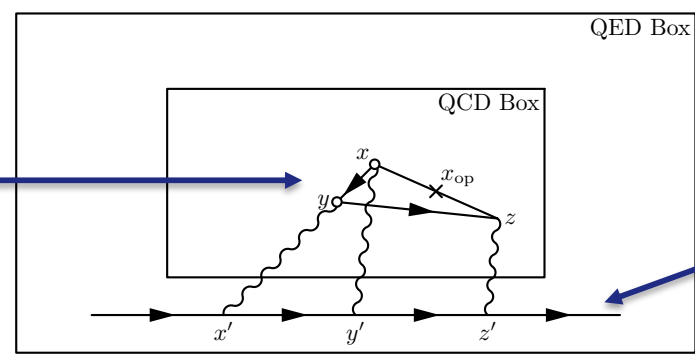
- Hadron part $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$ has following features due to the mass gap :
  - ▷ *For large distance separation, the 4pt Green function is exponentially suppressed:* $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \sim \exp[-m_\pi \times \textit{dist}(x,y,z,x_{\rm op})]$
  - ▷ *For fixed* $(x,y,z,x_{\rm op})$*, FV error (wraparound effect etc.) is exponentially suppressed:* $\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}|_V - \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}|_\infty \sim \exp[-m_\pi \times L]$

- By using QED$_\infty$ weight function $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$, which is not exponentially growing, asymptotic FV correction is exponentially suppressed

$$\Delta_V \left[\sum_{x,y,z,x_{\rm op}} \mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})] \right] \sim \exp[-m_\pi L]$$

($x_{\rm ref} = (x+y)/2$ is at middle of QCD box using transnational invariance)

$\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$



$\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$

Subtraction using current conservation

- From current conservation, $\partial_\rho V_\rho(x) = 0$, and mass gap, $\langle x V_\rho(x) \mathcal{O}(0) \rangle \sim |x|^n \exp(-m_\pi |x|)$

$$\sum_x \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) = \sum_x \langle V_\rho(x) V_\sigma(y) V_\kappa(z) V_\nu(x_{\rm op}) \rangle = 0$$

$$\sum_z \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) = 0$$

at $V \to \infty$ and $a \to 0$ limit (we use local currents).

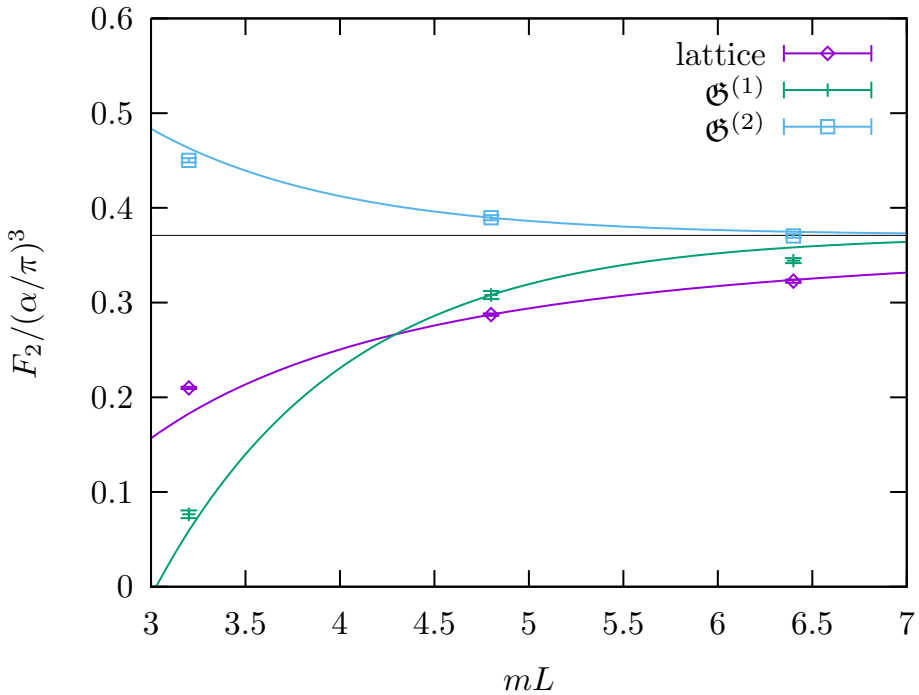
- We could further change QED weight

$$\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(x,y,z) = \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,y) + \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,y)$$

without changing sum $\sum_{x,y,z} \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$.

- Subtraction changes discretization error and finite volume error.
- Similar subtraction is used for HVP case in TMR kernel, which makes FV error smaller.
- Also now $\mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(z,z,x) = \mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(y,z,z) = 0$, so short distance $\mathcal{O}(a^2)$ is suppressed.
- The 4 dimensional integral is calculated numerically with the CUBA library cubature rules. (x,y,z) is represented by 5 parameters, compute on N^5 grid points and interpolates. ($|x-y| < 11$ fm).

Results, QED case, Finite Volume Error



- QED weight : QED$_L$ (purple diamond), QED$_\infty$ without subtraction (green plus), with subtraction (blue square)
- Curves correspond to expected finite volume scaling ($0.371 + k/L^2$) and infinite volume scaling ($0.371 + ke^{-mL}$), where the coefficient k is chosen to match the data at $mL = 4.8$.
- The right most point for the finite volume weighting function lies a bit off its scaling curve because the discretization error has not been completely removed, and the coefficient k does not contain any possible volume dependence.

$(g\text{-}2)_\mu$ SM Theory vs experiment

- QED, EW, Hadronic contributions

K. Hagiwara et al. , J. Phys. G: Nucl. Part. Phys. 38 (2011) 085003

$$a_\mu^{\rm SM} = (11\ 659\ 182.8\ \pm 4.9\ ) \times 10^{-10}$$

$$a_\mu^{\rm QED} = (11\ 658\ 471.808\ \pm 0.015\ ) \times 10^{-10}$$
$$a_\mu^{\rm EW} = (\ 15.4\ \pm 0.2\ ) \times 10^{-10}$$
$$a_\mu^{\rm had,LOVP} = (\ 694.91\ \pm 4.27\ ) \times 10^{-10}$$
$$a_\mu^{\rm had,HOVP} = (\ -9.84\ \pm 0.07\ ) \times 10^{-10}$$
$$a_\mu^{\rm had,lbl} = (\ 10.5\ \pm 2.6\ ) \times 10^{-10}$$

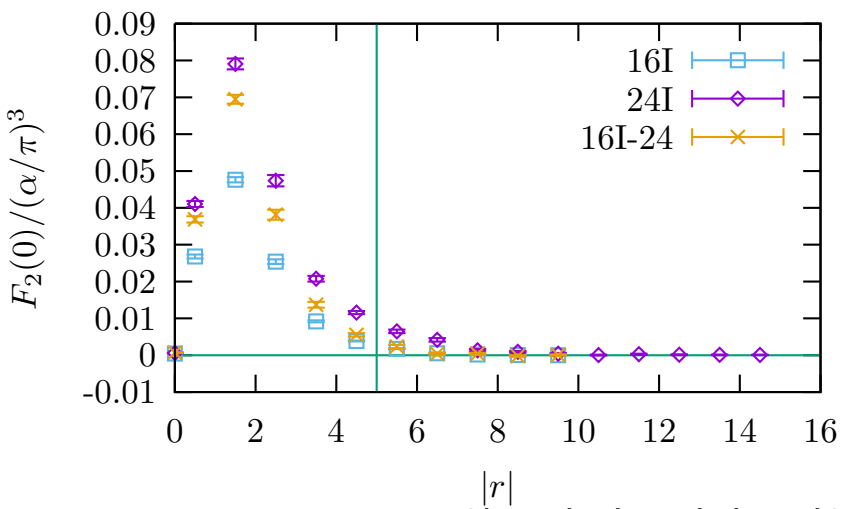
$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 28.8(6.3)_{\rm exp}(4.9)_{\rm SM} \times 10^{-10} \quad [3.6\sigma]$$

- Discrepancy between EXP and SM is larger than EW!
- Currently the dominant uncertainty comes from HVP, followed by HLbL
- x4 or more accurate experiment FNAL , J-PARC
- Our Goal : sub 1% accuracy for HVP, and
  → **10% accuracy for HLbL**

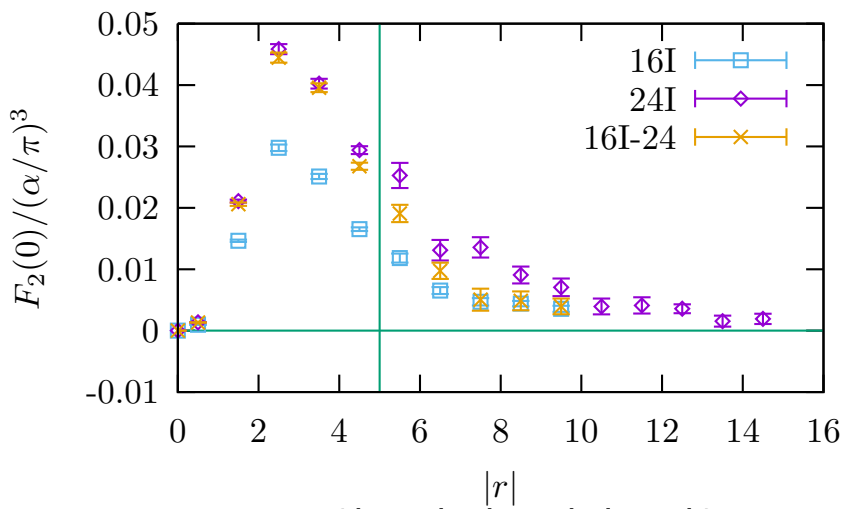
QED box in QCD box (contd.)

- Mπ=420 MeV, mμ=330 MeV, 1/a=1.7 GeV
- $(16)^3$ = $(1.8\ \text{fm})^3$ QCD box in $(24)^3$= $(2.7\ \text{fm})^3$ QED box

Ensemble	$m_\pi L$	QCD Size	QED Size	$\frac{F_2(q^2=0)}{(\alpha/\pi)^3}$
16I	3.87	$16^3 \times 32$	$16^3 \times 32$	$0.1158(8)$
24I	5.81	$24^3 \times 64$	$24^3 \times 64$	$0.2144(27)$
16I-24		$16^3 \times 32$	$24^3 \times 64$	$0.1674(22)$



r = min {|x-y|, |y-z|,|z-x|}



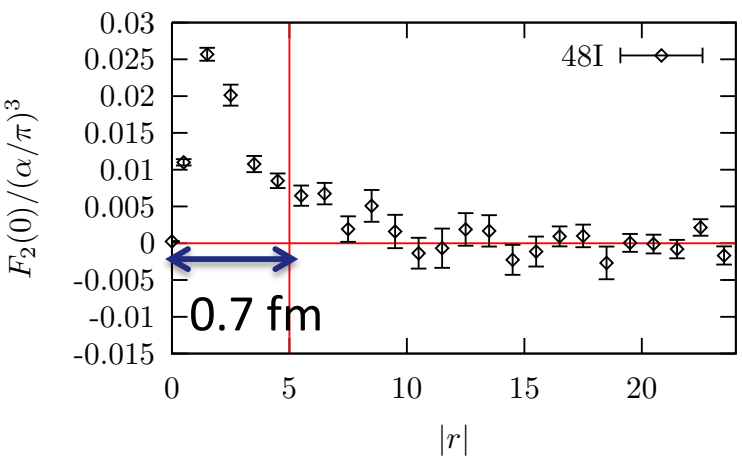
r = max{|x-y|, |y-z|,|z-x|}

physical M_π=140 MeV cHLbL result

[Luchang Jin et al. , Phys.Rev.Lett. 118 (2017) 022005]

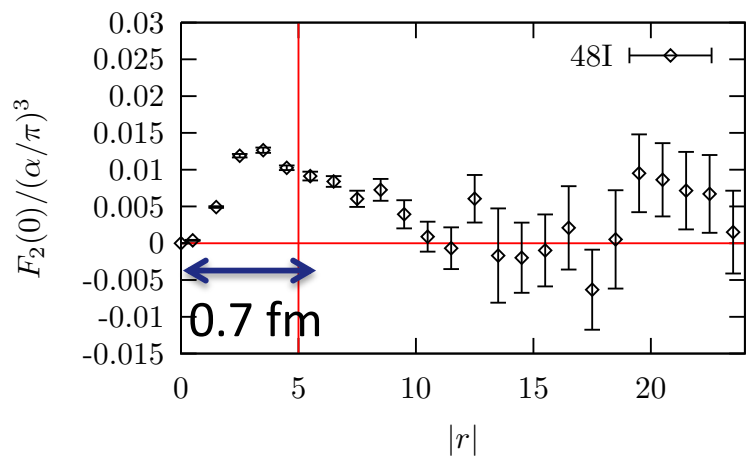
- V=(5.5 fm)^3, a = 0.11 fm, m_μ=106 MeV, 69 conf [RBC/UKQCD]
- Two stage AMA (2000 eigV, 200CG and 400 CG) using zMobius, ~4500 meas/conf
- 160 BG/Q Rack-days

0.6 fm

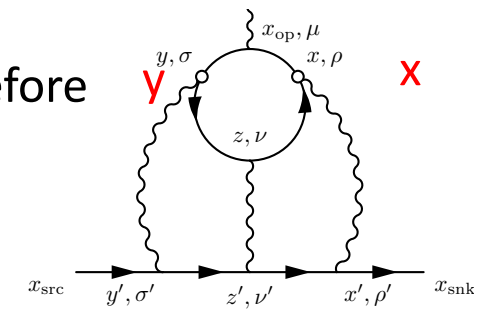


r = min {|x-y|, |y-z|,|z-x|}

integrand safely suppressed before reaching r ~ L/2



r = max{|x-y|, |y-z|,|z-x|}

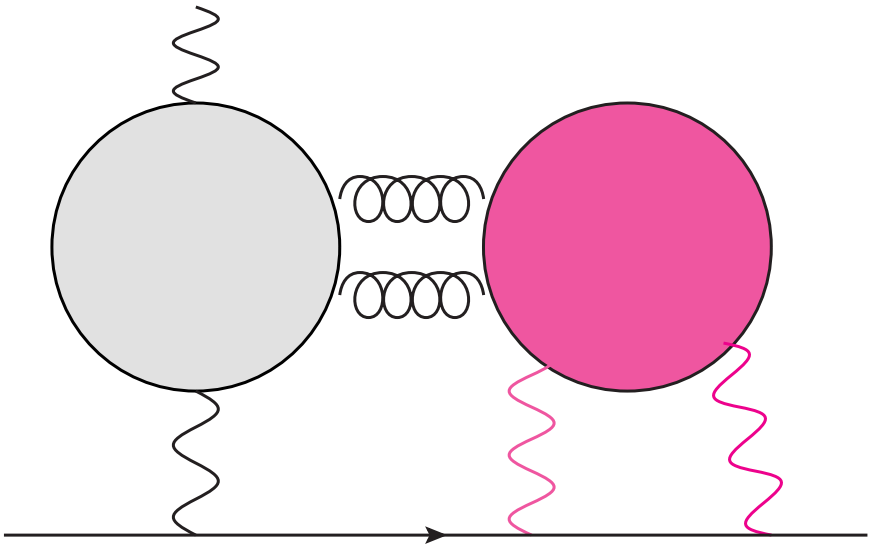


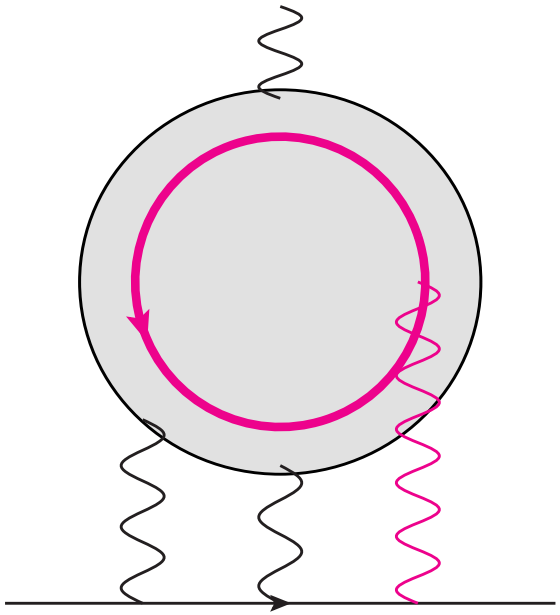
$$a_\mu^{\rm LbL,\ con} = (0.0926 \pm 0.0077) \times \left(\frac{\alpha}{\pi}\right)^3 = (11.60 \pm 0.96) \times 10^{-10},$$

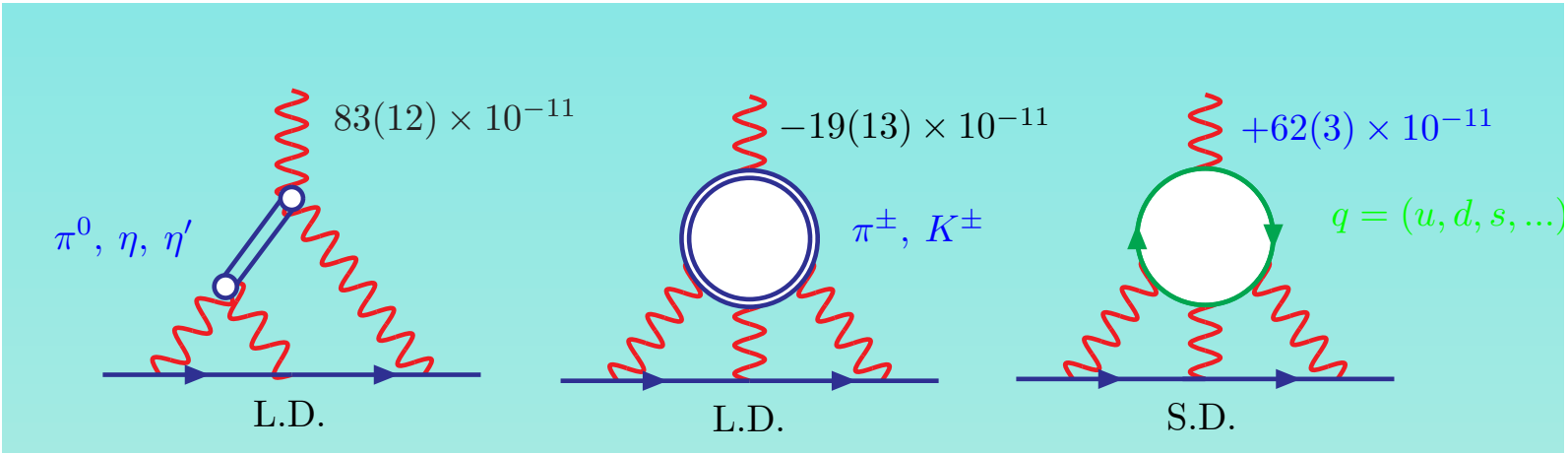
(preliminary, stat err only)

Disconnected diagrams in HLbL

- Disconnected diagrams







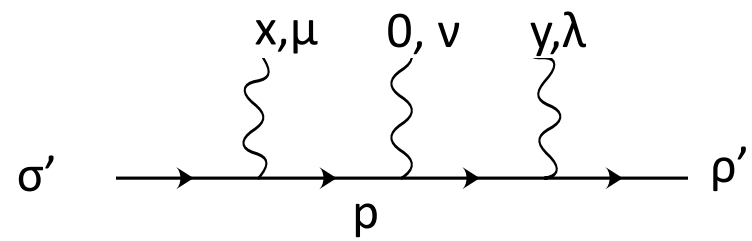
Continuum Infinite Volume (a.k.a HVP way)

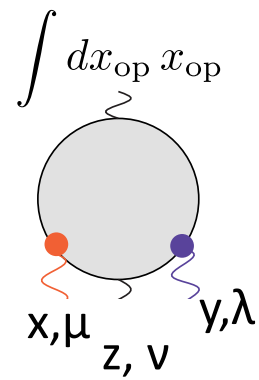
$$a_\mu^{\rm HVP} = \sum_t w(t) C(t), \quad w(t) \propto t^4 \cdots$$

- One could also use infinite volume/continuum lepton&photon diagram in coordinate space

  [J. Green et al. Mainz group, LAT16 proceedings]

$$\mathcal{L}_{\mu\nu\lambda\sigma\rho}(x, y; p)$$





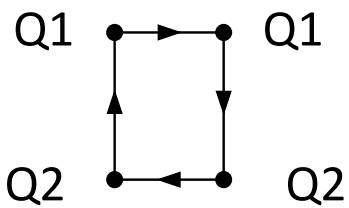
- Techniques in continuum model calculation [Knect Nyffeler 2002; Jegerlehner Nyffeler 2009] : angle average over muon momentum, and carry out angle of two virtual photons

Direct 4pt calculation for selected kinematical range

[J. Green et al. Mainz group, Phys. Rev. Lek 115, 222003(2015)]

- Compute connected contribution of 4 pt function in momentum space
- Forward amplitudes related to ▫*(Q1)▫*(Q2) -> hadron cross section via dispersion relation

$$\mathcal{M}_{\rm had}\left(\gamma^*(Q_1)\gamma^*(Q_2) \to \gamma^*(Q_1)\gamma^*(Q_2)\right)$$
$$\leftrightarrow \quad \sigma_{0,2}\left(\gamma^*(Q_1)\gamma^*(Q_2) \to {\rm had.}\right)$$



- solid curve: model prediction
- π0 exchange is seen to be not dominant, possibly due to heavy quark mass in the simulation (Mπ = 324 MeV)
- disconnected quark diagram loop in progress in 2016

$$\nu = -Q_1 \cdot Q_2$$

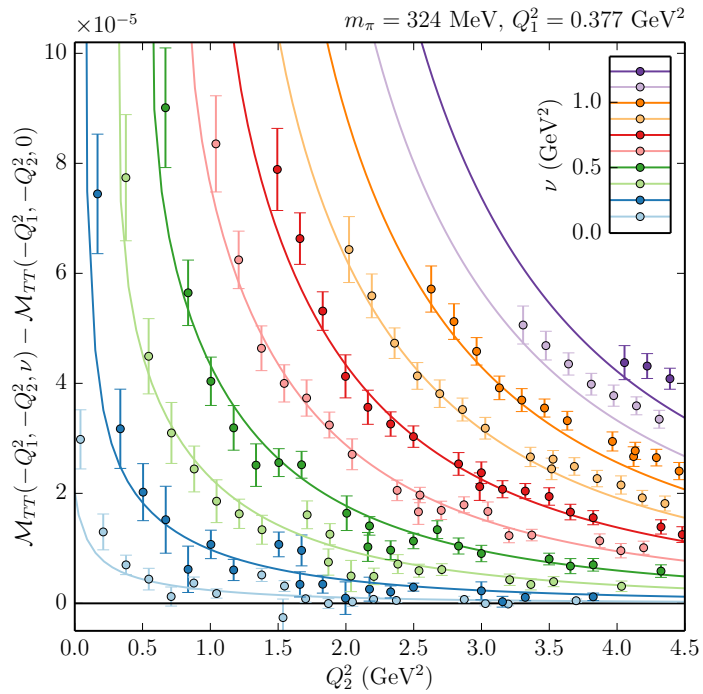


FIG. 3. The forward scattering amplitude $\mathcal{M}_{\rm TT}$ at a fixed virtuality $Q_1^2 = 0.377{\rm GeV}^2$, as a function of the other photon virtuality Q_2^2, for different values of ν. The curves represent the predictions based on Eq. (10), see the text for details.

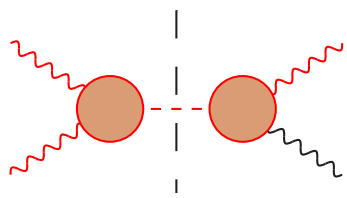
Dispersive approach for HLbL

[Colangelo et al. 2014, 2015, Pauk&Vanderhaeghen 2014]

- Using crossing symmetry, gauge invariance, 138 form factors are reduced 12 relevant for HLbL

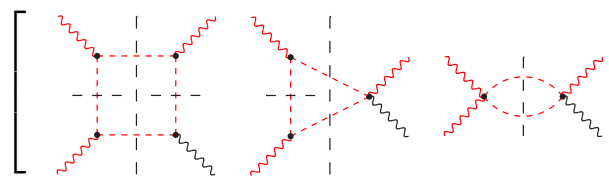
$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{1}{q_1^2 q_2^2 (q_1+q_2)^2}\frac{1}{(p+q_1)^2-m_\mu^2}\frac{1}{(p-q_2)^2-m_\mu^2}$$
$$\times \sum_{j=1}^{12} \xi_j \hat{T}_{i_j}(q_1,q_2;p)\hat{\Pi}_{i_j}(q_1,q_2,-q_1-q_2),$$

- π0, ▫,▫’ exchange, pion-loop (exactly scalar QED with pion Form factor)



$$= \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)F_{\pi^0\gamma^*\gamma^*}(q_3^2,q_4^2)}{s-M_\pi^2}$$

$$\Pi^{\mathrm{FsQED}}_{\mu\nu\lambda\sigma} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2) \times \Big[$$

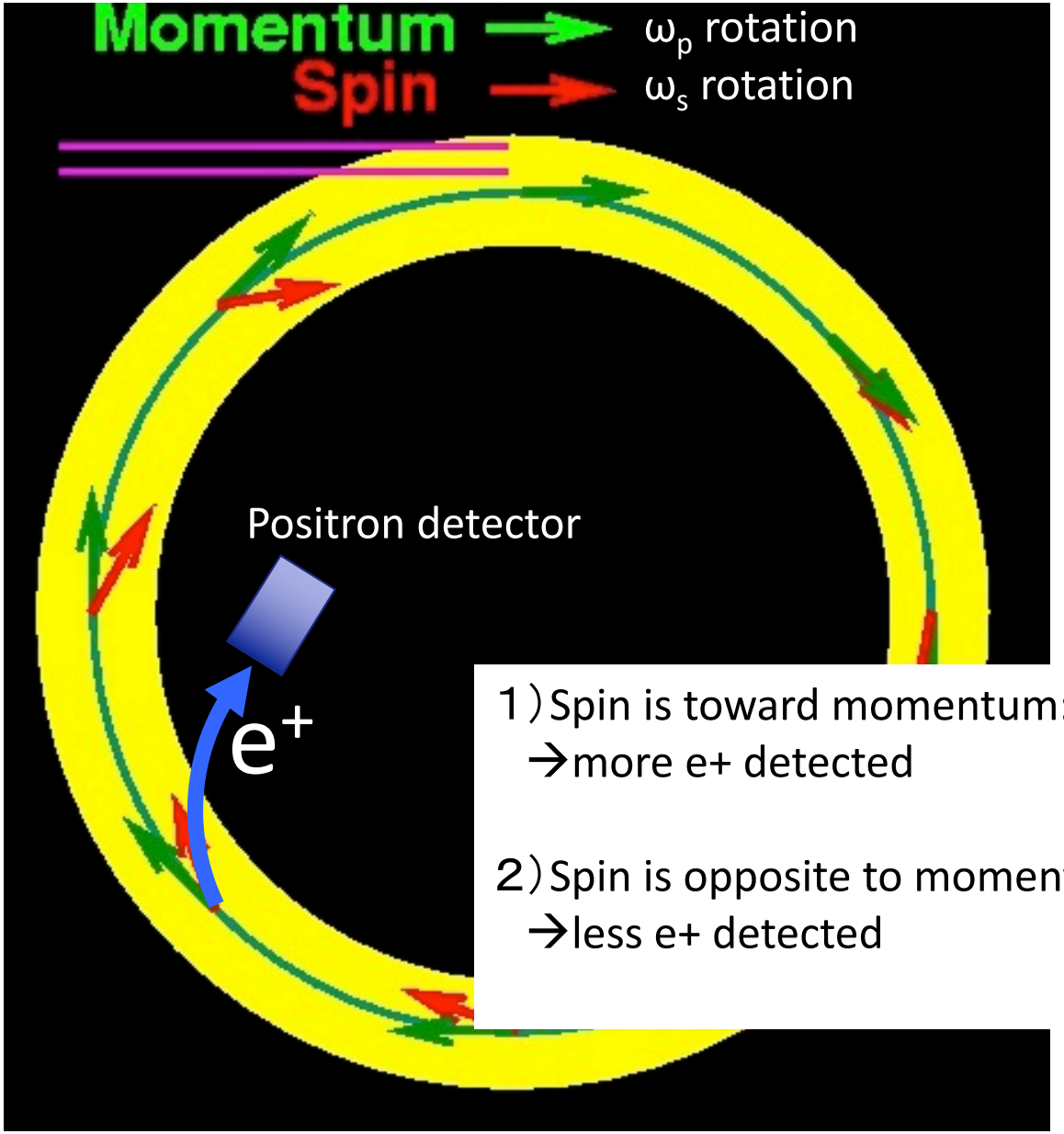


$$\Big]$$

- other contribution is neglected

Measurement of decay positron

Uniform B-field

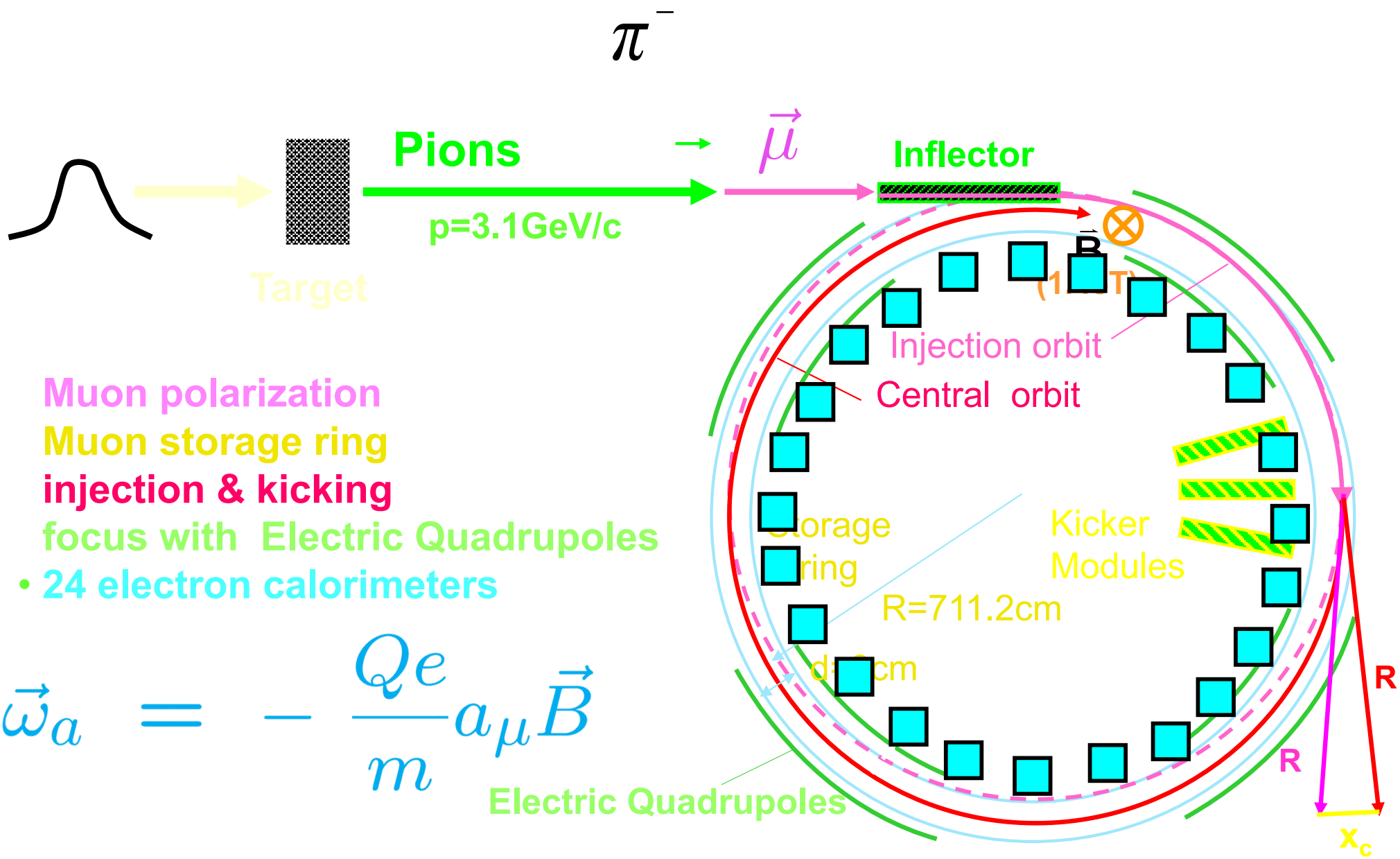


1）Spin is toward momentum:
→more e+ detected

2）Spin is opposite to momentum:
→less e+ detected

Experimental Technique

[Slide from L. Roberts]



Muon polarization
Muon storage ring
injection & kicking
focus with Electric Quadrupoles
• 24 electron calorimeters

$$\vec{\omega}_a = -\frac{Qe}{m} a_\mu \vec{B}$$

[T. Blum PRL91 (2003) 052001]

HVP from Lattice

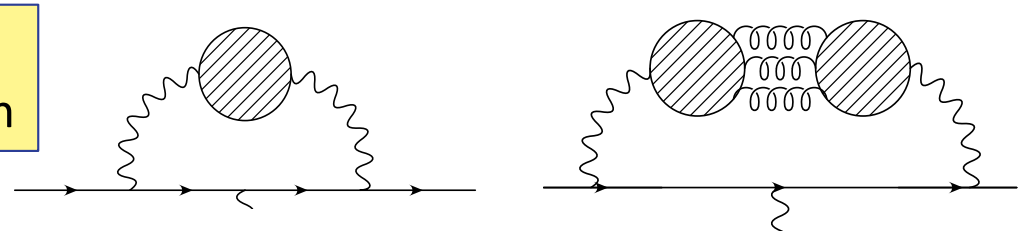
- Analytically continue to Euclidean/space-like momentum $K^2 = -q^2 > 0$
- Vector current 2pt function

$$a_\mu = \frac{g-2}{2} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2 f(K^2)\hat{\Pi}(K^2)$$

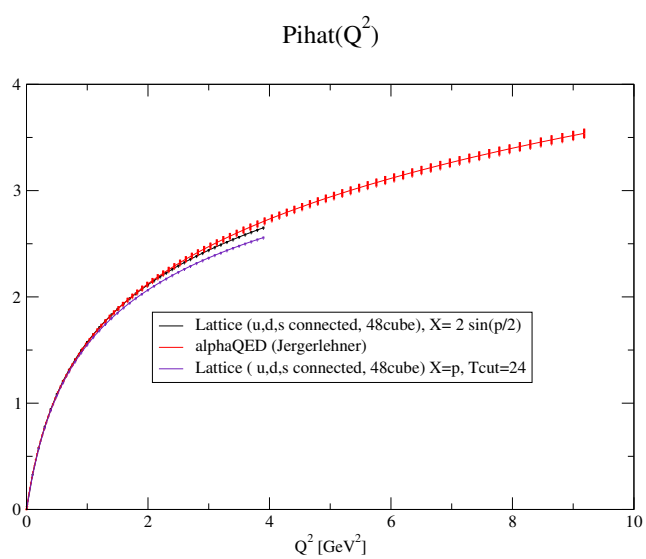
$$\Pi^{\mu\nu}(q) = \int d^4x e^{iqx} \langle J^\mu(x) J^\nu(0)\rangle$$

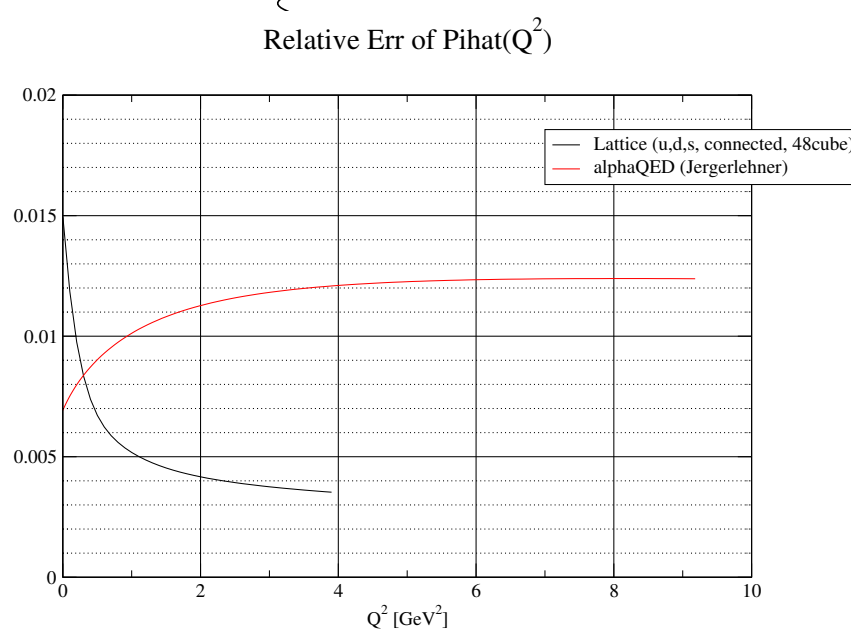
- Low Q2, or long distance, part of Π (Q2) is relevant for g-2

connected contribution



disconnected contribution





Current conservation, subtraction, and coordinate space representation

- Current conservation => transverse tensor

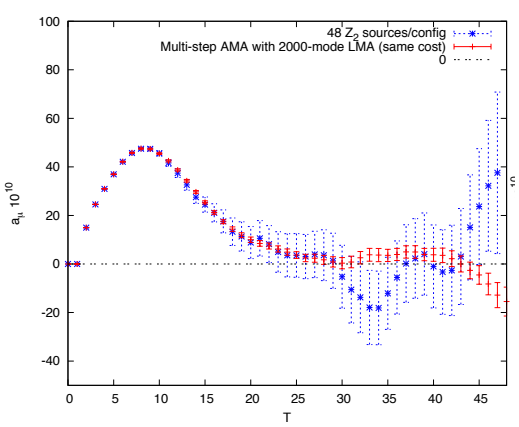
$$\sum_x e^{iQx}\langle J_\mu(x)J_\nu(0)\rangle = (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi(Q^2)$$

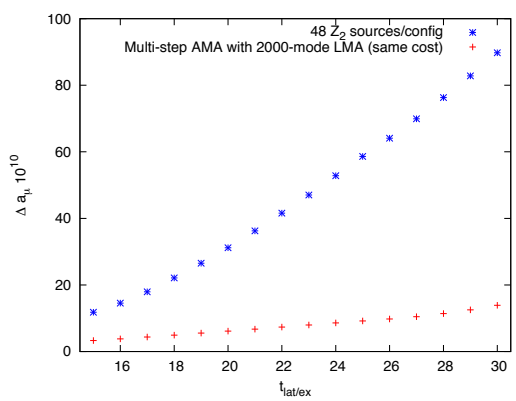
- Coordinate space vector 2 pt Green function C(t) is directly related to subtracted Π (Q2) [Bernecker-Meyer 2011, ...]

$$\Pi(Q^2) - \Pi(0) = \sum_t \left(\frac{\cos(qt)-1}{Q^2} + \frac{t^2}{2}\right) C(t)$$

- g-2 value is also related to C(t) with know kernel w(t) from QED.

$$a_\mu^{\mathrm{HVP}} = \sum_t w(t)C(t), \quad w(t) \propto t^4 \cdots$$





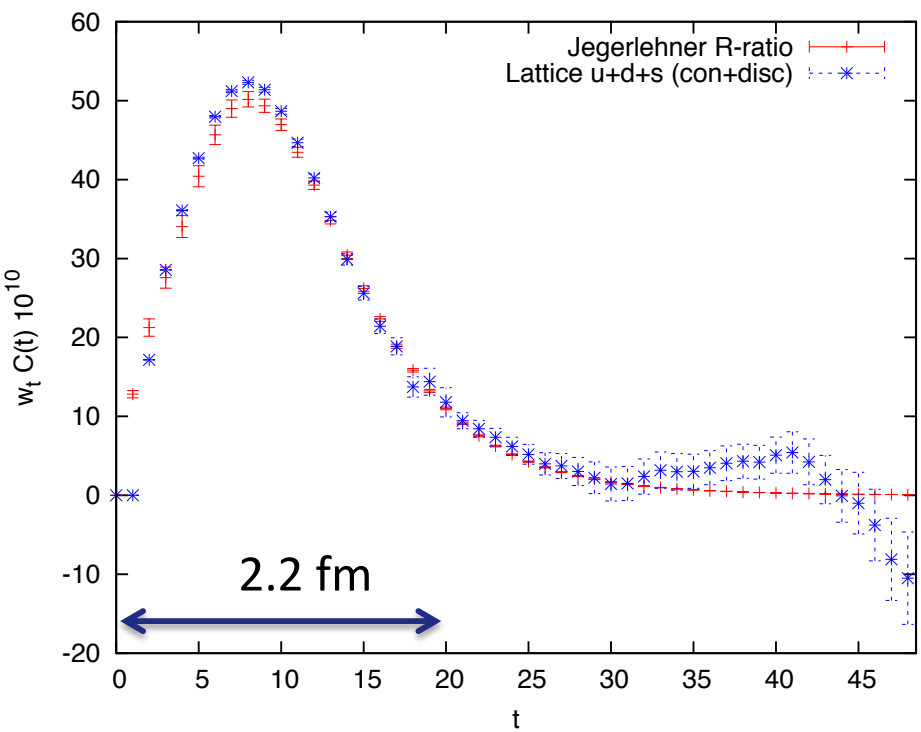
RBC/UKQCD
Chiral Lattice quark DWF
physical point
Quark Propagator Low Mode (A2A)
using All-Mode Averaging (AMA)

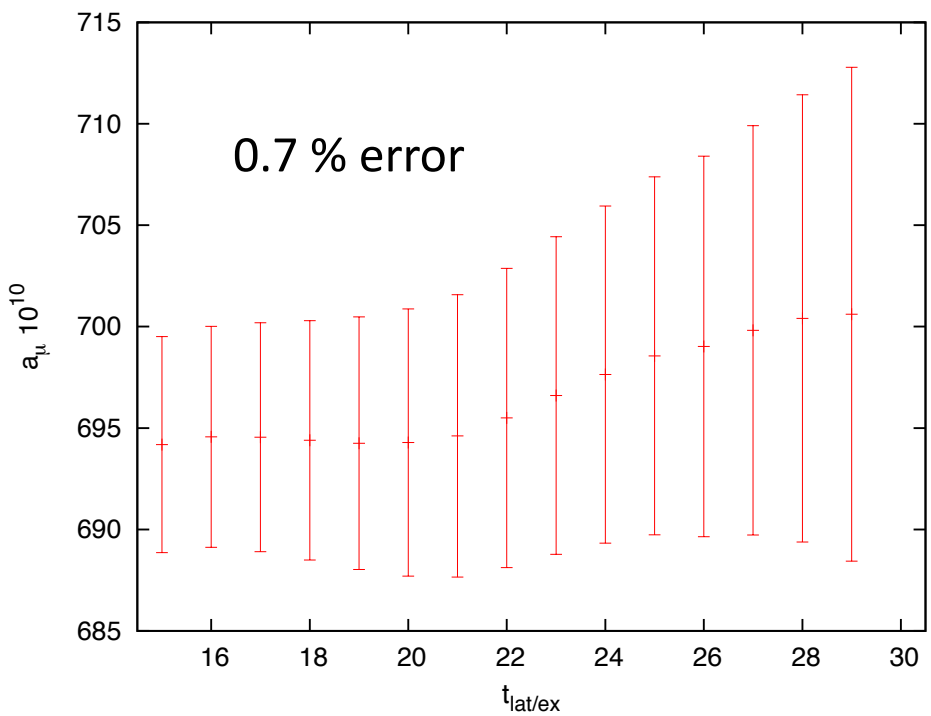
(plan B) Interplay between Lattice and Experiment

- Check consistency between Lattice and R-ratio
- Short distance from Lattice, Long distance from R-ratio :

  error <= 1 % at $t_{lat/exp}$ = 2fm

$$a_\mu^{\rm HVP} = \left[\sum_{t=0}^{t_{\rm lat/exp}} w(t)C(t)\right]^{\rm LAT} + \left[\int_{t_{\rm lat/exp}}^{\infty} dt\, w(t)C(t)\right]^{\rm EXP}$$





2016 : Disconnected, charm, QED, isospin breaking effects are being included
(RBC/UKQCD C. Lehner et al, also other collaborations)

Anomalous magnetic moment

- Fermion's energy in the external magnetic field:

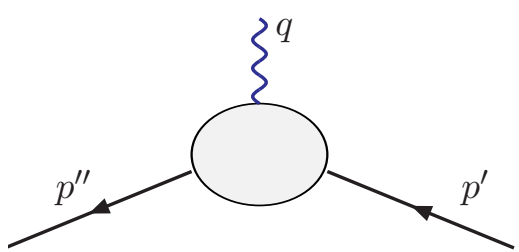
$$V(x) = -\vec{\mu}_l \cdot \vec{B}$$

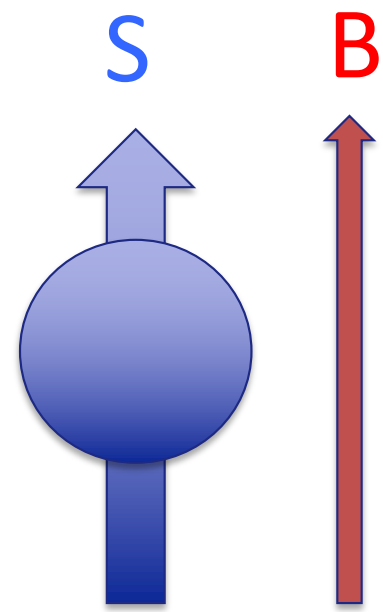
- Magnetic moment and spin g_l : Lande g-factor

g_l's deviation from tree level value, 2 :

$$\vec{\mu}_l = g_l \frac{e}{2m_l} \vec{S}_l \qquad a_l = \frac{g_l - 2}{2}$$

- Form factor : $\Gamma_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2\, m_l} \boxed{F_2(q^2)}$

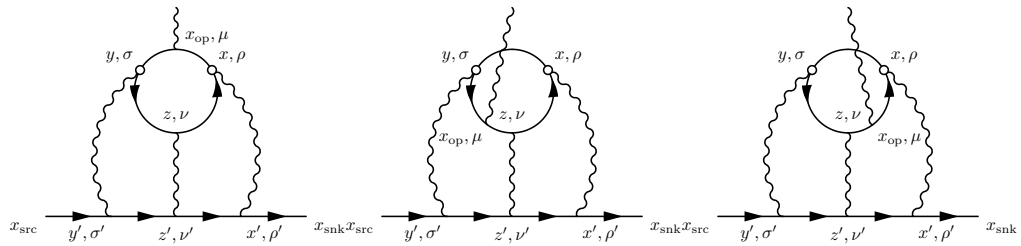




After quantum correction $\Rightarrow a_l = F_2(0)$

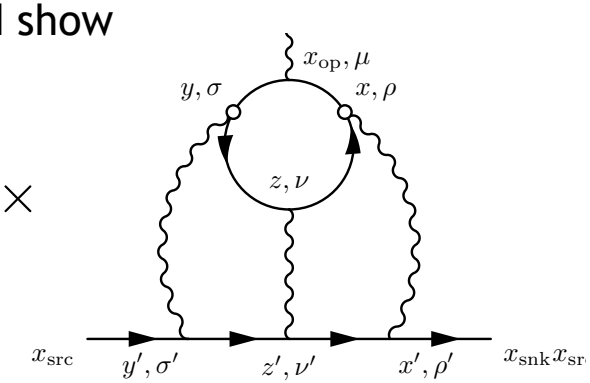
Conserved current & moment method

- [conserved current method at finite q2] To tame UV divergence, one of quark-photon vertex (external current) is set to be conserved current (other three are local currents). All possible insertion are made to realize conservation of external currents config-by-config.



- [moment method , q2→0] By exploiting the translational covariance for fixed external momentum of lepton and external EM field, q->0 limit value is directly computed via the first moment of the relative coordinate, xop – (x+y)/2, one could show

$$\frac{\partial}{\partial q_i}\mathcal{M}_\nu(\vec{q})|_{\vec{q}=0} = i \sum_{x,y,z,x_{\rm op}} (x_{\rm op} - (x+y)/2)_i \times$$



to directly get $F_2(0)$ without extrapolation.

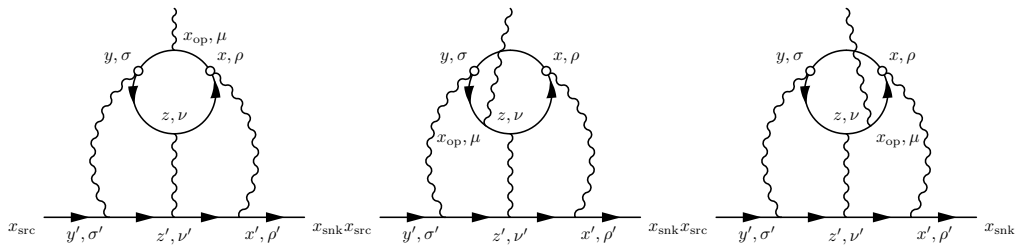
$$\text{Form factor}: \ \Gamma_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2\, m_l} F_2(q^2)$$

M_π=170 MeV cHLbL result (contd.)

"Exact" … q = 2pi / L,

"Conserved (current)" … q=2pi/L, 3 diagrams

"Mom" … moment method q->0, with AMA



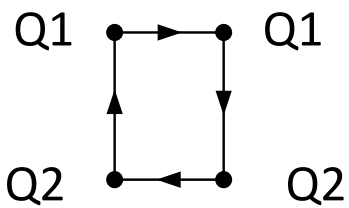
Method	$F_2/(\alpha/\pi)^3$	$N_{\rm conf}$	$N_{\rm prop}$	$\sqrt{\rm Var}$	$r_{\rm max}$	SD	LD	ind-pair
Exact	0.0693(218)	47	$58+8\times16$	2.04	3	−0.0152(17)	0.0845(218)	0.0186
Conserved	0.1022(137)	13	$(58+8\times16)\times7$	1.78	3	0.0637(34)	0.0385(114)	0.0093
Mom. (approx)	0.0994(29)	23	$(217+512)\times2\times4$	1.08	5	0.0791(18)	0.0203(26)	0.0028
Mom. (corr)	0.0060(43)	23	$(10+48)\times2\times4$	0.44	2	0.0024(6)	0.0036(44)	0.0045
Mom. (tot)	0.1054(54)	23						

Direct 4pt calculation for selected kinematical range

[J. Green et al. Mainz group, Phys. Rev. Lek 115, 222003(2015)]

- Compute connected contribution of 4 pt function in momentum space
- Forward amplitudes related to ▫*(Q1)▫*(Q2) -> hadron cross section via dispersion relation

$$\mathcal{M}_{\rm had}\left(\gamma^*(Q_1)\gamma^*(Q_2) \to \gamma^*(Q_1)\gamma^*(Q_2)\right)$$
$$\leftrightarrow \quad \sigma_{0,2}\left(\gamma^*(Q_1)\gamma^*(Q_2) \to {\rm had.}\right)$$



- solid curve: model prediction
- π0 exchange is seen to be not dominant, possibly due to heavy quark mass in the simulation (Mπ = 324 MeV)
- disconnected quark diagram loop in progress in 2016

$$\nu = -Q_1 \cdot Q_2$$

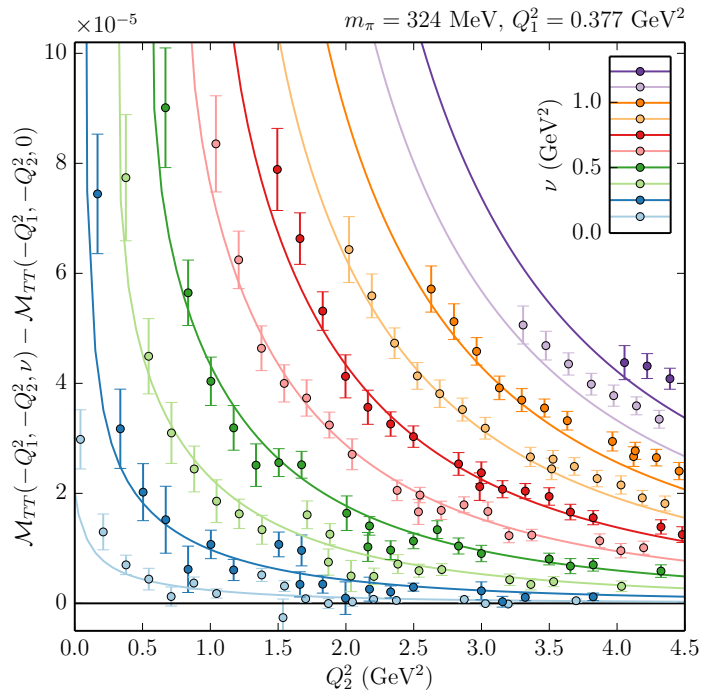


FIG. 3. The forward scattering amplitude $\mathcal{M}_{\rm TT}$ at a fixed virtuality $Q_1^2 = 0.377{\rm GeV}^2$, as a function of the other photon virtuality Q_2^2, for different values of ν. The curves represent the predictions based on Eq. (10), see the text for details.

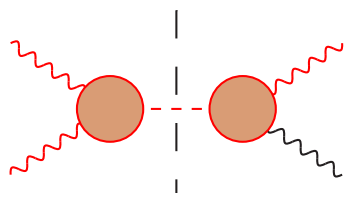
Dispersive approach for HLbL

[Colangelo et al. 2014, 2015, Pauk&Vanderhaeghen 2014]

- Using crossing symmetry, gauge invariance, 138 form factors are reduced 12 relevant for HLbL

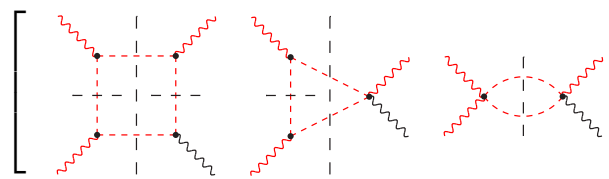
$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{1}{q_1^2 q_2^2 (q_1+q_2)^2}\frac{1}{(p+q_1)^2-m_\mu^2}\frac{1}{(p-q_2)^2-m_\mu^2}$$
$$\times \sum_{j=1}^{12} \xi_j \hat{T}_{i_j}(q_1,q_2;p)\hat{\Pi}_{i_j}(q_1,q_2,-q_1-q_2),$$

- π0, ▫,▫’ exchange, pion-loop (exactly scalar QED with pion Form factor)



$$= \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)F_{\pi^0\gamma^*\gamma^*}(q_3^2,q_4^2)}{s-M_\pi^2}$$

$$\Pi^{\mathrm{FsQED}}_{\mu\nu\lambda\sigma} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2) \times \Big[$$



$$\Big]$$

- other contribution is neglected

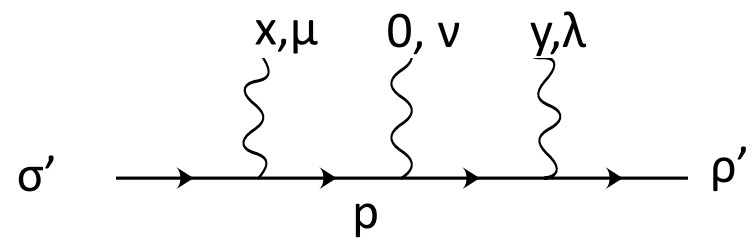
Continuum Infinite Volume (a.k.a HVP way)

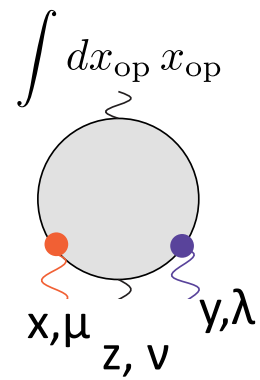
$$a_\mu^{\rm HVP} = \sum_t w(t) C(t), \quad w(t) \propto t^4 \cdots$$

- One could also use infinite volume/continuum lepton&photon diagram in coordinate space

  [J. Green et al. Mainz group, LAT16 proceedings]

$$\mathcal{L}_{\mu\nu\lambda\sigma\rho}(x, y; p)$$





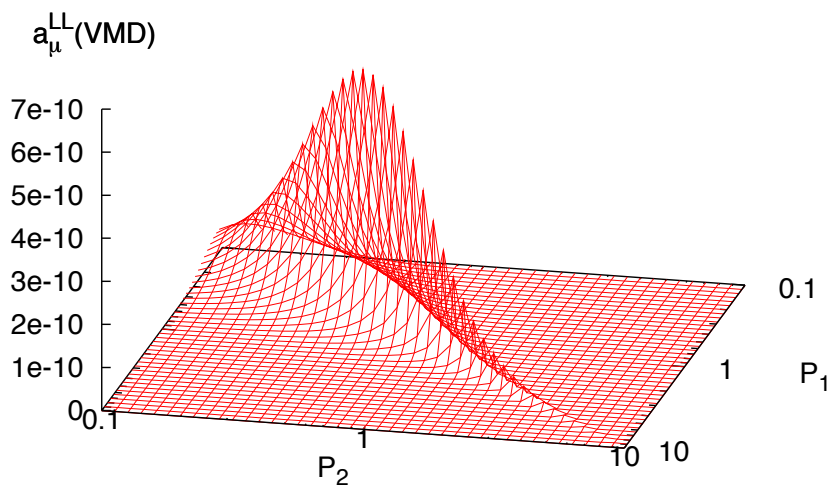
- Techniques in continuum model calculation [Knect Nyffeler 2002; Jegerlehner Nyffeler 2009] : angle average over muon momentum, and carry out angle of two virtual photons

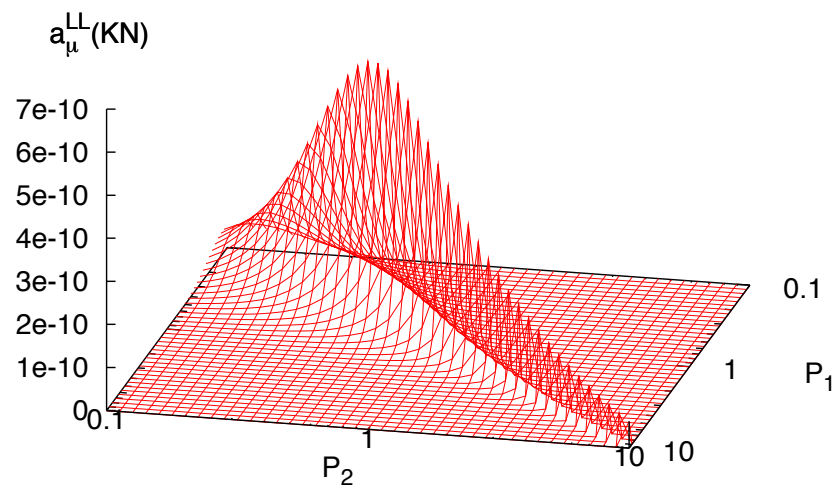
$$L(x_1, x_2) = \sum_{m,l} \sum_{\substack{k=|l-m| \\ \text{step}=2}}^{l+m} (-1)^k C_k(\hat{x}_1 \hat{x}_2)$$
$$\times \int dQ_1 dQ_2 \frac{4 Z_1 Z_2}{m^2 Q_1 Q_2 X_1 X_2} \frac{(-Z_1 Z_2)^l}{l+1} J_{k+1}(Q_1 X_1) J_{k+1}(Q_2 X_2)$$
$$\times \left[\frac{\theta(1 - Q_2/Q_1)}{Q_1^2} \left(\frac{Q_2}{Q_1}\right)^m + \frac{\theta(1 - Q_1/Q_2)}{Q_2^2} \left(\frac{Q_1}{Q_2}\right)^m \right]$$

Can Lattice produce a counter part ?

[J. Bijnens]

- Which momentum regimes important studied: JB and J. Prades, Mod. Phys. Lett. A **22** (2007) 767 [hep-ph/0702170]
- $a_\mu = \int dl_1 dl_2 a_\mu^{LL}$ with $l_i = \log(P_i/GeV)$





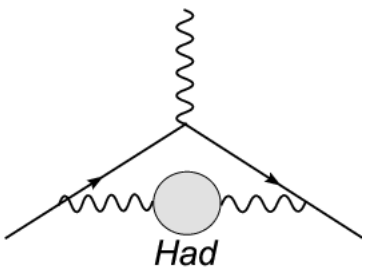
Which momentum regions do what:

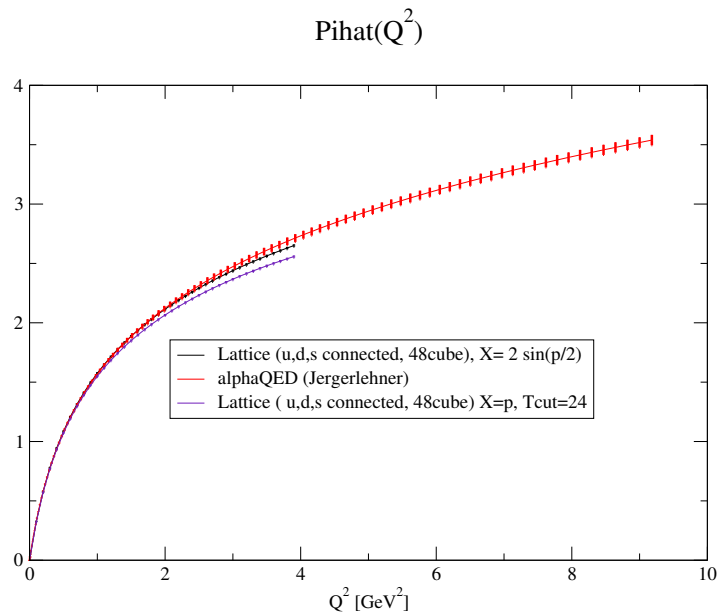
volume under the plot $\propto a_\mu$

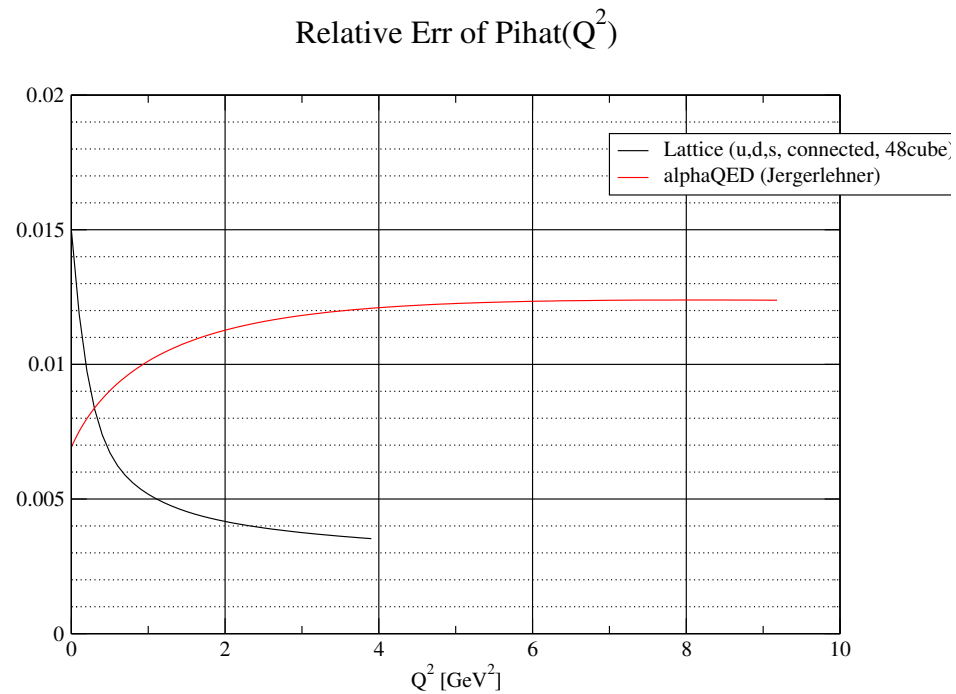
(plan B) Interplays between lattice and dispersive approach g-2

- R-Ratio error ~ 0.6%, HPQCD error ~ 2%
- Goal would be ~ 0.2 %
- Dispersive approach from R-ratio R(s)

$$\hat{\Pi}(Q^2) = \frac{Q^2}{3} \int_{s_0} ds \frac{R(s)}{s(s+Q^2)}$$







also [ETMC, Mainz, ...]

- Can we combine dispersive & lattice and get more precise (g-2)HVP than both ? [2011 Bernecker Meyer]
- Inverse Fourier trans to Euclidean vector correlator
- Relevant for g-2 $Q^2 = (m_\mu/2)^2 = 0.0025\ \mathrm{GeV}^2$
- It may be interesting to think

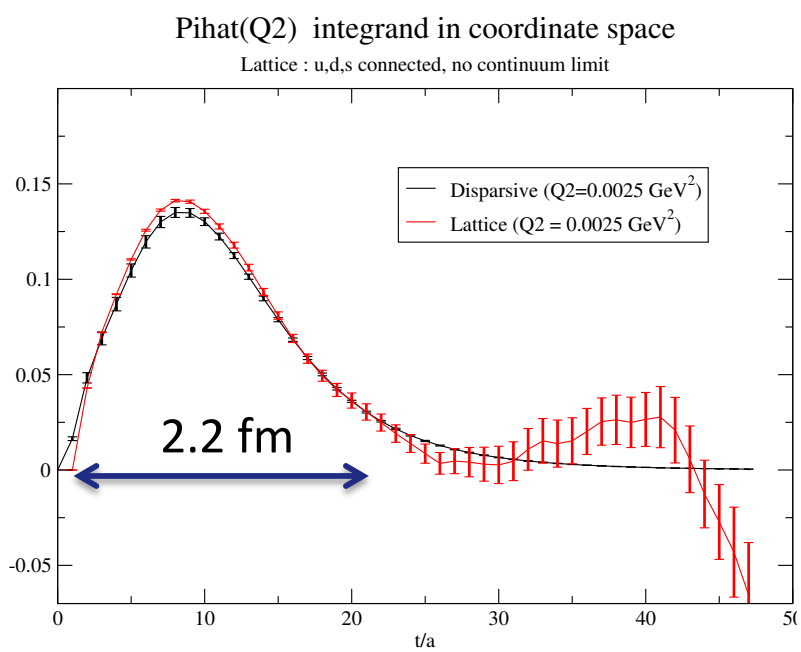
$$\frac{\hat{\Pi}(Q^2)}{Q^2} = \left[\frac{\hat{\Pi}(Q^2)}{Q^2} - \frac{\hat{\Pi}(P^2)}{P^2}\right]^{\mathrm{Exp}} + \left[\frac{\hat{\Pi}(P^2)}{P^2}\right]^{\mathrm{Lat}}$$

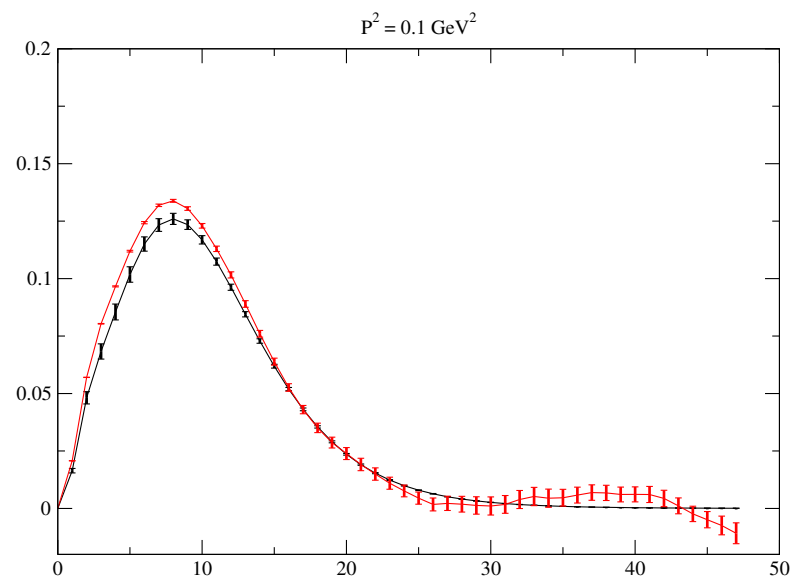
$$a_\mu^{\mathrm{HVP}} = \sum_t w(t)C(t), \quad w(t) \propto t^4 \cdots$$

$$\hat{\Pi}(Q^2) = \frac{Q^2}{3}\int_{s_0} ds \frac{R(s)}{s(s+Q^2)}$$

Black : R-ratio , alpha QED (Jegerlehner)

Red : Lattice (DWF)



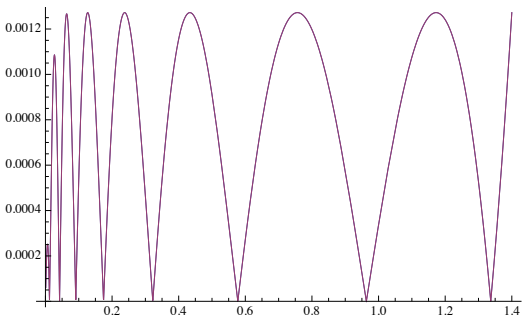


AMA+MADWF(fastPV)+zMobius accelerations

- We utilize complexified 5d hopping term of Mobius action [Brower, Neff, Orginos], zMobius, for a better approximation of the sign function.

$$\epsilon_L(h_M) = \frac{\prod_s^L(1+\omega_s^{-1}h_M) - \prod_s^L(1-\omega_s^{-1}h_M)}{\prod_s^L(1+\omega_s^{-1}h_M) + \prod_s^L(1-\omega_s^{-1}h_M)}, \qquad \omega_s^{-1} = b + c \in \mathbb{C}$$

- 1/a~2 GeV, Ls=48 Shamir ~ Ls=24 Mobius (b=1.5, c=0.5) ~ Ls=10 zMobius (b_s, c_s complex varying) ~5 times saving for cost AND memory



Ls	|eps(48cube) – eps(zMobius)|
6	0.0124
8	0.00127
10	0.000110
12	8.05e-6

- The even/odd preconditioning is optimized (sym2 precondition) to suppress the growth of condition number due to order of magnitudes hierarchy of b_s, c_s [also Neff found this]

$$\texttt{sym2}: \quad 1 - \kappa_b M_4 M_5^{-1} \kappa_b M_4 M_5^{-1}$$

- Fast Pauli Villars (mf=1) solve, needed for the exact solve of AMA via MADWF (Yin, Mawhinney) is speed up by a factor of 4 or more by Fourier acceleration in 5D [Edward, Heller]
- All in all, sloppy solve compared to the traditional CG is 160 times faster on the physical point 48 cube case. And ~100 and 200 times for the 32 cube, Mpi=170 MeV, 140, in this proposal (1,200 eigenV for 32cube) .

$$\underbrace{\frac{20,000}{600}}_{\text{MADWF+zMobius+deflation}} \times \underbrace{\frac{600 * 32/10}{300}}_{\text{AMA+zMobius}} = 33.3 \times 6.4 = \underline{210 \text{ times faster}}$$

[Blum, TI, Shintani PRD 88 (2013) 094503]

Covariant Approximation Averaging (CAA) a new class of Error reduction techniques

Original

$$\mathcal{O} = \mathcal{O}^{(\mathrm{appx})} + \mathcal{O}^{(\mathrm{rest})}$$

unbiased imporved

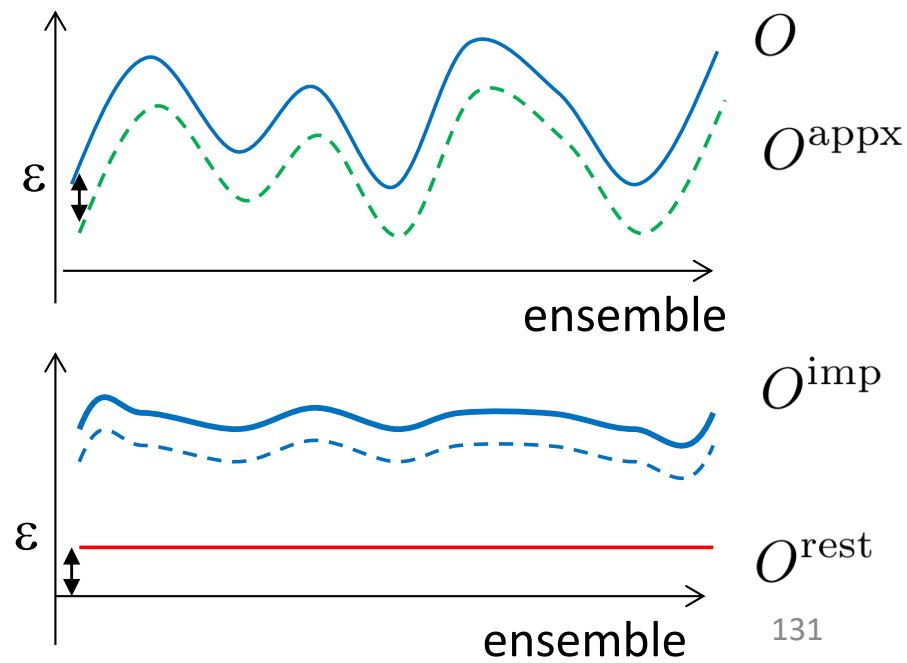
Lattice Symmetry

$$\mathcal{O}^{(\mathrm{imp})} = \mathcal{O}^{(\mathrm{rest})} + \frac{1}{N_G} \sum_{g \in G} \mathcal{O}^{(\mathrm{appx}),g}$$

Expensive : infrequently measured

Cheap : frequently measured

- $O^{(\mathrm{imp})}$ has smaller error
  $O^{(\mathrm{appx})}$ need to be cheap & not to be too accurate
  N_G suppresses the bulk part of noise cheaply



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Examples of Covariant Approximations (contd.)

- All Mode Averaging AMA

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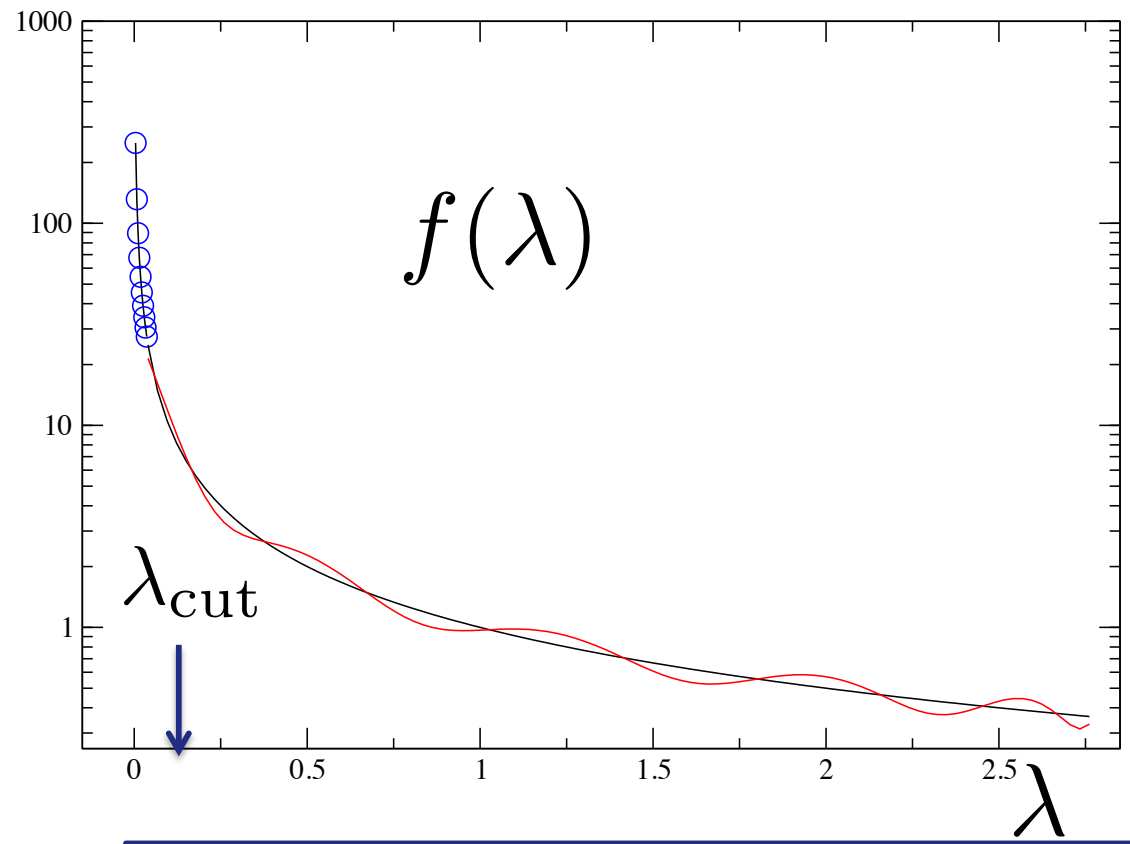


If quark mass is heavy, e.g. ~ strange, low mode isolation may be unneccesary

accuracy control :

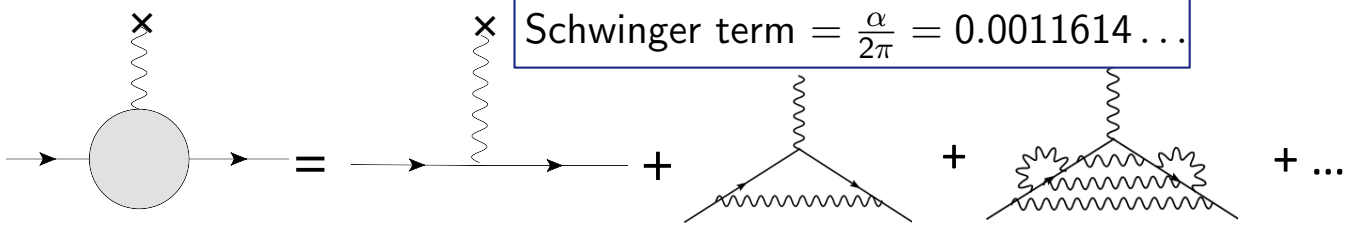
- low mode part : # of eig-mode
- mid-high mode : degree of poly.

SM Theory

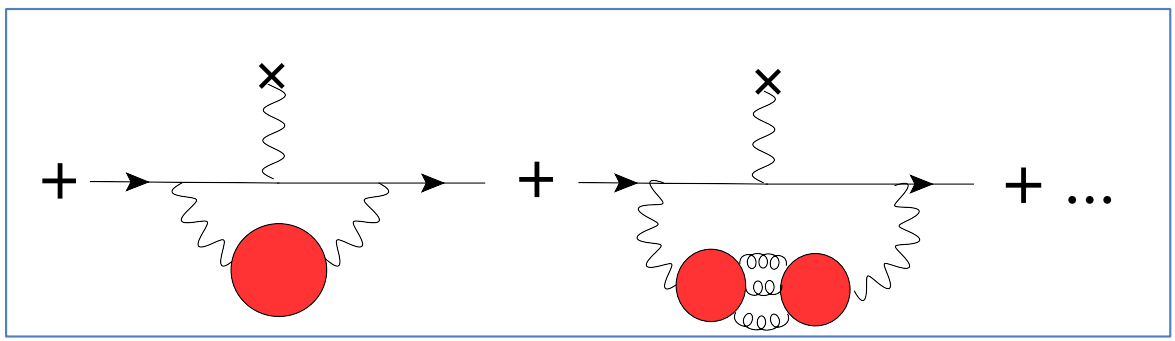
$$\gamma^\mu \;\rightarrow\; \Gamma^\mu(q) = \left(\gamma^\mu F_1(q^2) + \frac{i\,\sigma^{\mu\nu}\,q_\nu}{2m} F_2(q^2)\right)$$

- QED, hadronic, EW contributions

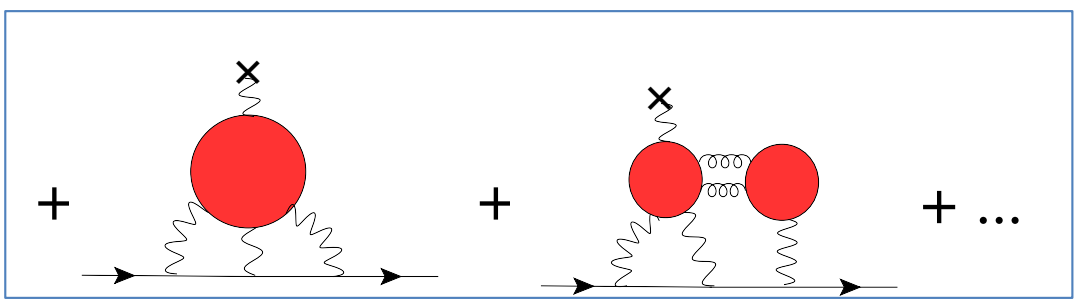
Schwinger term $= \frac{\alpha}{2\pi} = 0.0011614\ldots$



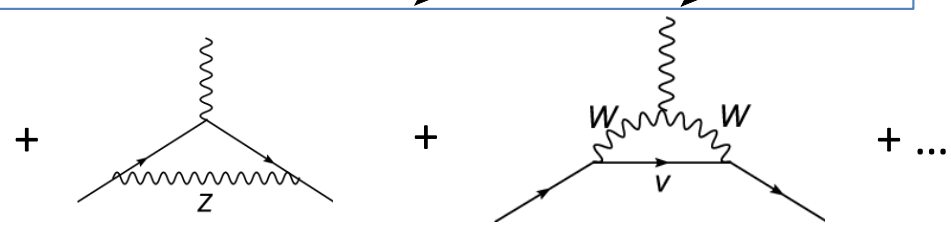
QED (5-loop)
Aoyama et al.
PRL109,111808 (2012)



Hadronic vacuum polarization (HVP)



Hadronic light-by-light (Hlbl)

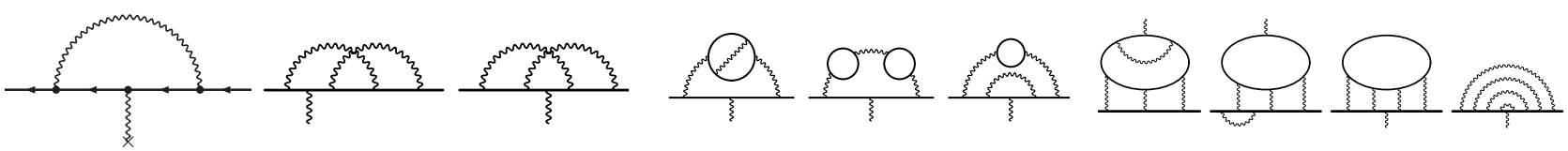


Electroweak (EW)
Knecht et al 02
Czarnecki et al. 02

QED calculations

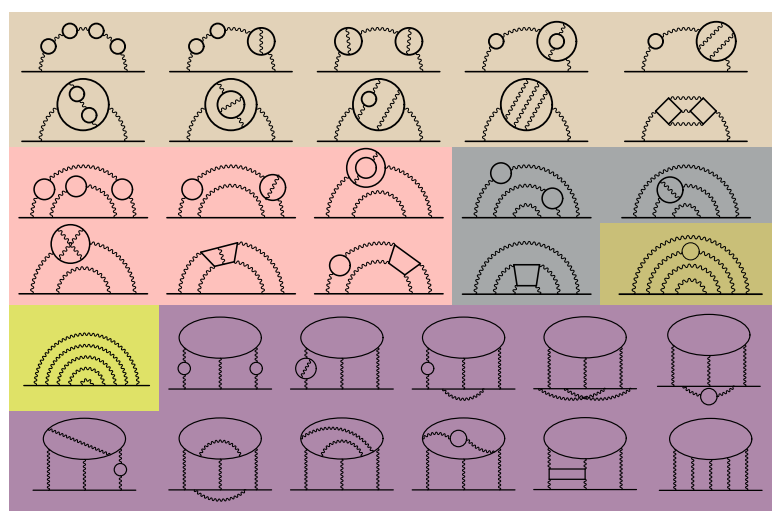
- Fine structure constant □
  Experimental input : anomalous magnetic moment of Electron
  a_e = 0.001 159 652 180 73(28) [0.24 ppb]
  [Hanneke, Fogwell Hoogerheide, Gabrielse, PRA83, 052122 (2011)]

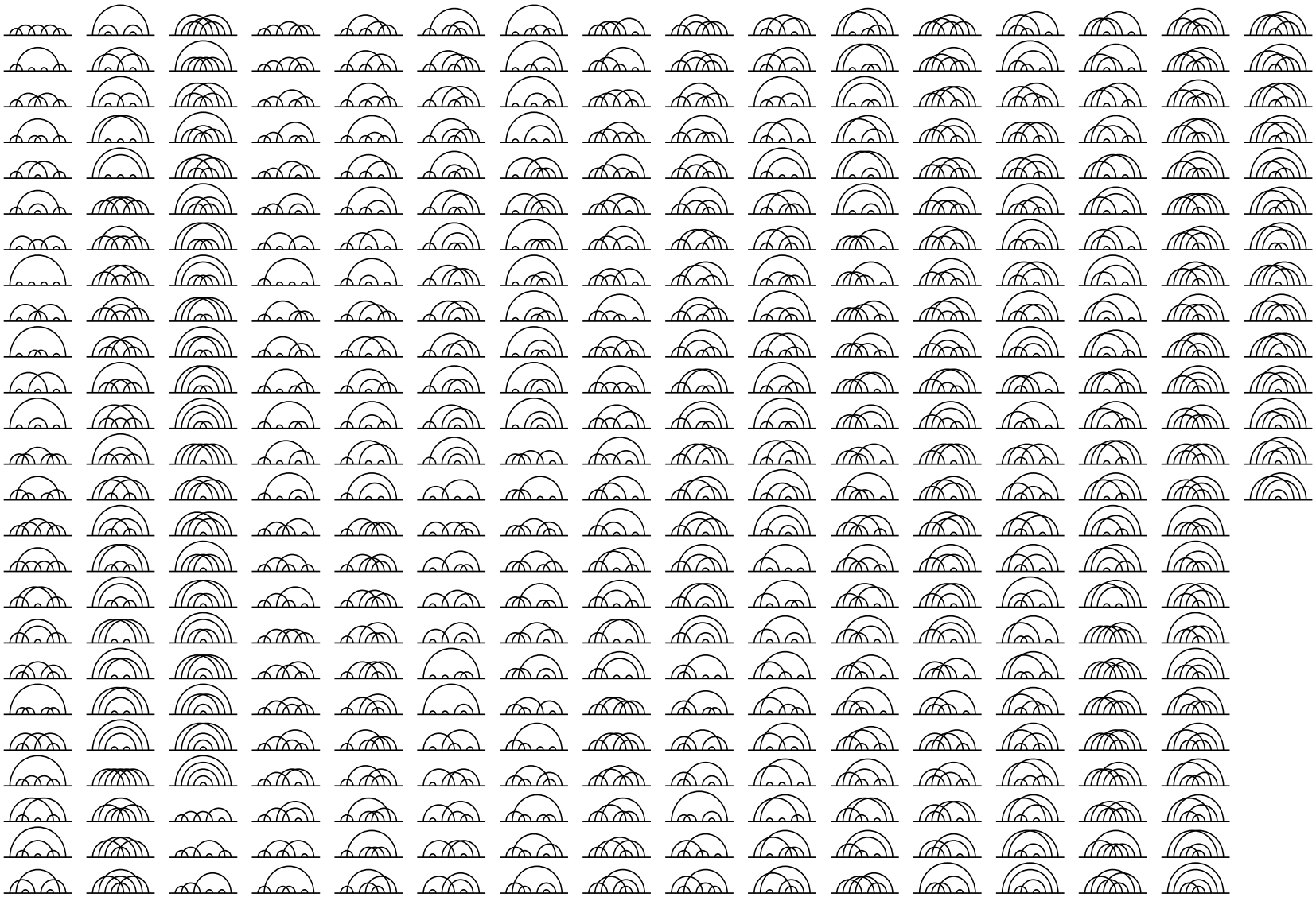
  Theory input: 10th order QED calculation (+ small had+EW)
  [Aoyama, Hayakawa, Kinoshita, Nio Phys. Rev. D 91, 033006 (2015)]
  $\square^{-1}$ = 137.035 999 1570 (334) [0.25 ppb]



$$\text{Schwinger term} = \frac{\alpha}{2\pi} = 0.0011614\ldots$$

- 1+7+72+891+12,672
  more than 13,000 diagrams !





$(g\text{-}2)_\mu$ SM Theory vs experiment

- QED, EW, Hadronic contributions

K. Hagiwara et al. , J. Phys. G: Nucl. Part. Phys. 38 (2011) 085003

$$a_\mu^{\rm SM} = (11\ 659\ 182.8\ \pm 4.9\ ) \times 10^{-10}$$

$$a_\mu^{\rm QED} = (11\ 658\ 471.808\ \pm 0.015\ ) \times 10^{-10}$$
$$a_\mu^{\rm EW} = (\ 15.4\ \pm 0.2\ ) \times 10^{-10}$$
$$a_\mu^{\rm had,LOVP} = (\ 694.91\ \pm 4.27\ ) \times 10^{-10}$$
$$a_\mu^{\rm had,HOVP} = (\ -9.84\ \pm 0.07\ ) \times 10^{-10}$$
$$a_\mu^{\rm had,lbl} = (\ 10.5\ \pm 2.6\ ) \times 10^{-10}$$

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 28.8(6.3)_{\rm exp}(4.9)_{\rm SM} \times 10^{-10} \quad [3.6\sigma]$$

- Discrepancy between EXP and SM is larger than EW!
- Currently the dominant uncertainty comes from HVP, followed by HLbL
- x4 or more accurate experiment FNAL , J-PARC
- Our Goal : sub 1% accuracy for HVP, and
  → **10% accuracy for HLbL**

G-2 from BSM sources

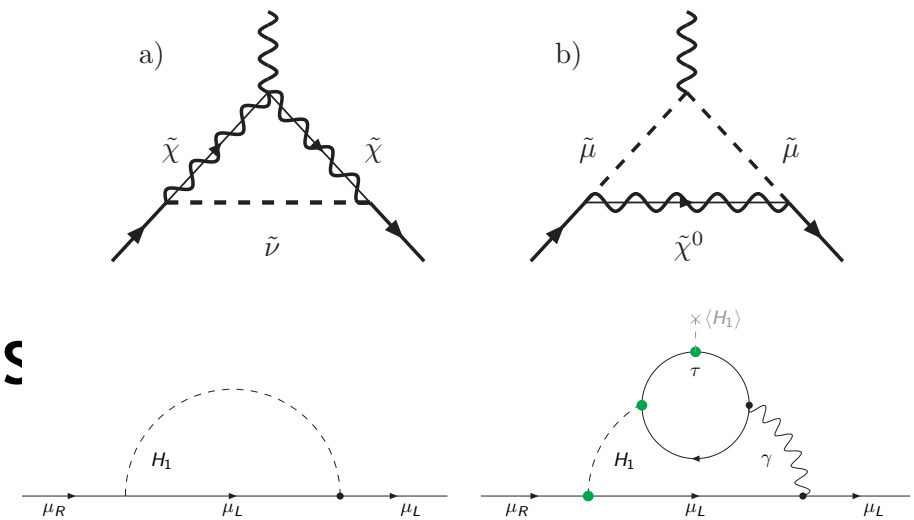
- Typical new particle contribute g-2

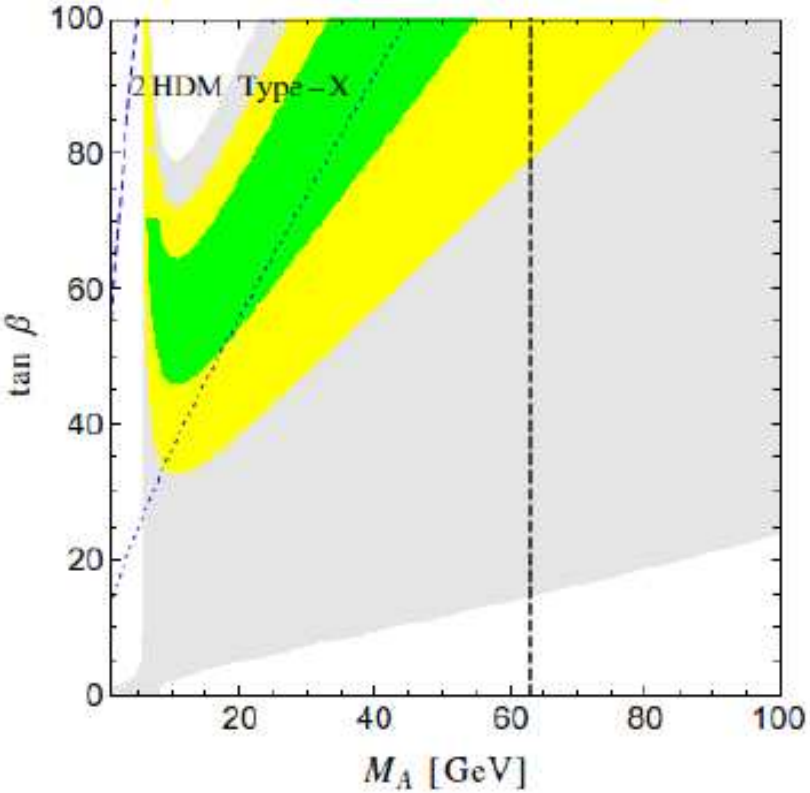
  g-2 ~ C $(m_\mu / m_{NP})^2$
- To explain current discrepancy

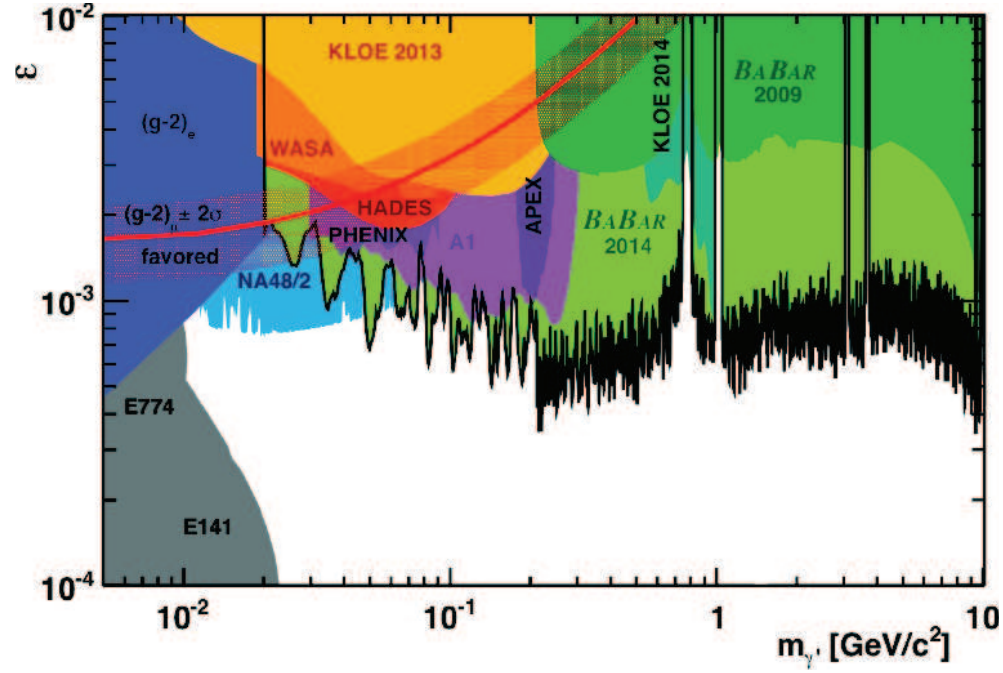
$\mathcal{C}$	1	$\frac{\alpha}{\pi}$	$(\frac{\alpha}{\pi})^2$
M_{NP}	$2.0^{+0.4}_{-0.3}$ TeV	100^{+21}_{-13} GeV	5^{+1}_{-1} GeV

- SUSY (scalar-lepton)
- 2 Higgs doublet models
  Type-X, ….
- Dark photons
  from kinematical mixings
  - □ $F_{\mu\square} F'_{\mu\square}$

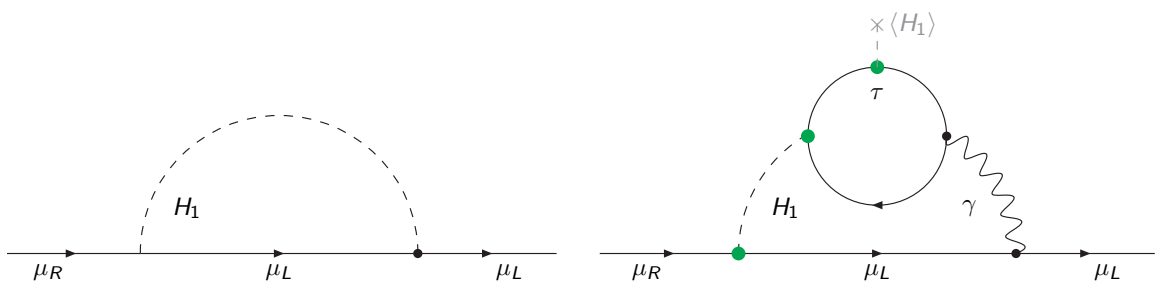
[A. Nyfler]







From: F. Curciarello, FCCP15, Capri, September 2015

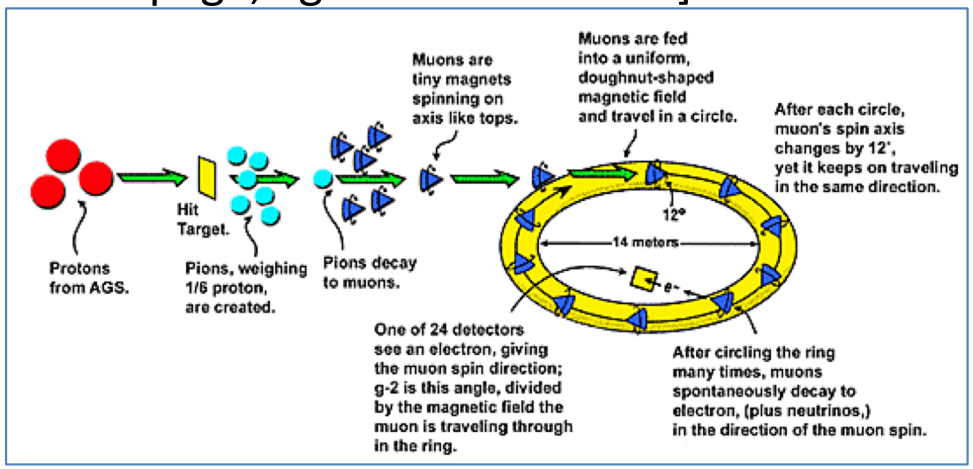


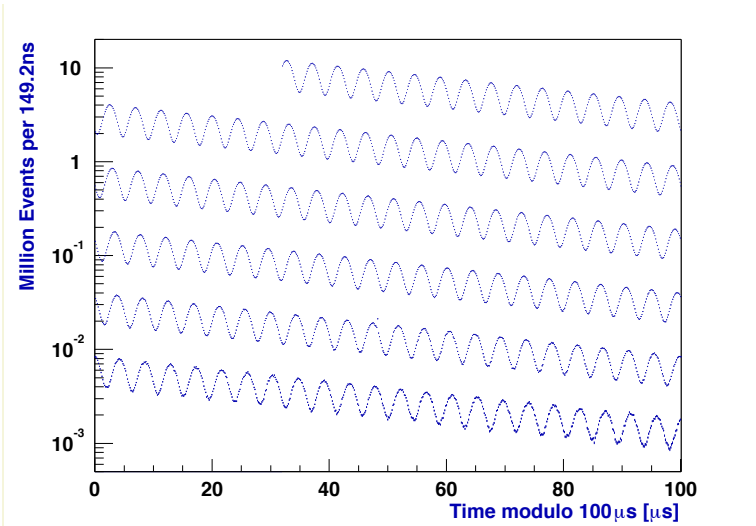
The Muon g-2 experiments
BNL E821 (-2004)

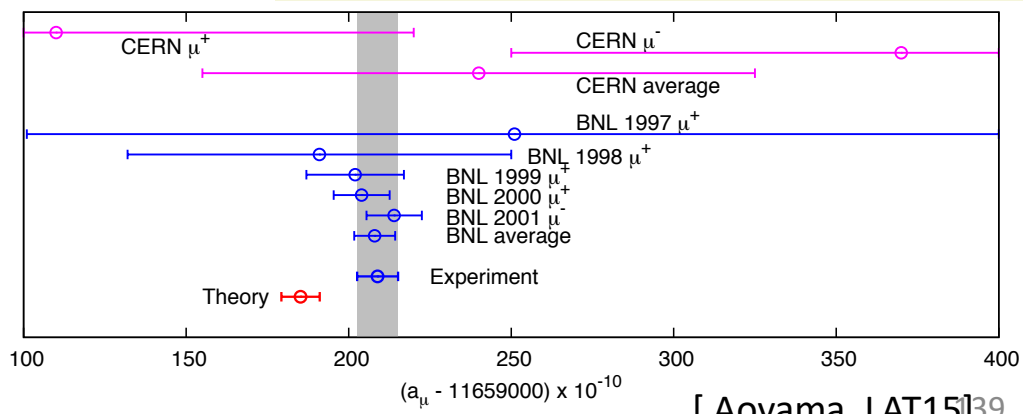
- measure precession of muon spin very accurately

$$N(t) = N_0(E)\ \exp\left(-t/\gamma\tau_\mu\right)\ [1 + A(E)\ \sin(\omega_a\, t + \phi(E))]$$

[BNL web page, g-2 collaboration]



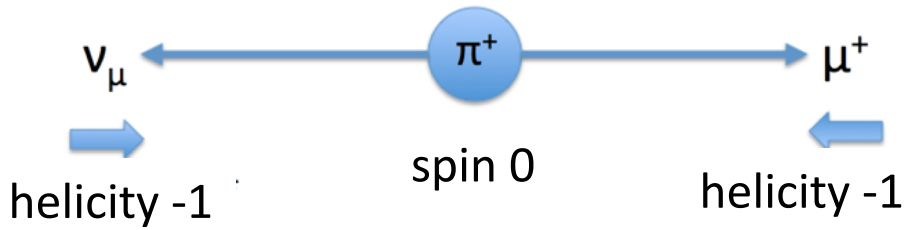




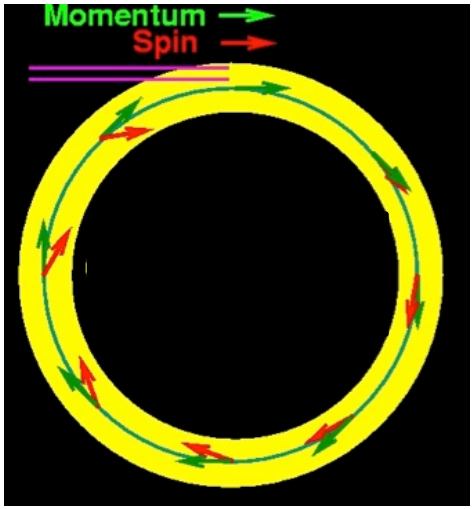
[Aoyama, LAT15]

Recipe of a g-2 measurement

1. Prepare a polarized muon beam from P-violating pion decay



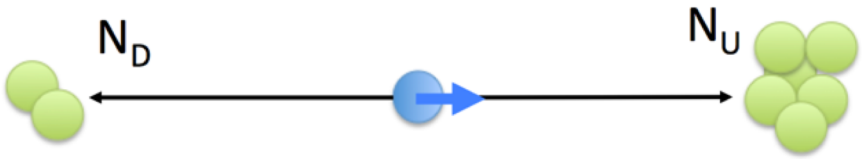
1. Store in a magnetic field (let muon spin precessed)



$$\vec{\omega} = -\frac{e}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\frac{\vec{\beta}\times\vec{E}}{c} + \frac{\eta}{2}\left(\vec{\beta}\times\vec{B} + \frac{\vec{E}}{c}\right)\right]$$

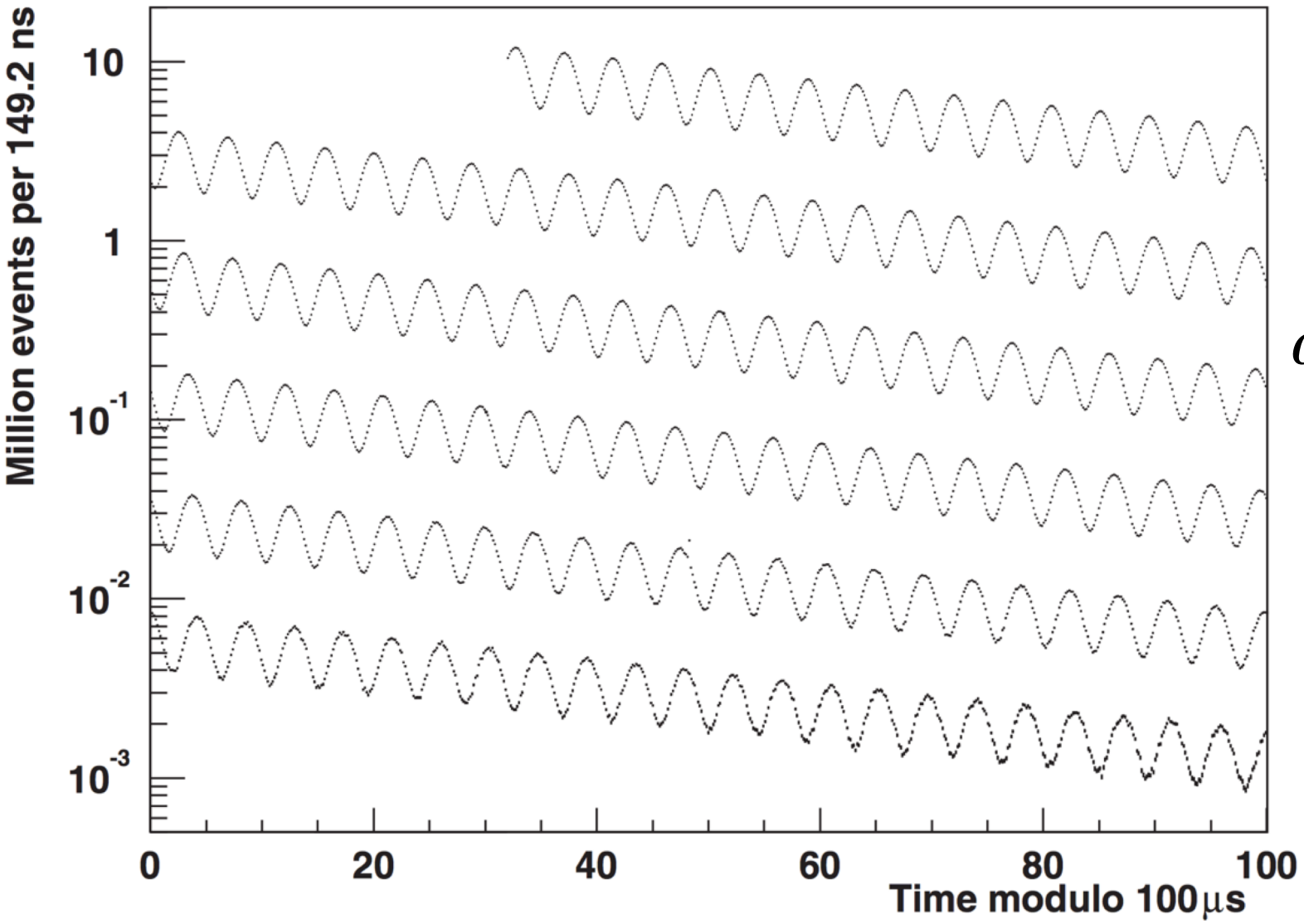
Magic momentum, γ=30 (p= 3 GeV/c),

2. Measure positron from P-violating muon decay



[Slide from T. Mibe, L. Roberts]

Positron time spectrum in BNL E821

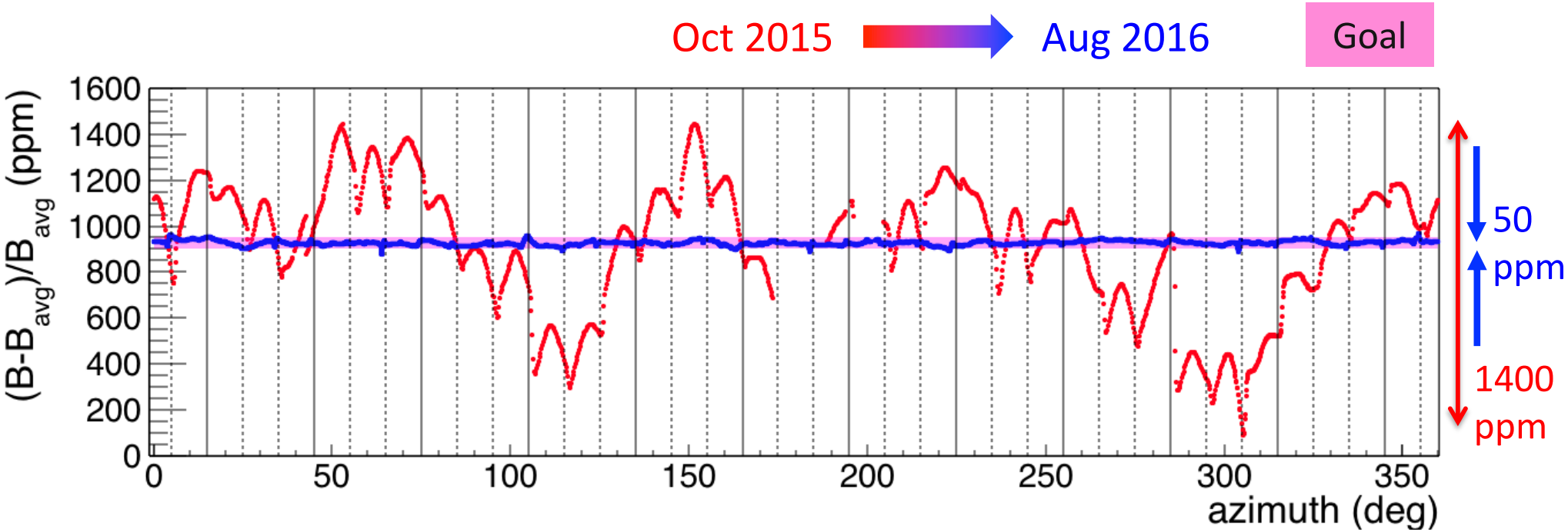


$$\vec{\omega}_a = -\frac{e}{m} a_\mu \vec{B}$$

Slide by P. Winter (ANL)

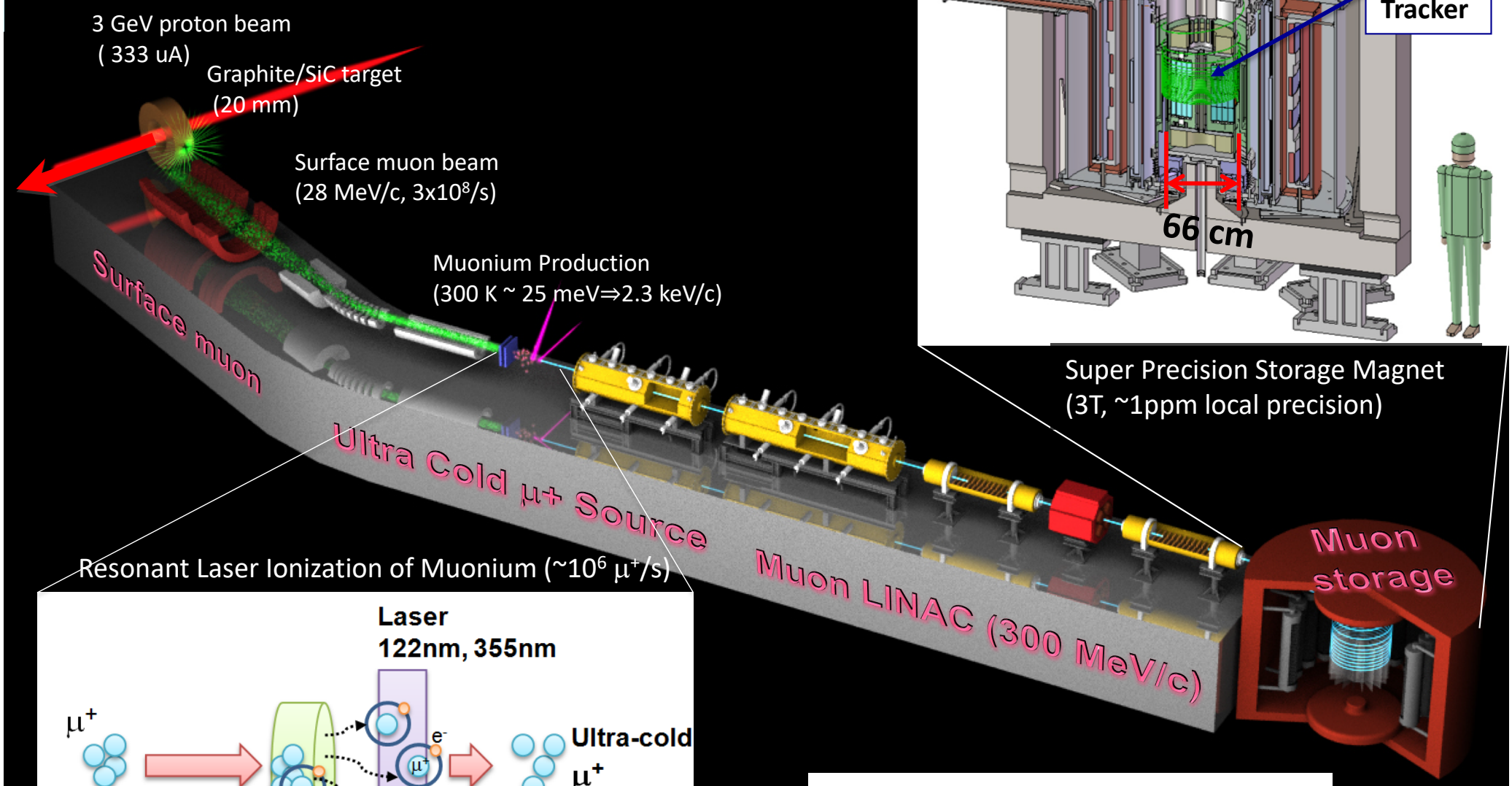
Shimming successfully completed in2016

- 10 months of align and optimize our shim knobs:
  - 72 pole pieces
  - 800 wedge shims
  - 9000 iron shim foils
  - ...

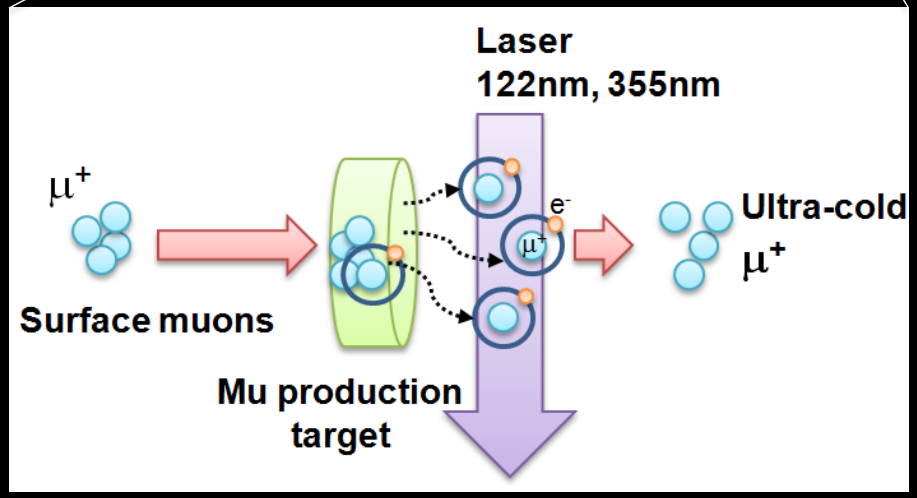


Shimming goal achieved with $\Delta B < \pm 25$ ppm ✓

New Muon g-2/EDM Experiment at J-PARC with Ultra-Cold Muon Beam

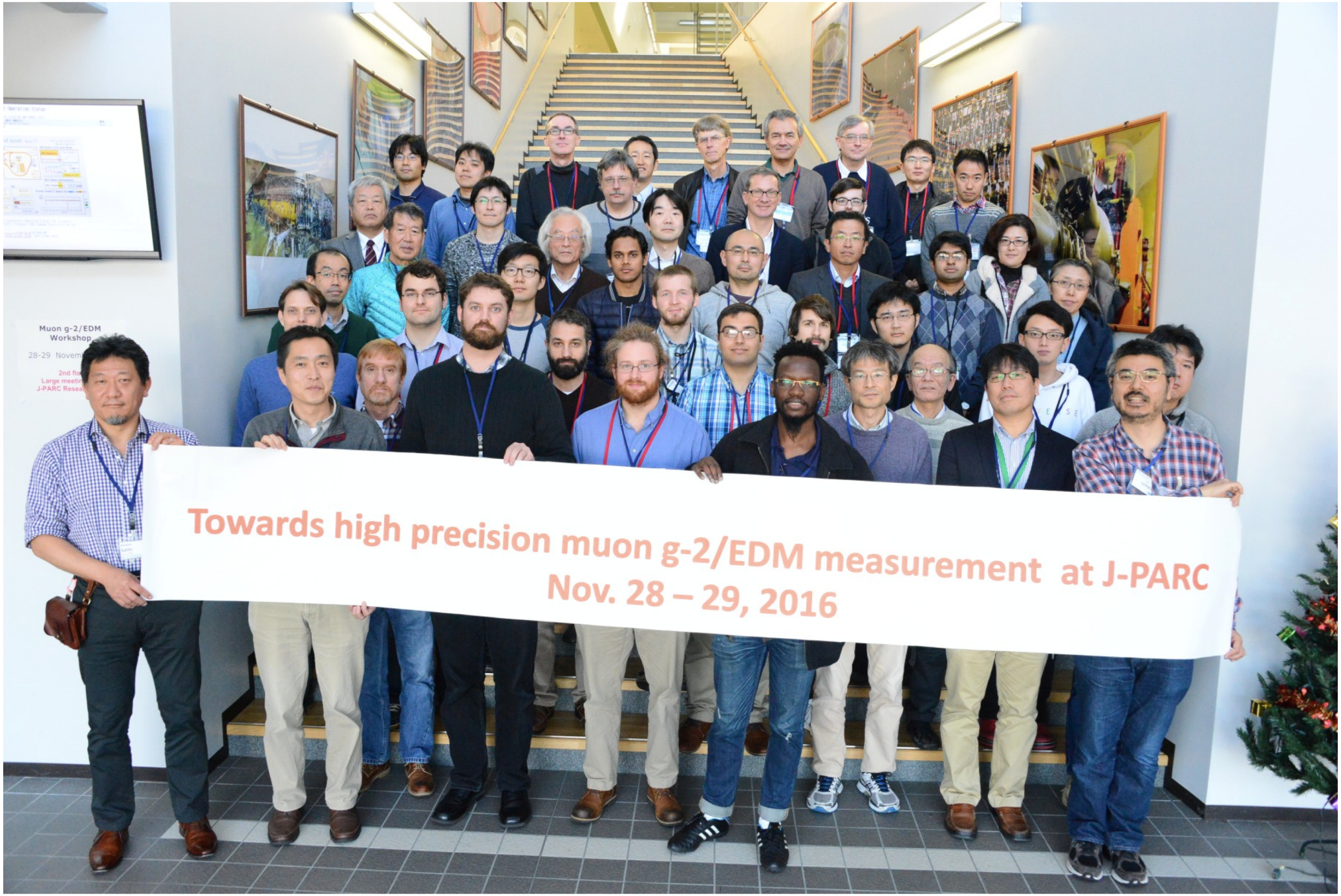


Resonant Laser Ionization of Muonium (~10^6 μ^+/s)



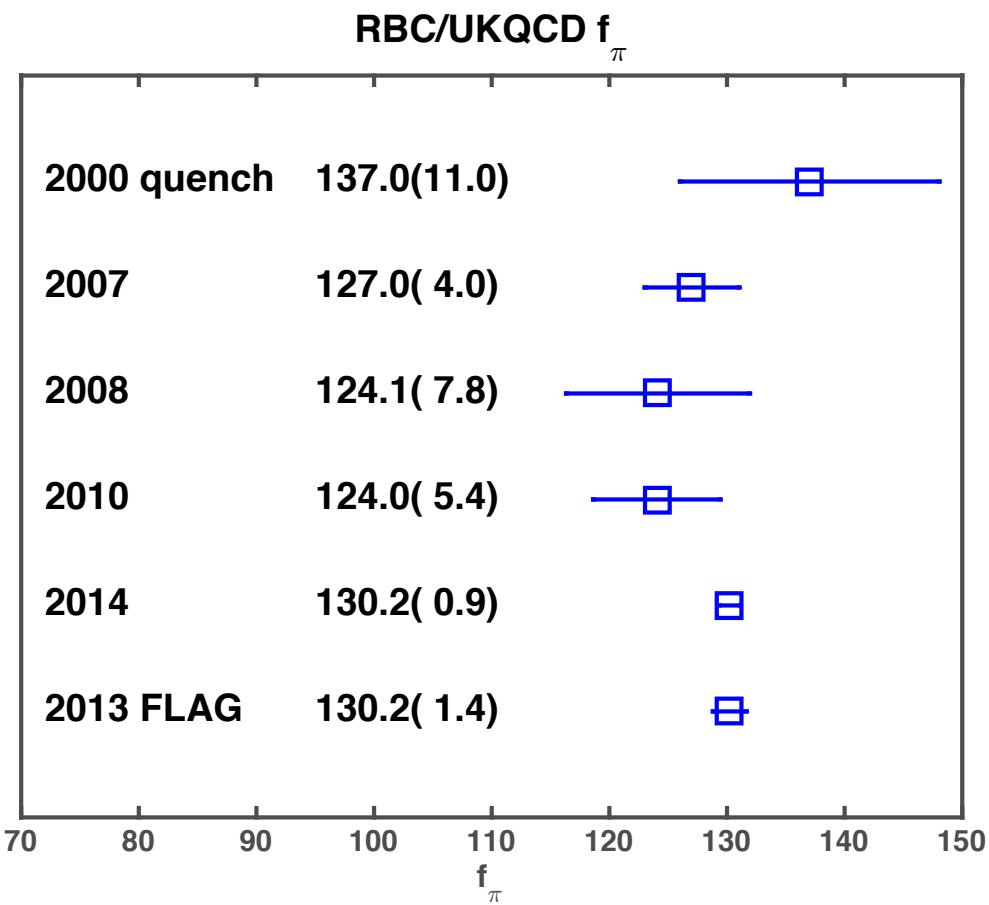
$\Delta(g-2) = 0.1$ ppm

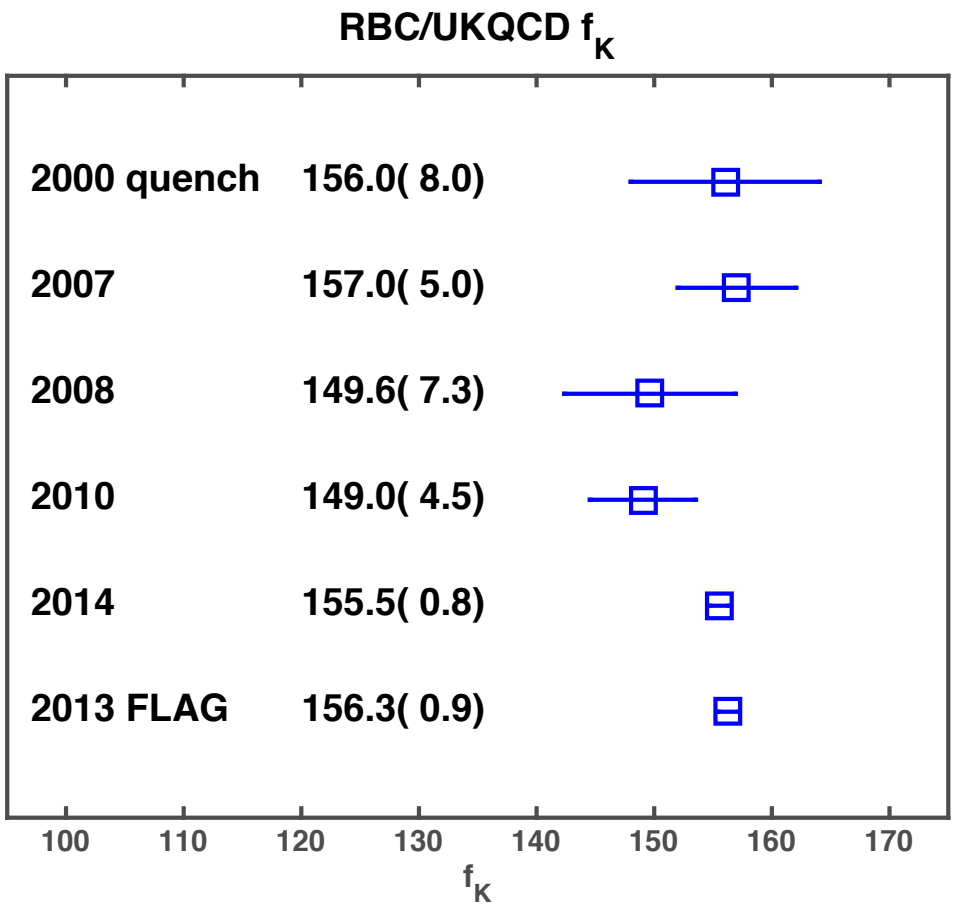
EDM $\sim 10^{-21}$ e·cm


Towards high precision muon g-2/EDM measurement at J-PARC
Nov. 28 – 29, 2016
Muon g-2/EDM
Workshop

Sub-percent accuracy on Physical point

- now adding on-physical point (M_π=135 MeV),
  2 lattice spacing a^{-1} = 1.7 and 2.4 GeV, V~(5.5 fm)^3 !

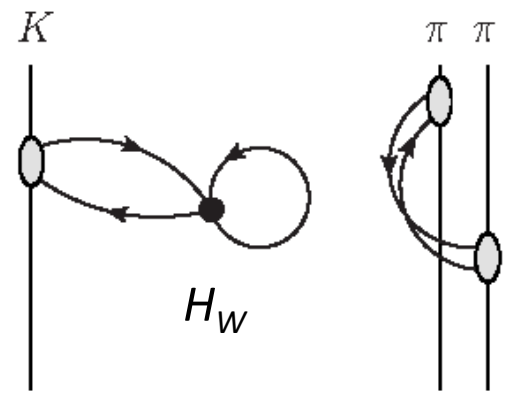




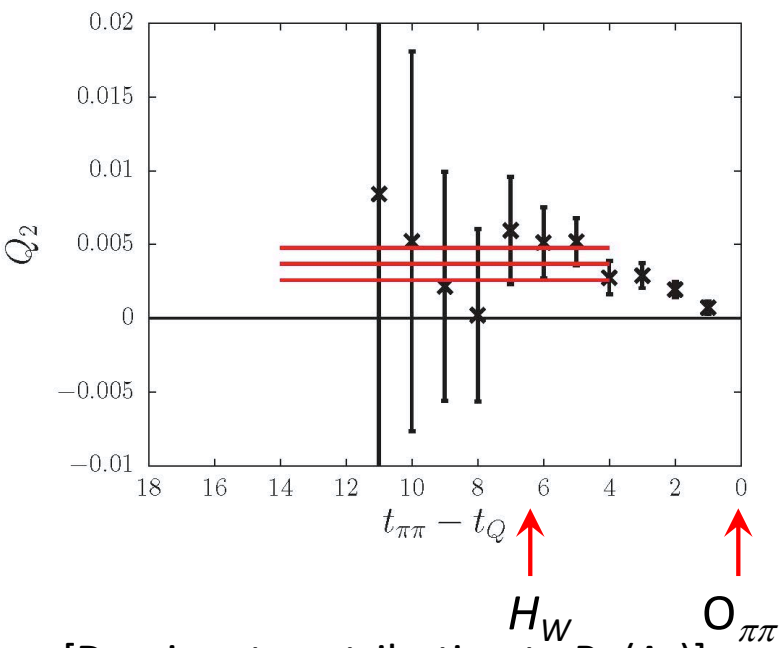
[R. Mawhinney]

$\Delta I = ½$ $K \rightarrow \pi\pi$ matrix elements

- Vary time separation between H_W and $\pi\pi$ operator.
- Show data for all K – H_W separations $t_Q - t_K \geq 6$ and $t_{\pi\pi} - t_K = 10, 12, 14, 16$ and 18.
- Fit correlators with $t_{\pi\pi} - t_Q \geq 4$
- Obtain consistent results for $t_{\pi\pi} - t_Q \geq 3$ or 5

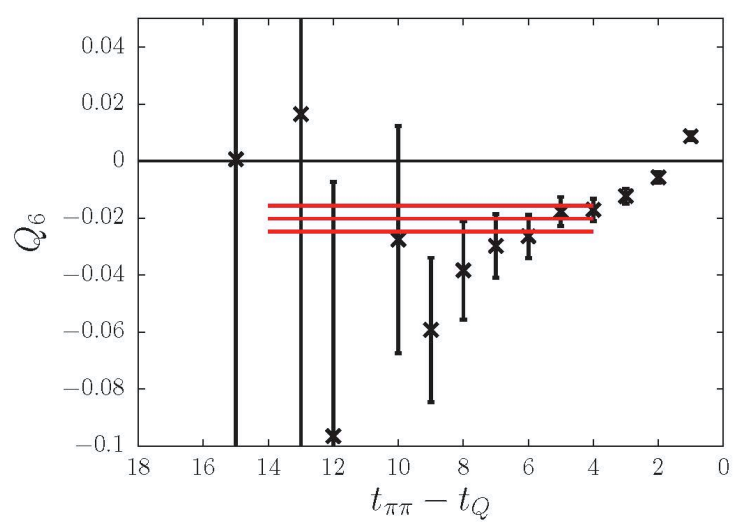


Q_2



[Dominant contribution to Re(A_0)]

Q_6



[Dominant contribution to Im(A_0)]

SM value of Re(ε'/ε)

$$\mathrm{Re}\left(\frac{\varepsilon'}{\varepsilon}\right) = \mathrm{Re}\left\{\frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\varepsilon}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right]\right\}$$

$= (1.38 \pm 5.15_{stat} \pm 4.59_{sys}) \times 10^{-4}$

Expt: $= (16.6 \pm 2.3) \times 10^{-4}$ [2.1 σ difference]

- Im(A_0), Im(A_2), δ_0 and δ_2 from lattice QCD
- Re(A_2) and Re(A_0) from measured decay rates
- $|\varepsilon| = 2.228(0.011) \times 10^{-3}$ from experiment
- arg(ε) = arctan($2\Delta M_K/\Gamma_S$) = 42.52^o (Bell-Steinberger relation)
- ¤$_0$ determined from phenomenology changes ¤'/¤ very small amount

[Blum, TI, Shintani PRD 88 (2013) 094503]

Covariant Approximation Averaging (CAA) a new class of Error reduction techniques

Original

$$\mathcal{O} = \mathcal{O}^{(\mathrm{appx})} + \mathcal{O}^{(\mathrm{rest})}$$

unbiased imporved

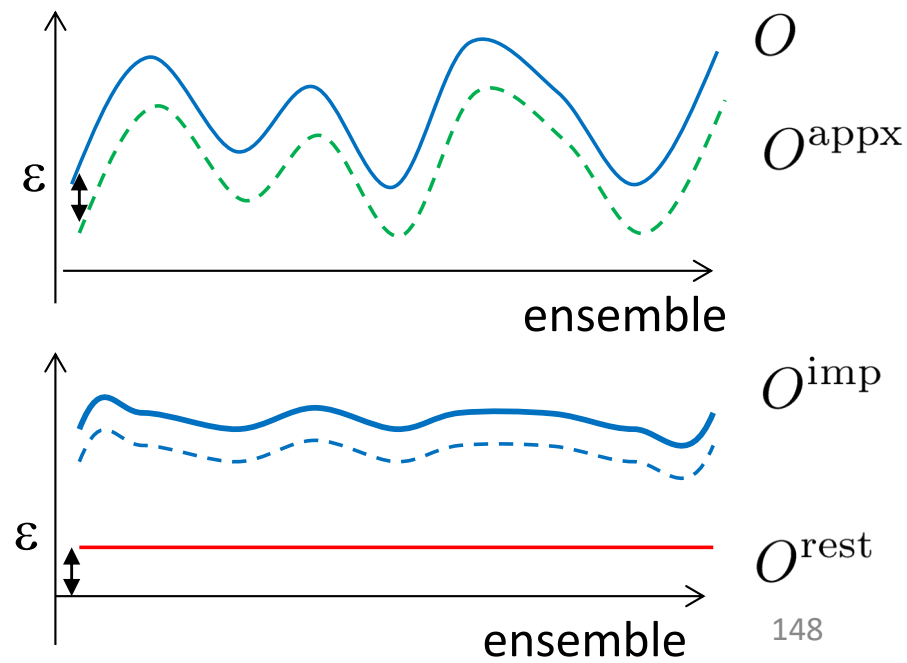
Lattice Symmetry

$$\mathcal{O}^{(\mathrm{imp})} = \mathcal{O}^{(\mathrm{rest})} + \frac{1}{N_G} \sum_{g \in G} \mathcal{O}^{(\mathrm{appx}),g}$$

Expensive : infrequently measured

Cheap : frequently measured

- $O^{(\mathrm{imp})}$ has smaller error
  $O^{(\mathrm{appx})}$ need to be cheap & not to be too accurate
  N_G suppresses the bulk part of noise cheaply



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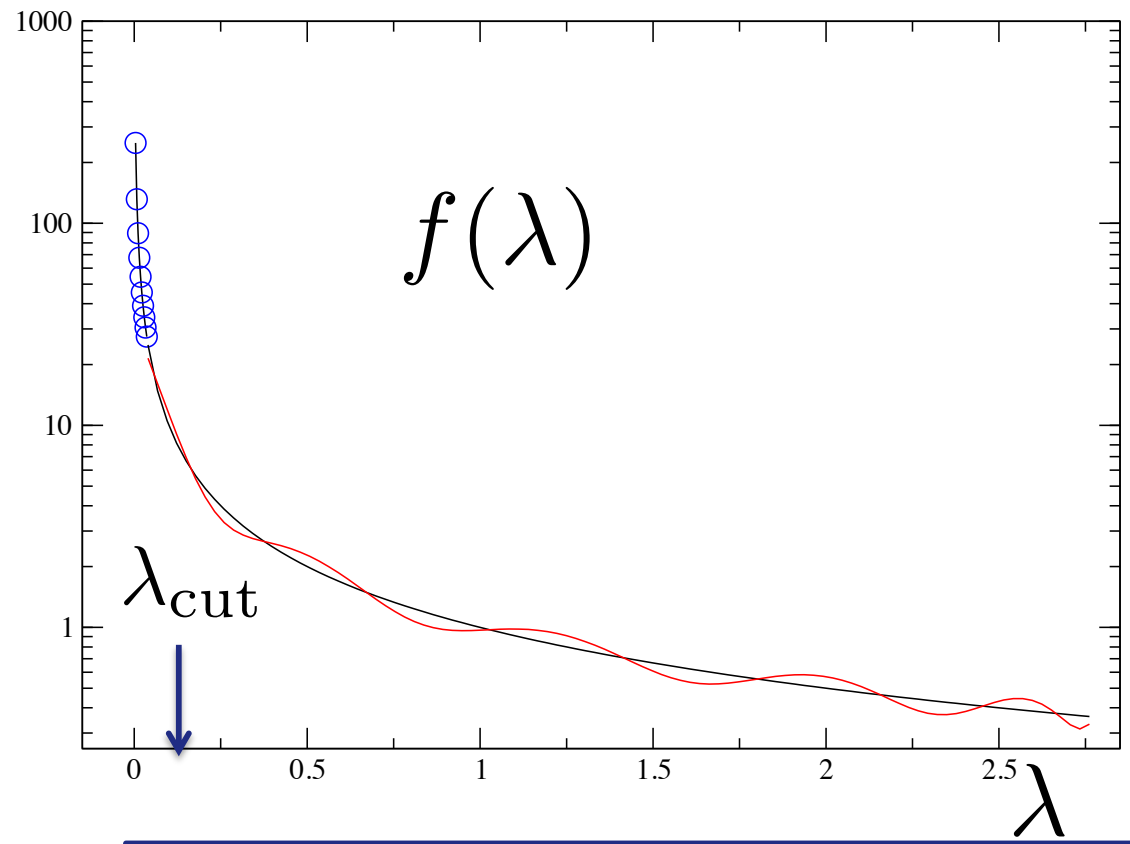

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